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PROXIMINALITY OF CERTAIN SPACES OF COMPACT OPERATORS

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Abstract. For any closed subspace X of ℓ_p, $1 < p < \infty$, $K(X)$ is proximinal in $L(X)$, and if X is a Banach space with an unconditional shrinking basis, then $K(X, c_0)$ is proximinal in $L(X, \ell_\infty)$.

1. Introduction

A closed subspace J of a Banach space X is said to be proximinal in X if for every $x \in X \setminus J$, there is $j_0 \in J$ such that

$$\|x - j_0\| = \inf\{\|x - j\| : j \in J\}.$$

An element $j_0 \in J$ satisfying the above equality is called a best approximation of x in J.

Obviously, every finite dimensional subspace of a Banach space X is proximinal in X and every closed subspace in a Hilbert space H is proximinal in H. It is known that a Banach space X is reflexive if and only if every closed subspace is proximinal in X [11].

Many authors have studied the problem of determining those Banach spaces X and Y for which $K(X, Y)$, the space of compact linear operators from X to Y is proximinal in $L(X, Y)$, the space of bounded linear operators from X to Y [2, 3, 4, 6, 8, 9, 13, 14]. We will write $L(X)$ for $L(X, X)$ and $K(X)$ for $K(X, X)$. It is known that $K(c_0)$, $K(X, c_0)$ for every Banach space X and $K(\ell_p, \ell_q)$ for $1 < p, q < \infty$ are M-ideals in the corresponding spaces of bounded linear operators [11, 13, 14], and hence are proximinal, while $K(X)$ is not proximinal in $L(X)$ if $X = \ell_\infty$ or $L_p(0,1)$, $1 \leq p \leq \infty$, $p \neq 2$ [4, 6].

An M-ideal has a very strong approximation property. If J is an M-ideal in a Banach space X and $x \in X \setminus J$, then the set of best approximations of x in J algebraically spans J [8]. The notion of an M-ideal was introduced

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by Alfsen and Effros [1]. They [1] also characterized an M-ideal by n-ball properties of open balls. In terms of closed balls, a closed subspace J of a Banach space X is said to have n-ball property in X if for any family $\{B(a_i, r_i)\}_{i=1}^n$ of closed balls in X with centers a_i and radii r_i such that $\cap_{i=1}^n B(a_i, r_i) \neq \emptyset$ and $J \cap B(a_i, r_i) \neq \emptyset$ $(i = 1, 2, \cdots, n)$ we have

$$J \cap (\cap_{i=1}^n B(a_i, r_i + \varepsilon)) \neq \emptyset \quad \text{for all } \varepsilon > 0.$$

One of the main results of Alfsen and Effros [1] is that J is an M-ideal in X if and only if it has 3-ball property (equivalently, n-ball property for all $n \geq 3$).

2-ball property with one of a_i in J is called $1\frac{1}{2}$-ball property. If a closed subspace J of a Banach space X has $1\frac{1}{2}$-ball property then J is proximinal in X [13, 15].

The purpose of this paper is to investigate the proximinality of spaces of compact operators. From Corollary 3, we obtain a fact that $K(X)$ is proximinal in $L(X)$ for every closed subspace of ℓ_p $(1 < p < \infty)$. In Theorem 4 we will also prove that $K(X, c_0)$ is proximinal in $L(X, \ell_\infty)$ if X is a Banach space with an unconditional shrinking basis.

2. Results

For a Banach space X, B_X will denote the closed unit ball of X. To prove part of our results we need the following theorem of Godini [7].

THEOREM 1 [7]. *Let M be a closed subspace of a Banach space X and $\pi : X \to X/M$ the natural projection onto the quotient space X/M. Then M is proximinal in X if and only if $\pi(B_X)$ is closed in X/M.*

Using the above theorem we can easily obtain a useful general fact about a proximinality.

