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WEYL SPECTRA OF THE χ-CLASS OPERATORS

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Abstract. In this paper we introduce a notion of the χ-class operators, which is a class including hyponormal operators and consider their spectral properties related to Weyl spectra.

Introduction

Throughout this paper let $\mathcal{H}$ denote an infinite dimensional separable Hilbert space. Let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$ and $\mathcal{K}(\mathcal{H})$ the closed ideal of compact operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$ write $N(T)$ and $R(T)$ for the null space and range of T; $\rho(T)$ for the resolvent set of T ; $\sigma(T)$ for the spectrum of T; $\pi_0(T)$ for the set of eigenvalues of T; $\pi_{0f}(T)$ for the eigenvalues of finite multiplicity; $\pi_{0i}(T)$ for the eigenvalues of infinite multiplicity. Recall ([**12**],[**13**]) that $T \in \mathcal{L}(\mathcal{H})$ is called *regular* if there is an operator $T' \in \mathcal{L}(\mathcal{H})$ for which $T = TT'T$. It is familiar that if $T \in \mathcal{L}(\mathcal{H})$ then T is regular if and only if T has closed range. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *upper semi-Fredholm* if it has closed range with finite-dimensional null space and *lower semi-Fredholm* if it has closed range with its range of finite co-dimension. If T is either upper or lower semi-Fredholm, we call it *semi-Fredholm* and if T is both upper and lower semi-Fredholm, we call it *Fredholm*. The *index* of a semi-Fredholm operator $T \in \mathcal{L}(\mathcal{H})$ is given by

$$\operatorname{ind}(T) = \dim N(T) - \dim R(T)^{\perp} \; (= \dim N(T) - \dim N(T^*)).$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *Weyl* if it is Fredholm of index zero. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *Browder* if it is Fredholm "of finite ascent and descent": equivalently ([**13**, Theorem 7.9.3]) if T is Fredholm and $T - \lambda I$ is

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invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\omega(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in \mathcal{L}(\mathcal{H})$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\};$$
$$\omega(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\};$$
$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\} :$$

then ([**13**])

$$\sigma_e(T) \subseteq \omega(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \operatorname{acc}\sigma(T) \quad \text{and} \quad \omega(T) \subseteq \eta\, \sigma_e(T), \tag{0.1}$$

where we write $\operatorname{acc} K$ and $\eta\, K$ for the *accumulation points* and the *polynomially-convex hull*, respectively, of $K \subseteq \mathbb{C}$. If we write $\operatorname{iso} K = K \setminus \operatorname{acc} K$, and $\partial\, K$ for the topological boundary of K, and

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \dim\,(T - \lambda I)^{-1}(0) < \infty\} \tag{0.2}$$

for the isolated eigenvalues of finite multiplicity, and ([**13**])

$$p_{00}(T) := \sigma(T) \setminus \sigma_b(T) \tag{0.3}$$

for the *Riesz points* of $\sigma(T)$, then by the punctured neighborhood theorem, i.e., $\partial\,\sigma(T) \setminus \sigma_e(T) \subseteq \operatorname{iso}\sigma(T)$ ([**13**], [**14**]),

$$\operatorname{iso}\sigma(T) \setminus \sigma_e(T) = \operatorname{iso}\sigma(T) \setminus \omega(T) = p_{00}(T) \subseteq \pi_{00}(T). \tag{0.4}$$

We say that *Weyl's theorem holds for* $T \in \mathcal{L}(\mathcal{H})$ if there is equality

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T). \tag{0.5}$$

If $T \in \mathcal{L}(\mathcal{H})$, write $r(T)$ for the spectral radius of T. It is familiar that $r(T) \leq ||T||$. An operator T is called *normaloid* if $r(T) = ||T||$ and *isoloid* if $\operatorname{iso}\sigma(T) \subseteq \pi_0(T)$. An operator T is said to satisfy condition (G_1) if $(T - \lambda I)^{-1}$ is normaloid for all $\lambda \notin \sigma(T)$. If $T \in \mathcal{L}(\mathcal{H})$, write $W(T)$ for the numerical range of T. It is also familiar that $W(T)$ is convex and $\operatorname{conv}\sigma(T) \subseteq \operatorname{cl} W(T)$. An operator T is called *convexoid* if $\operatorname{conv}\sigma(T) = \operatorname{cl} W(T)$. Let P be a property of operators. We say that an operator T is *restriction-P* if the restriction of T to every invariant subspace has property P and that T is *reduction-P* if every direct summand of T has property P. Evidently, restriction-$P \Longrightarrow$ reduction-P. It is known ([**3**]) that if $T \in \mathcal{L}(\mathcal{H})$ then we have:

(0.6) $(G_1) \Longrightarrow$ convexoid and isoloid;
(0.7) seminormal $\Longrightarrow$ reduction-$(G_1) \Longrightarrow$ reduction-isoloid;
(0.8) hyponormal $\Longrightarrow$ restriction-convexoid $\Longrightarrow$ reduction-isoloid.

