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ESTIMATES IN EXIT PROBABILITY FOR SOLUTIONS OF NUCLEAR SPACE-VALUED SDE

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Abstract. We consider a solution process of stochastic differential equation(SDE) driven by $\mathcal{S}'(R^d)$-valued Wiener process and study a large deviation type of estimates for the process. We get an upper bound in exit probability for such a process to leave a ball of radius r before a finite time t. We apply the Ito formula to the SDE under the structure of nuclear space.

1. Introduction

Let $\beta_t = (\beta_t^1, ..., \beta_t^d)$ be a standard Brownian motion in R^d and consider the following $d-$dimensional Ito process:

$$v_t = \int_0^t f_s(\omega)ds + \int_0^t \sigma_s(\omega)d\beta(s),$$

where f_t and σ_t are $\mathcal{F}_t$-progressively measurable process with values in R^d and $R^d \times R^d$, respectively. Suppose that $|f_t| \leq k$ and the trace $tr(\sigma_t^* \sigma_t) \leq m$, for all $t > 0$. Then the following exponential estimate holds[10]:

$$P\{\sup_{0\leq s\leq t} |v_s| \geq r\} \leq 2d\exp\{-\frac{(r - k\sqrt{d}t)^2}{2mdt}\}, \tag{1.1}$$

for $r > k\sqrt{d}t$, which depends on d. This estimate is useful in studying the large deviation problem and path properties for a diffusion process. Chow and Menaldi[2] extended the above estimate into a Hilbert space valued Ito process using a different approach independent of the dimension.

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In this paper we want to get a similar exit probability for a diffusion process in $\mathcal{S}'(R^d)$. Here $\mathcal{S}'(R^d)$ is the dual of Schwartz space $\mathcal{S}(R^d)$. Since $\mathcal{S}'(R^d)$ is not a Hilbert space the proofs can not be extended to this conuclear space-valued SDE in a straightforward manner.

We are going to review some structure of the Schwartz space as a nuclear space in the next section. For the reader's convenience we restrict ourselves to the Schwartz space and its dual space rather than we consider a general nuclear space. This $\mathcal{S}(R^d), \mathcal{S}'(R^d)$ can be replaced by any countably Hilbertian nuclear space and its dual space, respectively under minor assumptions.

Let $W(t)$ be a standard $\mathcal{S}'(R^d)$–valued Wiener process, i. e., it is a Wiener process with

$$E\exp(iW(t)[\xi]) = \exp -t(|\xi|_0^2/2), \text{ for all } \xi \in \mathcal{S}(R^d),$$

where $|\cdot|_0$ is the usual L_2–norm in R^d. Thus the covariance functional Q of $W(t)$ is given by $Q = I$.

We consider the following $\mathcal{S}'(R^d)$-valued stochastic differential equation(SDE):

$$X(t) = X(0) + \int_0^t F(s, X(s))ds + \int_0^t G(s, X(s))dW(s), \tag{1.2}$$

where $F : R^+ \times \mathcal{S}'(R^d) \to \mathcal{S}'(R^d)$ and $G : R^+ \times \mathcal{S}'(R^d) \to L(\mathcal{S}'(R^d), \mathcal{S}'(R^d))$ are two measurable functions, and $L(B, C)$ denotes the space of operators from B into C. SDEs of the type (1.2) have been studied by several authors. For instance, Kallianpur, Mitoma and Wolpert[4], Kallianpur, et al.[5], Tuckwell[11], and Walsh[12] have researched the linear and quasi-linear equations.

The existence and uniqueness of solution of equation (1.2) is well known, which is stated in the next section. It is also known that under some conditions there exist some Hilbert spaces which are embedded in $\mathcal{S}'(R^d)$ such that X_t has regular versions in these Hilbert spaces for $0 \le t \le T$. Our method is based on this fact.

