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ON THE LAW OF LARGE NUMBERS FOR WEIGHTED SUMS OF PAIRWISE NEGATIVELY QUADRANT DEPENDENT RANDOM VARIABLES

TAE-SUNG KIM AND HYUN-CHULL KIM

ABSTRACT. In this paper, we derive a general strong law of large numbers and a general weak law of large number for normed weighted sums of pairwise negative quadrant dependent random variables with the common distribution function.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables and let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants. Then $\{a_n X_n, n \geq 1\}$ is said to obey the general strong law of large numbers(SLLN) with norming constant $\{b_n, n \geq 1\}$ if the normed weighted sum $\sum_{j=1}^{n} a_j X_j / b_n$ converges to 0 almost surely, where $a_n \neq 0$ and $0 < b_n \uparrow \infty$ and $\{a_n X_n, n \geq 1\}$ is said to obey the general weak law of large number (WLLN) with centering $\{\nu_n, n \geq 1\}$ and norming constants $\{b_n, n \geq 1\}$ if the normed and centered weighted sum $\left(\sum_{j=1}^{n} a_j X_j - \nu_n\right) / b_n$ converges in probability to 0, where $\{\nu_n, n \geq 1\}$ is a suitable sequence of constants.

Feller(1946) proved that if $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed(i.i.d.) random variables such that $\sum_{n=1}^{\infty} P(|X_n| > b_n) < \infty$ and $\{b_n, n \geq 1\}$ is a sequence of positive constants with $b_n/n \uparrow \infty$ then $\left(\sum_{j=1}^{n} X_j / b_n\right) \to 0$ almost surely. Rosalsky(1987) improved Feller's SLLN to the sequence of pairwise independent and identically distributed random variables and Adler and Rosalsky(1989) generalized Feller's SLLN to the weighted sums of i.i.d. random variables. Furthermore, Adler, Rosalsky, and Taylor(1992) extended Adler and Rosalskys' theorem to the

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normed weighted sums of independent random variables which are stochastically dominated by a random variable X.

On the other hand, Chow and Teicher(1988) derived a classical WLLN for independent and identically distributed random variables under the condition $nP\{|X_1| > n\} = o(1)$, Adler and Rosalsky(1991) investigated a WLLN for the independent and identically distributed random variables and Adler, Rosalsky and Taylor(1991) generalized the earlier work of Adler and Rosalsky(1991), that is, they obtained a general WLLN for the normed weighted sums of independent random variables which are stochastically dominated by a random variable X.

In this paper, we study the SLLN and the WLLN for normed weighted sum of pairwise negative quadrant dependent(NQD) random variables with the same distribution $F(x)$.

In section 2, we derive some conditions for a general SLLN of the form $\left(\sum_{j=1}^{n} a_j X_j\right)/b_n \to 0$ a.s., where $\{a_n\}$ and $\{b_n\}$ are sequences of constants with $a_n > 0$ and $0 < b_n \uparrow \infty$ and in section 3, we also investigate a general WLLN of the form $\left(\sum_{j=1}^{n} a_j X_j - \nu_n\right)/b_n \xrightarrow{\mathcal{P}} 0$, where $\nu_n = \sum_{j=1}^{n} a_j X_{nj}$, $X_{nj} = X_j I(|X_j| < c_n) + c_n I(X_j > c_n) - c_n I(X_j < -c_n)$, $c_n = b_n/a_n > 0$.

2. A strong law of large numbers

In this section, for normed weighted sums of pairwise NQD random variables, we obtain conditions to establish a general strong law of large numbers of the form $(\sum_{j=1}^{n} a_j X_j)/b_n \to 0$ a.s., where $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ are sequences of constants with $a_n > 0$, $0 < b_n \uparrow$ and $b_n/a_n \uparrow$.

LEMMA 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables with the same distribution function and let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants satisfying $a_n > 0$, $0 < b_n \uparrow \infty$, $b_n/a_n \uparrow$,*

$$\sum_{j=n}^{\infty} 1/b_j^2 = O\ (1/b_n^2), \tag{1}$$

and

$$(\frac{b_n}{a_n})^2 \sum_{j=n}^{\infty} (\frac{a_j}{b_j})^2 = O(n). \tag{2}$$

Put

$$(3) \qquad X'_n = X_n I(|a_n X_n| \le b_n) + c_n I(a_n X_n > b_n) - c_n I(a_n X_n < -b_n),$$