Note that seminormal operators are reduction-convexoid, but they may not be restriction-convexoid: for example consider the backward shift U^* on ℓ_2, where U is the unilateral shift ([**4**]). Thus the replacement of "reduction-" by "restriction-" is very stringent. Now we shall say that an operator $T \in \mathcal{L}(\mathcal{H})$ *is in the χ-class* if T is restriction-convexoid and is reduced by each of its eigenspaces corresponding to isolated eigenvalues. Evidently, T hyponormal $\Rightarrow T \in \chi$.

1. The χ-class operators

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *reguloid* ([**15**]) if $T - \lambda I$ is regular for each $\lambda \in \operatorname{iso}\sigma(T)$. We begin with:

LEMMA 1.1. *If $T \in \mathcal{L}(\mathcal{H})$ then*

(1.1.1) $$(G_1) \Longrightarrow \textit{reguloid} \Longrightarrow \textit{isoloid}$$

and

(1.1.2) $$\textit{restriction-convexoid} \Longrightarrow \textit{restriction-reguloid.}$$

Proof. (1.1.1) is [**15**, Theorem 14]. For (1.1.2), suppose T is restriction-convexoid and $\mathfrak{M}$ is an invariant subspace of T. Write $S := T|\mathfrak{M}$. Then S is also restriction-convexoid. Suppose $\lambda \in \operatorname{iso}\sigma(S)$. Observe that if T is convexoid then so is $aT + bI$ for any $a, b \in \mathbb{C}$. Thus we may write S in place of $S - \lambda I$ and assume $\lambda = 0$. Using the spectral projection at $0 \in \mathbb{C}$ we can write $S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$, where $\sigma(S_1) = \{0\}$ and $\sigma(S_2) = \sigma(S) \setminus \{0\}$. Since by assumption, S_1 is convexoid it follows that $W(S_1) = \operatorname{conv}\sigma(S_1) = \{0\}$, and hence $S_1 = 0$. Thus we have

$$S = \begin{pmatrix} 0 & 0 \\ 0 & S_2 \end{pmatrix} = SS'S \quad \text{with } S' = \begin{pmatrix} 0 & 0 \\ 0 & S_2^{-1} \end{pmatrix},$$

which says that S is regular, and therefore T is restriction-reguloid. □

It was shown in ([**24**]) that Weyl's theorem holds for restriction-convexoid operators. We can prove more:

THEOREM 1.2. *Let $T \in \mathcal{L}(\mathcal{H})$. If either T or T^* is restriction-convexoid then Weyl's theorem holds for T.*

Proof. If T is restriction-convexoid then it follows from [**24**, Theorem 2.1] that Weyl's theorem holds for T. Now suppose T^* is restriction-convexoid. Let $\lambda \in \pi_{00}(T)$. Then $\overline{\lambda} \in \text{iso}\,\sigma(T^*)$. Since T^* is restriction-convexoid, it follows from Lemma 1.1 that $T-\lambda$ has closed range. Therefore it follows from the punctured neighborhood theorem that $\lambda \in \sigma(T)\omega(T)$. Conversely, suppose $\lambda \in \sigma(T)\backslash\omega(T)$. Then $\overline{\lambda} \in \sigma(T^*)\backslash\omega(T^*)$. Since T^* is restriction-convexoid, Browder's theorem holds for T^*. Therefore $\overline{\lambda} \in p_{00}(T^*)$. It follows from the fact $p_{00}(T^*) = p_{00}(T)^*$ that $\lambda \in \pi_{00}(T)$. This completes the proof. □

In 1970, S. Berberian ([**5**]) raised the following question: if T is restriction-convexoid and $\sigma(T)$ is countable, is T normal? We now give a partial answer.