2. Preliminaries

Let $\mathcal{S}(R^d)$ be the Schwartz space on R^d consisting of C^∞ functions which together with all their derivatives vanish at infinity faster than any power of $|x|$. $\mathcal{S}(R^d)$ is a separable Frechet space whose topology is given by the following increasing sequence $\{|\cdot|_p; p = 1, 2, ...\}$ of Hilbertian norms;

$$|\phi|_p^2 = \int (S^p\phi)^2 ds, \quad \text{where } S\phi = |x|^2\phi - \Delta\phi.$$

Let $\mathcal{S}_p$ denote the completion of $\mathcal{S}(R^d)$ with respect to the norm $|\cdot|_p$. For each p there exists $q > p$ (in fact, $q > p+\frac{d}{2}$) such that the canonical inclusion from $\mathcal{S}_q$ into $\mathcal{S}_p$ is a Hilbert-Schmidt operator. That is $|\cdot|_p <_{\mathrm{HS}} |\cdot|_q$.

Let $\mathcal{S}'(R^d)$ and $\mathcal{S}_{-p}$ denote the dual spaces of $\mathcal{S}(R^d)$ and $\mathcal{S}_p$, respectively. Let $|\cdot|_{-p}$ denote the norm on $\mathcal{S}_{-p}$. In fact we have $\mathcal{S}_0 = L_2(R^d)$ and the following continuous inclusions

$$\cap_{p=1}^{\infty}\mathcal{S}_p = \mathcal{S}(R^d) \subset \mathcal{S}_q \subset \mathcal{S}_p \subset \mathcal{S}_0 \subset \mathcal{S}_{-p} \subset \mathcal{S}_{-q} \subset \mathcal{S}'(R^d) = \cup_{p=1}^{\infty}\mathcal{S}_{-p}.$$

It is known that (e.g. see [1] or [12]) there exists a sequence $\{\xi_j; j = 1, 2....\}$ in $\mathcal{S}(R^d)$ such that $\{\xi_j\}$ is a CONS for $\mathcal{S}_0$ and is a complete orthogonal system in $\mathcal{S}_p$ for any p, $p = 0, 1, 2....$ For each positive integer p, let $\xi_j^{(p)} \equiv |\xi_j|_p^{-1}\xi_j$. Then $\{\xi_j^{(p)}; j \geq 1\}$ is a CONS for $\mathcal{S}_p$.

Define θ_p to be the isometric linear operator

$$\theta_p : \mathcal{S}_{-p} \to \mathcal{S}_p$$

such that $\theta_p\xi_j^{(-p)} = \xi_j^{(p)}$ for all $j \geq 1$. For each p the restriction of θ_p to $\mathcal{S}(R^d)$ is a continuous linear operator from $\mathcal{S}(R^d)$ into itself.

Now, we introduce a set of assumptions and state a theorem from [4] on the existence and uniqueness of solution of SDE(1.2).

Assumption (A). *For any $T > 0$, there exists $p_0 = p_0(T) \geq 0$ such that for any $p \geq p_0$ we can find $q = q(p) \geq p$ and $K = K(p,q) \in L^1([0,T])$ satisfying the following conditions;*

A1) *(Continuity) $F(t,\cdot) : \mathcal{S}_{-p} \to \mathcal{S}_{-q}$ and $G(t,\cdot) : \mathcal{S}_{-p} \to L(\mathcal{S}_0, \mathcal{S}_{-p})$ are continuous for all $t \in [0,T]$. Here $L(B,C)$ denotes the space of Hilbert-Schmidt operators from B into C.*

A2) *(Coercivity) For all $t \in [0,T]$ and $\xi \in \mathcal{S}_0$, we have*

$$2F(t,\xi)[\theta_p\xi] \leq K(t)(1 + |\xi|_{-p}^2).$$

A3) *(Growth) For all $t \in [0,T]$ and $\xi \in \mathcal{S}_{-p}(R^d)$, we have*

$$|F(t,\xi)|_{-q}^2 + \|G(t,\xi)\|_{L(\mathcal{S}_0,\mathcal{S}_{-p})}^2 \leq K(t)(1 + |\xi|_{-p}^2).$$