where $c_0 = 0,\ c_n = b_n/a_n$.
Assume that

$$(4) \qquad \sum_{n=1}^{\infty} P\{|a_n X_n| > b_n\} < \infty$$

and

$$(5) \qquad \frac{\sum_{j=1}^{n} a_j E X'_j}{b_n} \longrightarrow 0.$$

Then

$$(6) \qquad \frac{\sum_{j=1}^{n} a_j X_j}{b_n} \longrightarrow 0 \ a.s.$$

Proof. First we are going to prove

$$(7) \qquad \frac{\sum_{j=1}^{n} a_j X'_j}{b_n} = \left[\frac{\sum_{j=1}^{n}(a_j X'_j - a_j E X'_j)}{b_n} + \frac{\sum_{j=1}^{n} a_j E X'_j}{b_n}\right] \to 0 \ a.s.$$

To prove (7), it is sufficient to show that the first term on the right-hand side of (7) converges to 0 a.s. since the second term on the right-hand side of (7) is o(1) by (5), that is, by the Borel-Cantelli lemma, it is enough to show that for $\epsilon > 0$

$$(8) \qquad \sum_{n=1}^{\infty} P\left(\left|\frac{\sum_{j=1}^{n}(a_j X'_j - a_j E X'_j)}{b_n}\right| > \epsilon\right) < \infty.$$

Since $(a_j X'_j - a_j E X'_j)$'s are pairwise NQD according to Lemma 2 in Matula (1992), we have

$$(9) \qquad \begin{aligned} &\sum_{n=1}^{\infty} E\left(\frac{\sum_{j=1}^{n}(a_j X_{j'} - a_j E X_{j'})}{b_n}\right)^2 \\ &\le \sum_{n=1}^{\infty} \frac{1}{b_{n^2}} \sum_{j=1}^{n} a_j^2 Var(X'_j) \\ &\le \sum_{n=1}^{\infty} \frac{1}{b_{n^2}} \sum_{j=1}^{n} a_j^2 E(X_{j'^2}). \end{aligned}$$

It follows from (1) that for some constant $d > 0$

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{b_n^2} \sum_{j=1}^{n} a_{j^2} E(X_{j'^2}) \\
&= \sum_{j=1}^{\infty} a_j^2 E(X_{j'^2}) \sum_{n=j}^{\infty} \frac{1}{b_n^2} \\
&\le d \sum_{j=1}^{\infty} \frac{1}{c_j^2} E(X_{j'^2}) \qquad (10) \\
&= d \sum_{j=1}^{\infty} P(|X_j| > c_j) + d \sum_{j=1}^{\infty} \frac{1}{c_j^2} E X_j^2 I(|X_j| \le c_j).
\end{aligned}
$$

Now we see that the first term on the right-hand side of (10) is finite by (4). Next, observe that (4) is equivalent to

$$
\sum_{n=1}^{\infty} n P(c_{n-1} < |X_1| \le c_n) < \infty. \qquad (11)
$$

(see, eg., Chow and Teicher [6], p.120)

Hence, by virtue of (2) and (11), we have

$$
\begin{aligned}
&\sum_{j=1}^{\infty} \frac{1}{c_j^2} E X_1^2 I(|X_1| \le c_j) \\
&= \sum_{j=1}^{\infty} \frac{1}{c_j^2} \sum_{n=1}^{j} E X_1^2 I(c_{n-1} < |X_1| \le c_n) \\
&= \sum_{n=1}^{\infty} E X_1^2 I(c_{n-1} < |X_1| \le c_n) \sum_{j=n}^{\infty} \frac{1}{c_j^2} \qquad (12) \\
&\le \sum_{n=1}^{\infty} c_n^2 P(c_{n-1} < |X_1| \le c_n) \sum_{j=n}^{\infty} \frac{1}{c_j^2} \\
&\le C \sum_{n=1}^{\infty} n P(c_{n-1} < |X_1| \le c_n) < \infty,
\end{aligned}
$$

where C is a positive constant. Therefore it follows from (12) that the second term on the right-hand side of (10) is finite and so by (9), (10), and Chebyshev's inequality (8) obtains, that is, (7) holds. Finally, from the definition of X_n, we have

$$\sum_{n=1}^{\infty} P(X_n \neq X'_n) = \sum_{n=1}^{\infty} P(|X_n| > c_n) < \infty \quad \text{(by (4))}.$$

Thus $P[X_n \neq X'_n \text{ i.o.}] = 0$ and so we conclude that the desired result (6) follows by (7).
The main result may now be established. It reduces to Theorem 6 of Adler, Rosalsky, and Taylor (1992) when $\{X_n, n \geq 1\}$ is a sequence of independent random variables which are stochastically dominated by a random variable X. □