THEOREM 1.3. *Let $T \in \chi$. If $\sigma(T)$ is countable then T is diagonal and normal.*

Proof. Suppose $T \in \chi$ and $\sigma(T)$ is countable. Let δ be the set of all normal eigenvalues of T, i.e.,

$$\delta = \{\lambda \in \pi_0(T) : \ N(T-\lambda I) = N(T^* - \overline{\lambda} I)\}.$$

We first claim that $\delta \neq \emptyset$. Since $\sigma(T)$ is countable, there exists a point $\lambda \in \text{iso}\,\sigma(T)$, so that $\lambda \in \pi_0(T)$ because by Lemma 1.1, T is isoloid. Using the spectral projection at $\lambda \in \mathbb{C}$ we can represent T as the direct sum

$$T = R \oplus S, \quad \text{where } \sigma(R) = \pi_0(R) = \{\lambda\} \text{ and } \sigma(S) = \sigma(T)\backslash\{\lambda\}.$$

Since by assumption R is convexoid, we have that $W(R) = \text{conv}\,\{\lambda\} = \{\lambda\}$ and thus $\lambda \in \pi_0(R) \cap \partial W(R)$. Then an argument of Bouldin [**6**, Lemma 1] shows that λ is a normal eigenvalue of R. By assumption we can write $T^* = R^* \oplus S^*$. But since $S^* - \overline{\lambda} I$ is invertible, it follows

$$N(T-\lambda I) = N(R-\lambda I) = N(R^* - \overline{\lambda} I) = N(T^* - \overline{\lambda} I),$$

which implies that $\delta \neq \emptyset$. Now if $\mathfrak{M}$ is the closed linear span of the eigenspaces $N(T-\lambda I)$ $(\lambda \in \delta)$, then $\mathfrak{M}$ reduces T. Write

$$T_1 := T|\mathfrak{M} \quad \text{and} \quad T_2 := T|\mathfrak{M}^{\perp}.$$

Then an argument of Berberian [**3**, Proposition 4.1] shows that (i) T_1 is normal and diagonal; (ii) $\pi_0(T_1) = \delta$; (iii) $\sigma(T_1) = \mathrm{cl}\,\delta$; (iv) $\pi_0(T_2) = \pi_0(T)\backslash\delta$. Thus it will suffice to show that $\mathfrak{M}^\perp = \{0\}$. Assume to the contrary that $\mathfrak{M}^\perp \neq \{0\}$. Then since $\sigma(T_2)$ is also countable, there exists a point $\mu \in \mathrm{iso}\,\sigma(T_2)$. Since by assumption T_2 is restriction-convexoid and hence isoloid, it follows that $\mu \in \pi_0(T_2)$ and $\mu \notin \delta$. Thus using the spectral projection at $\mu \in \mathbb{C}$, we can decompose T_2 as the direct sum

$$T_2 = T_3 \oplus T_4,$$

where $\sigma(T_3) = \pi_0(T_3) = \{\mu\}$ and $\sigma(T_4) = \sigma(T_2)\backslash\{\mu\}$. Since again T_3 is convexoid, the same argument as the above gives that μ is an isolated normal eigenvalue of T_3 and further by assumption $T_2^* = T_3^* \oplus T_4^*$. But since $T_1 - \mu I$ and $T_4 - \mu I$ are both one-one we have

$$N(T - \mu I) = N(T_3 - \mu I) = N(T_3^* - \overline{\mu} I).$$

Further since $\pi_0(T_1^*) = \overline{\delta}$ and $\overline{\mu} \notin \sigma(T_4^*)$, it follows that $N(T^* - \overline{\mu} I) = N(T_3^* - \overline{\mu} I)$, and therefore $N(T - \mu I) = N(T^* - \overline{\mu} I)$, which implies that $\mu \in \delta$, giving a contradiction. This completes the proof. □

We have been unable to answer if restriction-convexoid operators are reduced by each of its eigenspaces corresponding to isolated eigenvalues. If the answer were affirmative then we would answer Berberian question affirmatively.

We recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is called a *Riesz* operator if $\sigma_e(T) = \{0\}$. We then have:

COROLLARY 1.4. *If $T \in \chi$ is Riesz then T is compact and normal.*

Proof. By Theorem 1.3, T is normal with pure point spectrum. Note that the nonzero eigenvalues are Riesz points, so that they are either finite or form a null sequence, which implies that T is compact. □

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *polynomially compact* if there exists a nonzero complex polynomial p such that $p(T)$ is compact. S. Berberian ([**3**]) considered a relationship between the polynomial compactness of the operator and the finiteness of its Weyl spectrum, and gave several sufficient conditions for the finiteness of the Weyl spectrum; for example, if T is a seminormal operator then T is polynomially compact if and only if $\omega(T)$ is finite. Observe

$$(1.4.1) \qquad T \text{ is polynomially compact } \Longrightarrow \omega(T) \text{ is finite}:$$

indeed if $p(T)$ is compact then $p(\sigma_e(T)) = \sigma_e(p(T)) = \{0\}$, so that $\sigma_e(T)$ is finite, which together with (0.1) implies that $\sigma_e(T) = \omega(T)$. Recently, the finiteness of the Weyl spectrum was characterized in ([**11**]).

LEMMA 1.5 ([11, LEMMA 3]). *If $\omega(T)$ is finite then $T \in \mathcal{L}(\mathcal{H})$ is decomposed into the finite direct sum*

$$(1.5.1) \qquad T = \bigoplus_{i=1}^{n} (N_i + K_i + \lambda_i I),$$

where the N_i are quasinilpotents, the K_i are compact, and $\{\lambda_1, \cdots, \lambda_n\} = \omega(T)$.

The following corollary provides a structure theorem for polynomially compact χ-calss operators (Compare with [**5**, Theorem 3]):

COROLLARY 1.6. *If $T \in \chi$ then the following statements are equivalent:*

(a) *T is polynomially compact;*
(b) *$\omega(T)$ is finite;*
(c) *T is the direct sum of finitely many thin normal operators, i.e.,*

$$(1.6.1) \qquad T = \bigoplus_{i=1}^{n} (R_i + \lambda_i I),$$

where the R_i are compact normal operators.

Proof. (a)⇒(b): This comes from (1.4.1).

(b)⇒(c): If $\omega(T)$ is finite then (1.5.1) holds with Riesz operators R_i. Thus if $T \in \chi$ then so is each R_i, and therefore it follows from Corollary 1.4 that each R_i is a compact normal operator.

(c)⇒(a): Suppose T satisfies (1.6.1). Then $p(T)$ is compact, where $p(z) = (z - \lambda_1) \cdots (z - \lambda_n)$, with λ_i as in (c). □

In [**10**, Solution 178], it was shown that if T is normal and if T^n is compact for some $n \in \mathbb{N}$ then T is compact. We can prove more:

COROLLARY 1.7. *If $T \in \chi$ and if T^n is compact for some $n \in \mathbb{N}$ then T is a diagonal compact operator.*

Proof. If $T \in \chi$ and if T^n is compact for some $n \in \mathbb{N}$ then it follows from Corollary 1.6 that $\sigma(T)$ is countable. Thus by Theorem 1.3, T is a diagonal normal operator with diagonal $\{\alpha_m\}_{m=1}^{\infty}$. But since T^n is a diagonal compact operator with diagonal $\{\alpha_m^n\}_{m=1}^{\infty}$, we can see that $\alpha_m^n \xrightarrow{m} 0$, so that $\alpha_m \xrightarrow{m} 0$. Therefore T is a diagonal compact operator. □

We consider here a relationship between convexoid operators and their spectral sets. Recall that a compact set σ in $\mathbb{C}$ is called a *spectral set* for $T \in \mathcal{L}(\mathcal{H})$ if $\sigma(T) \subseteq \sigma$ and if $||f(T)|| \leq ||f||_\sigma := \max_{z\in\sigma} |f(z)|$ for every rational function f with poles off σ. The following results are well-known:

(i) The closed unit disk $\mathbb{D}$ is a spectral set for every contraction operator ([**26**]).
(ii) The spectrum of a subnormal operator is a spectral set ([**1**], [**18**]).
(iii) There exists a hyponormal operator whose spectrum contains a disk and is not a spectral set ([**27**]).

