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EXISTENCE OF A MILD SOLUTION OF A FUNCTIONAL INTEGRODIFFERENTIAL EQUATION WITH NONLOCAL CONDITION

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ABSTRACT. In this paper we prove the existence and uniqueness of a mild solution of a functional differential equation with nonlocal condition using the semigroup theory and the Banach fixed point principle.

1. Introduction

Byszewski [4] studied the problem of existence of solutions of semilinear evolution equations with nonlocal conditions in Banach spaces. Byszewski and Acka [6] established the existence and uniqueness and continuous dependence of a mild solution of a semilinear functional differential equation with nonlocal condition of the form

$$\frac{du(t)}{dt} + Au(t) = f(t, u_t), \qquad t \in [0, a],$$
$$u(s) + [g(u_{t_1}, .., u_{t_p})](s) = \phi(s), \qquad s \in [-r, 0]$$

where $0 < t_1 < ... < t_p \leq a \ \ (p \in N)$, $-A$ is the infinitesimal generator of a C_0 semigroup of operators on a general Banach space, f, g and ϕ are given functions and $u_t(s) = u(t+s)$ for $t \in [0, a]$, $s \in [-r, 0]$.

In this paper, we shall prove the existence and uniqueness of a mild solution for a functional integrodifferential equation with nonlocal conditions of the form

$$\frac{du(t)}{dt} + Au(t) = f(t, u_t, \int_0^t k(t, \tau, u_\tau)d\tau), \quad t \in [0, a], \tag{1}$$

$$u(s) + [g(u_{t_1}, .., u_{t_p})](s) = \phi(s), \qquad s \in [-r, 0] \tag{2}$$

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where $-A$ is the infinitesimal generator of C_0 semigroup of operators $T(t)$, $t \geq 0$ on a Banach space E and $\phi \in C([-r, 0], E)$ and the nonlinear operators f, k, g are given functions satisfying some assumptions.

Theorems about the existence, uniqueness and stability of solutions of differential, integrodifferential equations and functional-differential abstract evolution equations with nonlocal conditions were studied by Byszewski [4,5], Balachandran and Chandrasekaran [2,3] and Lin and Liu [10]. This paper is a generalization of the results of Byszewski and Akca [6].

In the case if the nonlocal condition, considered in the paper, is reduced to the classical initial condition then the result of the paper is reduced to some results of Hale [7], Thompson [12], and Akca, Shakhmurow and Aralan [1] on the existence, uniqueness and continuous dependence of functional differential evolution equations.

2. Preliminaries

Here we assume that E is a Banach space with norm $\|.\|$, $-A$ is the infinitesimal generator of a C_0 semigroup $\{T(t)\}_{t\geq 0}$ on E and $M = \sup_{t\in[0,a]} \|T(t)\|_{B(E)}$. In this sequel the operator norm $\|.\|_{B(E)}$ will be donoted by $\|.\|$. To simplify the notation let us take $I_0 = [-r, 0]$, $I = [0, a]$ and $X = C([-r, 0] : E)$, $Y = C([-r, a] : E)$, $Z = C([0, a] : E)$. For a continuous function $w : [-r, a] \to E$, we denote w_t a function belonging to X and defined by $w_t = w(t+s)$ for $t \in I$, $s \in I_0$. Let $f : I \times X \times X \to E$, $k : I \times I \times X \to X$ and $\phi \in X$.

We make the following assumptions:

(A_1) For every $u, w \in X$ and $t \in I$, $f(., u_t, w_t) \in Z$.

(A_2) There exists a constant $L > 0$ such that

$$\|f(t, x_t, w_t) - f(t, y_t, u_t)\| \leq L[\|x - y\|_{C([-r,t]:E)} + \|w - u\|_{C([-r,t]:E)}]$$

for $x, y, w, u \in Y$, $t \in I$.

(A_3) There exists a constant $K > 0$ such that

$$\|k(t, s, x_s) - k(t, s, y_s)\| \leq K\|x - y\|_{C([-r,s]:E)} \text{ for } x, y \in Y, \quad s \in I.$$

(A_4) Let $g : X^p \to X$ and there exists a constant $G > 0$ such that

$$\|[g(w_{t_1}, .., w_{t_p})](s) - [g(u_{t_1}, .., u_{t_p})](s)\| \leq G\|w - u\|_X$$

for $w, u \in Y$, $s \in I_0$.