THEOREM 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise* NQD *random variables with the same distribution and let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants satisfying $a_n > 0, 0 < b_n \uparrow \infty$,*

$$b_n/a_n \uparrow, \tag{13}$$

$$\frac{b_n}{na_n} \to \infty, \ \frac{b_n}{na_n} = O\left(inf_{j\geq n}\frac{b_j}{ja_j}\right), \tag{14}$$

and

$$\sum_{j=1}^{n} a_j = O(na_n). \tag{15}$$

If (1) *and* (4) *hold then the* SLLN (6) *obtains.*

Proof. Let $c_0 = 0, c_n = b_n/a_n$, $n \geq 1$. Define

$$X'_n = X_n I[|X_n| \leq c_n] + c_n I[X_n > c_n] - c_n I[X_n < -c_n].$$

As in the proof of Theorem 6 of Adler, Rosalsky, and Taylor (1992), it follows from (14) that

$$\left(\frac{b_n}{a_n}\right)^2 \sum_{j=n}^{\infty} \left(\frac{a_j}{b_j}\right)^2 \leq Cn^2 \sum_{j=n}^{\infty} \frac{1}{j^2} = O(n), \tag{16}$$

that is, (2) obtains and that (14) and (15) ensure that

$$\sum_{j=1}^{n} a_j = o(b_n). \tag{17}$$

It remains to obtain (5). Let $n > N \geq 1$. Now

$$\begin{aligned} &\frac{1}{b_n}\left|\sum_{j=1}^{n} a_j EX'_j\right| \\ (18) \qquad &\leq \frac{1}{b_n}\sum_{j=1}^{n} a_j E|X_j| I(|X_j| \leq c_j) \\ &+\frac{1}{b_n}\sum_{j=1}^{n} b_j P(|X_j| > c_j). \end{aligned}$$

The second term on the right-hand side of (18) is o(1) by (4) and the Kronecker lemma. Now recalling (13), it follows from (14) and (15) that the first term on the right-hand side of (18) is majorized by

$$(19) \qquad \frac{c_N}{b_n}\sum_{j=1}^{n} a_j + C\sum_{k=N+1}^{n} kP\{c_{k-1} < |X_1| \leq c_k\},$$

where $1 \leq N \leq n$. (see [5, p.352]).
Obviously, the first term on the right-hand side of (19) converges to 0 as $n \to \infty$ by (17) and the second term converges to 0 as $N \to \infty$ by (4).
Thus (5) obtains, and by Theorem 2.2, it follows from (4) and (16) that the desired result follows. □

3. A weak law of large numbers

In this section, we derive a general weak law of large numbers for pairwise negative quadrant dependent random variables with the common distribution function.

LEMMA 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables with the same distribution function. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants with $a_n > 0$, $0 < b_n \to \infty$. Put $c_n = b_n/a_n$ and define*

$$X_{nj} = X_j I(|X_j| \leq c_n) + c_n I(X_j > c_n) - c_n I(X_j < -c_n),\ 1 \leq j \leq n,\ n \geq 1.$$

If

$$(20) \qquad nP\left\{|X_1| > \frac{b_n}{a_n}\right\} = o(1)$$

then the WLLN

$$(21) \qquad \frac{\sum_{j=1}^{n} a_j (X_j - X_{nj})}{b_n} \xrightarrow{\mathcal{P}} 0$$

obtains.

Proof. For arbitrary $\epsilon > 0$,

$$\begin{aligned} P\left\{\frac{\left|\sum_{j=1}^{n} a_j(X_j - X_{nj})\right|}{b_n} > \epsilon\right\} &\le P\left\{\bigcup_{j=1}^{n}[X_j \ne X_{nj}]\right\} \\ &\le \sum_{j=1}^{n} P\{|X_j| > c_n\} \\ &\le nP\{|X_1| > c_n\} = o(1) \end{aligned}$$

by (20). Hence the desired result follows. □

By modifying methods of proof of Theorem 6 in Adler, Rosalsky, and Taylor (1991), we obtain the following result.