We now have:

THEOREM 1.8. *If $\operatorname{conv}\sigma(T)$ is a spectral set for $T \in \mathcal{L}(\mathcal{H})$ then T is convexoid.*

Proof. Suppose $\operatorname{conv}\sigma(T)$ is a spectral set for T. Thus $||f(T)|| \leq ||f||_{\operatorname{conv}\sigma(T)}$ for every rational function f with poles off $\operatorname{conv}\sigma(T)$. If K is a convex subset of $\mathbb{C}$, write $\operatorname{Ext} K$ for the set of extreme points of K. Observe that if K is a compact convex set in $\mathbb{C}$, then $||z||_K$ occurs on $\operatorname{Ext} K$. But by the Krein-Milman theorem,

$$\operatorname{conv}\sigma(T) = \overline{\operatorname{conv}}\left(\operatorname{Ext}\operatorname{conv}\sigma(T)\right) \quad \text{and} \quad \operatorname{Ext}\left(\operatorname{conv}\sigma(T)\right) \subseteq \sigma(T),$$

where $\overline{\operatorname{conv}}$ denotes the closed convex-hull. Thus for every $\lambda \in \mathbb{C}$,

$$\begin{aligned} r(T-\lambda I) \leq ||T-\lambda I|| \leq ||z-\lambda||_{\operatorname{conv}\sigma(T)} &= ||z||_{\operatorname{conv}\sigma(T-\lambda I)} \\ = ||z||_{\operatorname{Ext}\operatorname{conv}\sigma(T-\lambda I)} &= ||z||_{\sigma(T-\lambda I)} = r(T-\lambda I), \end{aligned}$$

which implies that $r(T-\lambda I) = ||T-\lambda I||$ for every $\lambda \in \mathbb{C}$. This says that $T-\lambda I$ is normaloid for every $\lambda \in \mathbb{C}$. It therefore follows that T is convexoid ([**4**]). □

Note that $\sigma(T)$ need not be a spectral set for T even though $\operatorname{conv}\sigma(T)$ is. For example if S is the bilateral shift on $L^2(\mathbb{T})$ of the unit circle $\mathbb{T}$, take

$$T = S \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{1.8.1}$$

where the second summand is a two-dimensional operator. Then $\sigma(T) = \mathbb{T} \cup \{0\}$. Choose $f(z) = (z - \frac{1}{2})^{-1}$. Then $||f||_{\sigma(T)} = 2$, but $||f(T)|| \geq ||\begin{pmatrix} -2 & -4 \\ 0 & -2 \end{pmatrix}|| > 2$, which implies that $\sigma(T)$ is not a spectral set for T. But since $\operatorname{conv}\sigma(T) = \mathbb{D}$ and T is a contraction operator it follows from the statement (i) in the remark above Theorem 1.8 that $\operatorname{conv}\sigma(T)$ is a spectral set for T. Also note that the conclusion of Theorem 1.8 cannot be strengthened by "reduction-convexoid": for example consider the operator T defined by (1.8.1).

2. Commutators

A *commutator* is an operator of the form $AB - BA$. Then Brown-Pearcy theorem [**7**, Theorem 3] says that $T \in \mathcal{L}(\mathcal{H})$ is a noncommutator if and only if T is of the form $K + \lambda I$, where $\lambda \neq 0$ and K is compact. Thus we have that

$$T \text{ is a noncommutator} \Longrightarrow \omega(T) = \{\lambda\},\ \lambda \neq 0. \tag{2.0.1}$$

But the converse of (2.0.1) is, in general, not true. We however have:

THEOREM 2.1. *If $T \in \chi$ and $\omega(T) = \{\lambda\}$, $\lambda \neq 0$, then T is a noncommutator.*

Proof. Suppose $\omega(T) = \{\lambda\}$, $\lambda \neq 0$. Then $\sigma_e(T) = \{\lambda\}$, and hence $\sigma_e(T - \lambda I) = \{0\}$. Thus if $T \in \chi$ and hence so is $T - \lambda I$, then it follows from Corollary 1.4 that $T - \lambda I$ is a compact operator. Therefore by the Brown-Pearcy theorem, T is a noncommutator. □

In Theorem 2.1, "restriction-convexoid in the definition of χ" cannot be replaced by "convexoid". To see this, let on ℓ_2

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \oplus \left[\begin{pmatrix} \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \otimes 1_\infty \right].$$

Then we have that (i) $\omega(T) = \{\frac{1}{3}\}$; (ii) T is convexoid because $\operatorname{conv}\sigma(T) = W(T)$, which is the equilateral triangle whose vertices are the three cube roots of 1; (iii) T is a commutator because T has a "large" kernel (see [**10**, Problem 234]); (iv) T is not reduction-convexoid because $\begin{pmatrix} \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ is not convexoid.