A4) *(Monotonicity) For all $t \in [0,T]$ and $\xi_1, \xi_2 \in \mathcal{S}_{-p}$, we have*

$$\begin{aligned} 2\langle F(t,\xi_1) - F(t,\xi_2), \xi_1 - \xi_2\rangle_{-q} + \|G(t,\xi_1) - G(t,\xi_2)\|_{L(\mathcal{S}_0,\mathcal{S}_{-q})}^2 \\ \leq K(t)(|\xi_1 - \xi_2|_{-q}^2). \end{aligned}$$

A5) *For all $t \in [0,T]$ and $\xi \in \mathcal{S}_{-p}$, we have*

$$\|G(t,\xi)\|_{L(\mathcal{S}_0,\mathcal{S}_{-p})}^2 \leq K(t).$$

Let $C_{\mathcal{S}_{-p_1}}[0,T]$ be the sapce of all continuous mappings of $[0,T]$ to $\mathcal{S}_{-p_1}$.

THEOREM 2.1. *Suppose F and G satisfy Assumption A1)-A4). Then the $\mathcal{S}'(R^d)$–valued Stochastic differential equation (1.2) has a unique solution X. Moreover, if $X(0) \in \mathcal{S}_{-r_0}$ and $p_1 \geq p_0 \vee r_0$ such that the inclusion map from $\mathcal{S}_{(-p_0 \vee r_0)}$ into $\mathcal{S}_{-p_1}$ is a Hilbert-Schmidt operator, then $X|_{[0,T]} \in C_{\mathcal{S}_{-p_1}}[0,T]$ a.s. and*

$$E[\sup_{0\leq t\leq T} |X(t)|^2_{-p_1}] < \infty.$$

This theorem implies that we can regard the solution process as a $\mathcal{S}_{-p}$-valued process for sufficiently large p. This kind way of regularizing is also used in [3] and [1].

3. Main theorem

We adapt the following lemma and its proof from [9].

LEMMA 3.1. *For any integer $p > 0$,*

$$\begin{aligned} |X(t)|^2_{-p} &= |X(0)|^2_{-p} + 2\int_0^t F(s,X(s))[\theta_p X(s)]ds \\ &\quad + 2\sum_{k=1}^{\infty}\int_0^t (G(s,X(s))\xi_k)[\theta_p X(s)]dW(s)[\xi_k] \qquad (3.1) \\ &\quad + \int_0^t \|G(s,X(s))\|^2_{L(\mathcal{S}_0,\mathcal{S}_{-p})}ds. \end{aligned}$$

Proof. As before, let $\{\xi_j : j \geq 1\}$ be a CONS for $\mathcal{S}_0$. Let $\xi_j^{(p)} \equiv |\xi_j|_p^{-1}\xi_j$. Then $\{\xi_j^{(p)} : j \geq 1\}$ forms a CONS for $\mathcal{S}_p$. For any $j \geq 1$, we have

$$\begin{aligned} X(t)[\xi_j^{(p)}] &= X(0)[\xi_j^{(p)}] + \int_0^t F(s,X(s))[\xi_j^{(p)}]ds \\ &\quad + \sum_{k=1}^{\infty}\int_0^t (G(s,X(s))\xi_k)[\xi_j^{(p)}]dW(s)[\xi_k]. \end{aligned}$$