(A_5) $M(Ka^2L + La + G) < 1$.

A function $u \in Y$ satisfying the conditions:

$$\begin{aligned}(3) \qquad \text{(i)}\ u(t) &= T(t)\phi(0) - T(t)[g(u_{t_1},..,u_{t_p})](0) \\ &+ \int_0^t T(t-\tau)f(\tau, u_\tau, \int_0^\tau k(\tau,\theta,u_\theta)d\theta)d\tau, \quad t \in I, \\ \text{(ii)}\ u(s) &+ [g(u_{t_1},..,u_{t_p})](s) = \phi(s), \quad s \in I_0\end{aligned}$$

is said to be a mild solution of the nonlocal Cauchy problem (1)-(2).

3. Existence of a mild solution

THEOREM 3.1. *Assume that the functions f and g satisfy assumptions $(A_1) - (A_5)$. Then the nonlocal Cauchy problem $(1) - (2)$ has a unique mild solution.*

Proof. Define an operator F on the Banach space Y by the formula

$$(Fu)(t) = \begin{cases} \phi(t) - [g(u_{t_1},..,u_{t_p})](t), & t \in I_0, \\ T(t)\phi(0) - T(t)[g(u_{t_1},..,u_{t_p})](0) & \\ \quad + \int_0^t T(t-\tau)f(\tau,u_\tau,\int_0^\tau k(\tau,\theta,u_\theta)d\theta)d\tau, & t \in I. \end{cases}$$

It is easy to see that F maps Y into itself. Now, we will show that F is a contraction on Y. Consider

$$(4) \qquad (Fw)(t) - (Fu)(t) = [g(w_{t_1},..,w_{t_p})](t) - [g(u_{t_1},..,u_{t_p})](t),$$

for $w, u \in Y$, $t \in [-r, 0)$ and

(5)

$$\begin{aligned}(Fw)(t) - (Fu)(t) &= T(t)[(g(w_{t_1},..,w_{t_p}))(0) - (g(u_{t_1},..,u_{t_p}))(0)] \\ &\quad + \int_0^t T(t-\tau)\Big[f(\tau,w_\tau,\int_0^\tau k(\tau,\theta,w_\theta)d\theta) \\ &\quad - f(\tau,u_\tau,\int_0^\tau k(\tau,\theta,u_\theta)d\theta)\Big]d\tau, \quad w,u \in Y, \quad t \in I.\end{aligned}$$

From (5) and (A_4),

$$(6) \qquad \|(Fw)(t) - (Fu)(t)\| \le G\|w-u\|_Y, \text{ for } w,u \in Y, \quad t \in I_0.$$

Moreover by (5), (A_2), (A_3) and (A_4),

$$(7) \quad \|(Fw)(t) - (Fu)(t)\|$$

$$
\begin{aligned}
&\leq \|T(t)\| \|(g(w_{t_1},..,w_{t_p}))(0) - (g(u_{t_1},..,u_{t_p}))(0)\| \\
&\quad + \int_0^t \|T(t-\tau)\| L[\|w-u\| + \int_0^\tau \|k(\tau,\theta,w_\theta) - k(\tau,\theta,u_\theta)\| d\theta] d\tau, \\
&\leq MG\|w-u\|_Y + ML\int_0^t [\|w-u\|_{C([-r,s]:E)} \\
&\quad + K\int_0^\tau \|w-u\|_{C([-r,\tau]:E)} d\tau] ds \\
&\leq MG\|w-u\|_Y + MLa\|w-u\|_Y + MLKa\int_0^t \|w-u\|_{C([-r,\tau]:E)} ds \\
&\leq M(Ka^2L + La + G)\|w-u\|_Y, \quad \text{for } w,u \in Y,\ 0 < s < \tau < t \leq a.
\end{aligned}
$$

From (6) and (7) we get

$$\|Fw - Fu\|_Y \leq q\|w-u\|_Y, \quad \text{for} \quad w,u \in Y, \tag{8}$$

where $q = M(Ka^2L + La + G)$. □

Since, $q < 1$ then (8) shows that F is a contraction on Y. Consequently, the operator F satisfies all the assumptions of the Banach contraction theorem. Therefore, in space Y there is a unique fixed point for F and this point is the mild solution of the nonlocal Cauchy problem (1)-(2).