THEOREM 2. *Suppose Y is a closed subspace of a Banach space Z. If Z_1 is a proximinal subspace of Z and Y_1 is a closed subspace of Y with $Y_1 \subseteq Z_1$, then Y_1 is proximinal in Y.*

Proof. Let $\pi_Z : Z \to Z/Z_1$ and $\pi_Y : Y \to Y/Y_1$ be natural projections onto corresponding quotient spaces. Obviously the map $\phi : Y/Y_1 \to Z/Z_1$ defined by $\phi(y + Y_1) = y + Z_1$ for $y \in Y$ is a norm decreasing linear map.

Since Z_1 is proximinal in Z, by Theorem 1, $\pi_Z(B_Z)$ is closed in Z/Z_1 and hence $\pi_Y(B_Y) = \phi^{-1}(\pi_Z(B_Z))$ is closed in Y/Y_1. Therefore, Y_1 is proximinal in Y. □

There are several Banach spaces X and Y for which $K(X,Y)$ is proximinal in $L(X,Y)$ [2, 3, 5, 8, 13, 14, 15]. It is known that if X is a closed subspace of ℓ_p $(1 < p < \infty)$, then $K(X,\ell_p)$ is an M-ideal in $L(X,\ell_p)$ [5]. By Theorem 2 we can find a larger class of spaces of compact operators which are proximinal in the corresponding spaces of bounded linear operators. More specifically we have the following corollary.

COROLLARY 3. *Suppose X and Y are Banach spaces for which $K(X,Y)$ is proximinal in $L(X,Y)$. If Z is a closed subspace in Y, then $K(X,Z)$ is proximinal in $L(X,Z)$. In particular, if X is a closed subspace of ℓ_p $(1 < p < \infty)$, then $K(X)$ is proximinal in $L(X)$.*

As mentioned earlier, $1\frac{1}{2}$-ball property is a sufficient condition for the proximinality. Recall that a closed subspace J of a Banach space X is said to have the $1\frac{1}{2}$-ball property in X if $x \in X$, $j \in J$, $r_1, r_2 > 0$, $B(x,r_1) \cap B(j,r_2) \neq \emptyset$ and $B(x,r_1) \cap J \neq \emptyset$ implies that $J \cap B(x,r_1) \cap B(j,r_2) \neq \emptyset$. Using $1\frac{1}{2}$-ball property we will prove the following theorem.

THEOREM 4. *If X is a Banach space with an unconditional shrinking basis, then $K(X,c_0)$, the space of all compact linear operators from X to c_0 is proximinal in $L(X,\ell_\infty)$, the space of all bounded linear operators from X to ℓ_∞.*

Proof. To simplify our proof, we will assume that X is reflexive. Let X have an unconditional shrinking basis $\{x_n\}_{n=1}^\infty$ with biorthogonal functional $\{x_n^*\}_{n=1}^\infty$ (i.e. $x_n^*(x_m) = \delta_{n,m}$). Then $X^* = [x_n^*]_{n=1}^\infty$, closed linear span of $\{x_n^*\}_{n=1}^\infty$. Let P_n be the natural projection associated with $\{x_n\}_{n=1}^\infty$ (i.e. $P_n x = \sum_{i=1}^n \alpha_i x_i$ if $x = \sum_{i=1}^\infty \alpha_i x_i$), then P_n^* is the natural projection associated with the basis $\{x_n^*\}_{n=1}^\infty$. By abuse of the notation P_n will also denote the norm-one projection on ℓ_∞ defined by $P_n a = (\alpha_1, \cdots, \alpha_n, 0, 0, \cdots)$ for $a = (\alpha_1, \cdots, \alpha_n, \cdots) \in \ell_\infty$.

We will prove that $K(X,c_0)$ has the $1\frac{1}{2}$-ball property in $L(X,\ell_\infty)$.