Apply the Ito formula to get

$$\begin{aligned} (X(t)[\xi_j^{(p)}])^2 &= (X(0)[\xi_j^{(p)}])^2 + 2\int_0^t X(s)[\xi_j^{(p)}]F(s,X(s))[\xi_j^{(p)}]ds \\ &\quad + \sum_{k=1}^{\infty}\int_0^t 2X(s)[\xi_j^{(p)}](G(s,X(s))\xi_k)[\xi_j^{(p)}]dW(s)[\xi_k] \qquad (3.2) \\ &\quad + \int_0^t |G(s,X(s))^*\xi_j^{(p)}|_0^2 d(s), \end{aligned}$$

where G^* denotes the adjoint of G. Note that $\theta_p X(s) = \sum_{j=1}^{\infty} X(s)\ [\xi_j^{(p)}]$ $\xi_j^{(p)}$ and

$$\begin{aligned}\|G(s,X(s))\|^2_{L(\mathcal{S}_0,\mathcal{S}_{-p})} &= \|G(s,X(s))^*\|^2_{L(\mathcal{S}_{-p},\mathcal{S}_0)}\\ &= \sum_{j=1}^{\infty} |G(s,X(s))^*\xi_j^{(p)}|_0^2.\end{aligned}$$

Hence if we sum up the equation (3.2) over j we obtain (3.1). □

For following estimations let

$$\phi_\lambda(x) = (1+\lambda x)^{\frac{1}{2}}, \quad \lambda, x \in R_+.$$

For convenience, we abuse some notations as $|\cdot|_{-p} = \|\cdot\|$ and $\|G(s,X(s))\|^2_{L(\mathcal{S}_0,\mathcal{S}_{-p})} = \|\cdot\|$. Also, we define

$$\begin{aligned}\eta_t^\lambda \equiv &\lambda \sum_{k=1}^{\infty} \int_0^t \phi_\lambda^{-1}(\|X(s)\|^2)(G(s,X(s))\xi_k)[\theta_p(X(s))]dW_s[\xi_k]\\ &-\frac{1}{2}\lambda^2 \int_0^t \phi_\lambda^{-2}(\|X(s)\|^2)|G(s,X(s))^*\theta_p(X(s))|_0^2 ds,\end{aligned} \tag{3.3}$$

$$\mu_t \equiv \int_0^t K(s)ds. \tag{3.4}$$

Lemma 3.2. *For every $\lambda \in R^+$, $Z_t^\lambda \equiv \exp\{\eta_t^\lambda\}$ is a local martingale so that $EZ_t^\lambda = 1$.*

Proof. It follows easily from the Ito formula. □

Theorem 3.3. *Suppose F and G in SDE(1.2) satisfy the Assumption (A). If $X(0) = 0$ then there exists a positive constant μ_t, independent of r, such that*

$$P\Big\{\sup_{0\le s\le t} |X(s)|^2_{-p} \ge r\Big\} \le \exp\Big\{1 - (\frac{e^{-2\mu_t}\cdot r}{6\mu_t} + \frac{3\mu_t}{2r})\Big\},$$

for any $t \in [0,T]$

Proof. We again apply the Ito formula to (3.1), then

$$\begin{aligned}
&\phi_\lambda(\|X(t)\|^2)\\
=&\phi_\lambda(\|X(0)\|^2) + \lambda \int_0^t \phi_\lambda^{-1}(\|X(s)\|^2)(F(s,X(s))[\theta_p X(s)]\\
&+ \frac{1}{2}\|G(s,X(s))\|^2)ds\\
&+ \lambda \sum_{k=1}^{\infty} \int_0^t \phi_\lambda^{-1}(\|X(s)\|^2)(G(s,X(s))\xi_k)[\theta_p(X(s))]dW_s[\xi_k] \qquad (3.5)\\
&- \lambda^2 \int_0^t \phi_\lambda^{-3}(\|X(s)\|^2)|G(s,X(s))^*(\theta_p X(s))|_0^2 ds\\
\leq&\phi_\lambda(\|X(0)\|^2) + \lambda \int_0^t \phi_\lambda^{-1}(\|X(s)\|^2)(K(s)(1+\|X(s)\|^2))ds + \eta_t^\lambda
\end{aligned}$$

by Assumption (A2) and (A3)

$$\begin{aligned}
&+ \frac{1}{2}\lambda^2(\int_0^t \phi_\lambda^{-2}(\|X(s)\|^2)|G(s,X(s))^*\theta_p(X(s))|_0^2 ds\\
&- 2\int_0^t \phi_\lambda^{-3}(\|X(s)\|^2)|G(s,X(s))^*\theta_p(X(s))|_0^2 ds) \qquad (3.6)
\end{aligned}$$