LEMMA 3.2. *Let $\{X_n, n \ge 1\}$ be a sequence of random variables with the same distribution function. Let $\{a_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$ be sequences of constants with $a_n > 0$, $0 < b_n \to \infty$ and suppose that either*

$$\frac{b_n}{a_n} \uparrow,\ \frac{b_n}{na_n} \downarrow,\ \sum_{j=1}^{n} a_j^2 = o(b_n^2),\ \text{and}\ \sum_{j=1}^{n} \frac{b_j^2}{j^2 a_j^2} = O\left(\frac{b_n^2}{\sum_{j=1}^{n} a_j^2}\right) \tag{22}$$

or

$$\frac{b_n}{a_n} \uparrow,\ \frac{b_n}{na_n} \to \infty,$$

$$\sum_{j=1}^{n} a_j^2 = O(na_n^2),\ \text{and}\ \sum_{j=1} n\frac{b_j^2}{j^2 a_j^2} = O\left(\frac{b_n^2}{\sum_{j=1}^{n} a_j^2}\right) \tag{23}$$

or

$$\frac{b_n}{na_n} \uparrow,\ \text{and}\ \sum_{j=1}^{n} a_j^2 = O(na_n^2) \tag{24}$$

holds. Then (20) entails that

$$\sum_{j=1}^{n} a_j^2 P\{|X_1| > c_n\} = o(a_n^2) \tag{25}$$

and

$$\sum_{j=1}^{n} a_j^2 EX_1^2 I(|X_1| \le c_n) = o(b_n^2) \tag{26}$$

hold, where $c_n = b_n/a_n$.

LEMMA 3.3. *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise* NQD *random variables with the same distribution function. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be constants with $a_n > 0,\ 0 < b_n \to \infty,\ n \geq 1$ and suppose that either* (22) *or* (23) *or* (24) *holds. If* (20) *holds then the* WLLN

$$\frac{\sum_{j=1}^{n} a_j(X_{nj} - EX_{nj})}{b_n} \xrightarrow{\mathcal{P}} 0 \tag{27}$$

obtains, where $X_{nj} - EX_{nj}$ is defined as in Lemma 3.1.

Proof. First note that $\{X_{nj} - EX_{nj}\}$'s are pairwise NQD by Lemma 2 of Matula (1992). It follows from Lemma 3.2 that for arbitrary $\epsilon > 0$,

$$\begin{aligned}
&P\left\{\frac{\left|\sum_{j=1}^{n} a_j(X_{nj} - EX_{nj})\right|}{b_n} > \epsilon\right\} \\
&\leq \frac{1}{\epsilon^2 b_n^2} E\left|\sum_{j=1}^{n} a_j(X_{nj} - EX_{nj})\right|^2 \\
&(by\ Chebyshev's\ inequality) \\
&\leq \frac{1}{\epsilon^2 b_n^2} \sum_{j=1}^{n} a_j^2 E(X_{nj} - EX_{nj})^2 \\
&(by\ pairwise\ negative\ quadrant\ dependence) \\
&\leq \frac{1}{\epsilon^2 b_n^2} \sum_{j=1}^{n} a_j^2 E(X_{nj}^2) \\
&\leq \frac{1}{\epsilon^2 b_n^2} \sum_{j=1}^{n} a_j^2 EX_j^2 I(|X_j| \leq c_n) + \frac{1}{\epsilon^2 b_n^2} \sum_{j=1}^{n} a_j^2 c_n^2 P\{|X_j| > c_n\} \\
&\leq \frac{1}{\epsilon^2 b_n^2} \sum_{j=1}^{n} a_j^2 E|X_1|^2 I(|X_1| \leq c_n) + \frac{1}{\epsilon^2 a_n^2} \sum_{j=1}^{n} a_j^2 P\{|X_1| > c_n\} \\
&= o(1)
\end{aligned}$$

by (25) and (26). Thus the desired result (27) follows. □

From Lemmas 3.1 and 3.3, we obtain the following result :

THEOREM 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise* NQD *random variables with the same distribution function. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be constants with $a_n > 0,\ 0 < b_n \to \infty,\ n \geq 1$, and suppose that either*

(22) *or* (23) *or* (24) *holds. If* (20) *holds then the* WLLN

$$\frac{\sum_{j=1}^{n} a_j(X_j - EX_{nj})}{b_n} \xrightarrow{\mathcal{P}} 0 \tag{28}$$

obtains, where $X_{nj} = X_j I(|X_j| \le c_n) + c_n I(X_j > c_n) - c_n I(X_j < -c_n)$, $1 \le j \le n,\ n \ge 1$ *and* $c_n = b_n/a_n$.

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TAE SUNG KIM, DIVISION OF MATHEMATICAL SCIENCE, WONKWANG UNIVERSITY, IKSAN, CHONBUK, 570-749, KOREA
E-mail: starkim@wonnms.wonkwang.ac.kr

HYUN CHULL KIM, DIVISION OF COMPUTER SCIENCE, DAEBUL UNIVERSITY, YOUNGNAM, CHONNAM, 526-800
E-mail: kimhc@daebull.daebull.ac.kr