Let $T \in L(X,\ell_\infty)$, $G \in K(X,c_0)$, $r_1, r_2 > 0$, $B(T,r_1) \cap B(G,r_2) \neq \emptyset$ and $B(T,r_1) \cap K(X,c_0) \neq \emptyset$.

The first step is expressing $\|T\|$ using the matrix representation of T. Since $T^* : \ell_\infty^* \to X^*$ is weak*-to-weak* continuous and B_{ℓ_1} is weak*-dense in $B_{\ell_1^{**}} (= B_{\ell_\infty^*})$, $\|T^*\| = \|T^*|_{\ell_1}\|$. Since $T^*|_{\ell_1} : \ell_1 \to X^*$ is a bounded linear operator, and since X is reflexive, $(T^*|_{\ell_1})^* : X \to \ell_\infty$ is a bounded linear operator. We can easily check that $(T^*|_{\ell_1})^* = T$.

Therefore, we have

$$\|T\| = \|(T^*|_{\ell_1})\| = \sup_{1 \leq j < \infty} \|T^* e_j\|,$$

where $\{e_j\}_{j=1}^{\infty}$ is the unit vector basis for ℓ_1. If X is not reflexive, we can simply replace $(T^*|_{\ell_1})^*$ by $(T^*|_{\ell_1})^*|_X$ in the above argument.

Observe that coordinates of T^*e_j form the j-th column of the matrix of $T^*|_{\ell_1}$ (rel. to base $\{e_n\}_{n=1}^{\infty}$ and $\{x_n^*\}_{n=1}^{\infty}$) and the j-th row of the matrix of $T \in L(X, \ell_\infty)$.

Let T have the matrix representation $T = (t_{ij})$. Then P_nT and $T - P_nT$ have matrix representations

$$P_nT = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1j} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ t_{n1} & t_{n1} & \cdots & t_{nj} & \cdots \\ \hline 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

$$(1) \qquad T - P_nT = \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \cdots \\ \hline t_{n+1\,1} & t_{n+1\,2} & \cdots & t_{n+1\,j} & \cdots \\ t_{n+2\,1} & t_{n+2\,2} & \cdots & t_{n+2\,j} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Given $0 < \varepsilon < \frac{r_1 - d}{2}$, we choose $K \in K(X, c_0)$ such that $\|T - K\| < d + \varepsilon$, where $d = \text{dist}(T, K(X, c_0))$.

If $K \in K(X, c_0)$ has the matrix representation $K = (k_{ij})$, then $T - P_nK$ is represented by

$$(2) \qquad T - P_nK = \begin{pmatrix} t_{11} - k_{11} & \cdots & t_{1j} - k_{1j} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ t_{n1} - k_{n1} & \cdots & t_{nj} - k_{nj} & \cdots \\ \hline t_{n+1\,1} & \cdots & t_{n+1\,j} & \cdots \\ t_{n+2\,1} & \cdots & t_{n+2\,j} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

From (1) and (2), we have $\|T - P_nT\| \le \|T - P_nK\|$.

Since c_0 has the unit vector basis $\{e_n\}_{n=1}^{\infty}$, for sufficiently large n, $\|K - P_nK\| < \varepsilon$. Thus

$$\begin{aligned} \|T - P_nT\| &\le \|T - P_nK\| \\ &\le \|T - K\| + \|K - P_nK\| \\ &\le d + 2\varepsilon < r_1. \end{aligned}$$

If $S \in B(T, r_1) \cap B(G, r_2)$, then

$$\begin{aligned} \|T - P_n S\| &= \max\{\|P_n(T - P_n S)\|, \|(I - P_n)(T - P_n S)\|\} \\ &= \max\{\|P_n(T - S)\|, \|(I - P_n)T\|\} \\ &< r_1. \end{aligned}$$

Similarly, $\|G - P_n S\| < r_2$.

Therefore, $P_n S \in B(T, r_1) \cap B(G, r_2) \cap K(X, c_0)$. This completes the proof. □

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