Note that $\phi_\lambda^{-1}(x) \leq 1$ and θ_p is an isometry from $\mathcal{S}_{-p}$ into $\mathcal{S}_p$.

In (3.5) and (3.6),

$$\begin{aligned}
&\lambda \int_0^t \phi_\lambda^{-1}(\|X(s)\|^2)(K(s)(1+\|X(s)\|^2))\, ds\\
\leq&\lambda \int_0^t K(s)ds + \int_0^t \phi_\lambda(\|X(s)\|^2)K(s)\, ds
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\lambda^2(\int_0^t \phi_\lambda^{-2}(\|X(s)\|^2)|G(s,X(s))^*\theta_p(X(s))|_0^2\, ds\\
\leq&\frac{1}{2}\lambda \int_0^t \|G(s,X(s))\|_{L(\mathcal{S}_0,\mathcal{S}_{-p})}^2\, ds\\
\leq&\frac{1}{2}\lambda \int_0^t K(s)\, ds
\end{aligned}$$

Put

$$g(t) = C + \frac{3}{2}\lambda \int_0^t K(s)ds + \eta_t^\lambda \quad \text{and } f(t) = \phi_\lambda(\|X(t)\|^2).$$

Then we have

$$f(t) \leq g(t) + \int_0^t K(s)f(s)ds.$$

Therefore, by the Gronwall's inequality

$$f(t) \leq g(t) \exp\{\int_0^t K(s)ds\}.$$

This implies that

$$e^{-\mu_t}\phi_\lambda(\|X(t)\|^2) \leq C + \frac{3}{2}\lambda\mu_t + \eta_t^\lambda.$$

Note that if $|X(t)|_{-p}^2 \geq r$ then

$$\exp\{e^{-\mu_t}(1+\lambda r)^{\frac{1}{2}}\} \leq \exp\{e^{-\mu_t}\phi_\lambda(|X(t)|_{-p}^2)\} \leq \exp\{C + \frac{3}{2}\lambda\mu_t + \eta_t^\lambda\}$$

and

$$\exp\{\eta_t^\lambda\} \geq \exp\{e^{-\mu_t}(1+\lambda r)^{\frac{1}{2}} - C - (\frac{3}{2})\lambda\mu_t\}.$$

Hence by the Doob's inequality and Lemma 3.2, we have

$$\begin{aligned} P\{\sup_{0\leq s\leq t} |X(s)|_{-p}^2 \geq r\} &\leq P\{\sup_{0\leq s\leq t} \phi_\lambda(|X(s)|_{-p}^2) \geq (1+\lambda r)^{\frac{1}{2}}\} \\ &\leq P\{\sup_{0\leq s\leq t} Z_s^\lambda \geq \exp\{e^{-\mu_t}(1+\lambda r)^{\frac{1}{2}} - C - \frac{3}{2}\lambda\mu_t\}\} \\ &\leq \exp\{-e^{-\mu_t}(1+\lambda r)^{\frac{1}{2}} + C + \frac{3}{2}\lambda\mu_t\}, \end{aligned} \tag{3.7}$$

The exponent in (3.7) is minimal when $\lambda = \lambda_0 = \frac{e^{-2\mu_t}\cdot r}{9\mu_t^2} - \frac{1}{r} > 0$. If we put $X(0) = 0$ and $\lambda = \lambda_0$ in (3.7), it gives the desired upper bound for each $t \in [0, T]$. □

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