THEOREM 2.2. *If either $\sigma(A)$ or $\sigma(B)$ is not a singleton set then $A \otimes B$ is a commutator. In particular if either A or B is a nonconstant convexoid operator then $A \otimes B$ is a commutator.*

Proof. Suppose either $\sigma(A)$ or $\sigma(B)$ is not a singleton set. Since [**16**, Theorem 4.2]

$$\omega(A \otimes B) = \omega(A) \cdot \sigma(B) \cup \sigma(A) \cdot \omega(B),$$

it follows that either $\omega(A \otimes B) = \{0\}$ or $\omega(A \otimes B)$ has at least two elements. Thus by (2.0.1), $A \otimes B$ is a commutator. This proves the first assertion. For the second assertion we suppose that A is nonconstant and convexoid. In view of the first assertion it suffices to show that $\sigma(A)$ is not a singleton set. Assume to the contrary that $\sigma(A) = \{\lambda\}$, $\lambda \in \mathbb{C}$. Then $A - \lambda I$ is convexoid and quasinilpotent. But since the only convexoid quasinilpotent is 0, it follows that $A = \lambda I$, giving a contradiction. This completes the proof. □

A *self-commutator* is an operator of the form $A^*A - AA^*$. Then Radjavi's theorem ([**25**]) says that a self-adjoint operator $T \in \mathcal{L}(\mathcal{H})$ is a self-commutator if and only if $0 \in \operatorname{conv}\omega(T)$. Thus the Radjavi's theorem gives the following:

THEOREM 2.3. *If $T \in \mathcal{L}(\mathcal{H})$ is a self-adjoint operator whose direct summands are nonconstant then T is a self-commutator if and only if either $0 \in \omega(T)$ or T is not semi-definite.*

Proof. If $0 \in \omega(T)$, then evidently T is a self-commutator. If instead T is not semi-definite, then there exist $\lambda, \mu \in \sigma(T)$ such that $\lambda > 0$ and $\mu < 0$. We now claim that $\lambda, \mu \in \omega(T)$. Assume to the contrary that $\lambda \notin \omega(T)$. Then it follows from Weyl's theorem that $\lambda \in \operatorname{iso}\sigma(T)$. Thus T should be of the form $T = \lambda I \oplus S$, which contradicts to our assumption. Therefore we have that $0 \in \operatorname{conv}\omega(T)$, and hence by the Radjavi's theorem, T is a self-commutator. The converse is evident. □

Theorem 2.3 is readily applicable for self-adjoint operators with no eigenvalues (e.g., Toeplitz operators with real-valued symbols).

An invertible operator $T \in \mathcal{L}(\mathcal{H})$ is called a *multiplicative commutator* if it is of the form $ABA^{-1}B^{-1}$. By contrast, a commutator $AB - BA$ is often called an additive commutator. It is known that if T is a multiplicative commutator of the form $K + \lambda I$, where K is compact and $\lambda \in \mathbb{C}$, then $|\lambda| = 1$ ([**8**, Theorem 1], [**10**, Problem 238]). It remains open whether a multiplicative commutator is not of the form $K + \lambda I$, where $|\lambda| \neq 1$ and K is compact. But another argument of Brown and Pearcy [**8**, Theorem 5] shows that an invertible normal operator $T \in \mathcal{L}(\mathcal{H})$ is a multiplicative commutator if and only if T is not of the form $K + \lambda I$, where $|\lambda| \neq 1$ and K is compact. Thus if $T \in \mathcal{L}(\mathcal{H})$ is invertible and normal then

(2.3.1) T is not a multiplicative commutator $\Longrightarrow \omega(T) = \{\lambda\},\ |\lambda| \neq 1.$

The following theorem shows that the converse of (2.3.1) is also true with a weaker condition:

THEOREM 2.4. *If $T \in \chi$ is invertible and $\omega(T) = \{\lambda\}$, then*

$$T \text{ is a multiplicative commutator} \iff |\lambda| = 1.$$

Proof. If $\omega(T) = \{\lambda\}$, then $\sigma_e(T - \lambda I) = \{0\}$. By Corollary 1.4, $T - \lambda I$ is a compact normal operator, say K. But then $T = K + \lambda I$ is invertible and normal. Therefore by the Brown-Pearcy characterization [**8**, Theorem 5], T is a multiplicative commutator if and only if $|\lambda| = 1$. □

Theorems 2.1 and 2.4 show that if T is an invertible χ-class operator and $\omega(T) = \{\lambda\}$, it is impossible that T is both a multiplicative and an additive commutator: for if $|\lambda| = 1$, then T is a multiplicative commutator but not an additive commutator, and if $|\lambda| \neq 1$, then T is neither a multiplicative nor an additive commutator.

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