4. Continuous dependence of a mild solution

THEOREM 4.1. *Suppose that the functions f and g satisfy assumptions (A_1)-(A_4). Then for each $\phi_1,\ \phi_2 \in X$ and for the corresponding mild solutions $u_1,\ u_2$ of the problems*

$$\frac{du(t)}{dt} + Au(t) = f(t, u_t, \int_0^t k(t,\tau,u_\tau)d\tau), \qquad t \in I \tag{9}$$

$$u(s) + [g(u_{t_1},..,u_{t_p})](s) = \phi_i(s), \quad s \in I_0,\ (i = 1,2), \tag{10}$$

the following inequality

$$\|u_1 - u_2\|_Y \leq Me^{aML(1+Ka)}[\|\phi_1 - \phi_2\|_X + G\|u_1 - u_2\|_Y] \tag{11}$$

is true. Additionally, if $G < \frac{1}{M}e^{-aML(1+Ka)}$ then,

$$\|u_1 - u_2\|_Y \leq \frac{Me^{aML(1+Ka)}}{1 - GMe^{aML(1+Ka)}}[\|\phi_1 - \phi_2\|_X] \tag{12}$$

Proof. Let ϕ_i $(i = 1, 2)$ be arbitrary functions belonging to X and let u_i $(i = 1, 2)$ be the mild solutions of problems (9)–(10). Consequently,

$$\begin{aligned} u_1(t) - u_2(t) &= T(t)[\phi_1(0) - \phi_2(0)] \\ &- T(t)[(g((u_1)_{t_1}, .., (u_1)_{t_p}))(0) - (g((u_2)_{t_1}, .., (u_2)_{t_p}))(0)] \\ &+ \int_0^t T(t-\tau)[f(\tau, (u_1)_\tau, \int_0^\tau k(\tau, \theta, (u_1)_\theta)d\theta) \\ &- f(\tau, (u_1)_\tau, \int_0^\tau k(\tau, \theta, (u_2)_\theta)d\theta)]d\tau, \; t \in I \end{aligned} \tag{13}$$

and

$$\begin{aligned} u_1(t) - u_2(t) &= [\phi_1(t) - \phi_2(t)] - [(g((u_1)_{t_1}, .., (u_2)_{t_p}))(t) \\ &- (g((u_2)_{t_1}, .., (u_2)_{t_p}))(t)], \text{ for } t \in [-r, 0). \end{aligned} \tag{14}$$

From our assumptions,

$$\begin{aligned} \|u_1(\theta) - u_2(\theta)\| &\le M\|\phi_1 - \phi_2\|_X + MG\|u_1 - u_2\|_Y \\ &\quad + ML\int_0^\theta [\|u_1 - u_2\|_{C([-r,\theta]:E)} \\ &\quad + K\int_0^\tau \|u_1 - u_2\|_{C([-r,\tau]:E)}]\, ds, \\ &\le M\|\phi_1 - \phi_2\|_X + MG\|u_1 - u_2\|_Y \\ &\quad + ML(1+aK)\int_0^t \|u_1 - u_2\|_{C([-r,s]:E)}\, ds, \\ &\quad \text{for } 0 \le \tau \le \theta \le t \le a. \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{\theta\in[0,t]} \|u_1(\theta) - u_2(\theta)\| \le M\|\phi_1 - \phi_2\|_X + MG\|u_1 - u_2\|_Y \\ + ML(1+aK)\int_0^t \|u_1 - u_2\|_{C([-r,s]:E)}\, ds, \end{aligned} \tag{15}$$

for $t \in [0, a]$. Simultaneously, by (14) and (A_4),

$$\|u_1(t) - u_2(t)\| \le M\|\phi_1 - \phi_2\|_X + MG\|u_1 - u_2\|_Y, \tag{16}$$

for $t \in [-r, 0)$. Since $M \ge 1$, (15) and (16) imply that

$$\begin{aligned} \|u_1(t) - u_2(t)\|_{C([-r,t]:E)} &\le M\|\phi_1 - \phi_2\|_X + MG\|u_1 - u_2\|_Y \\ &+ ML(1+aK)\int_0^t \|u_1 - u_2\|_{C([-r,s]:E)}\, ds, \end{aligned} \tag{17}$$

for $t \in I$. By Gronwall's inequality,

$$\|u_1(t) - u_2(t)\|_Y \le [M\|\phi_1 - \phi_2\|_X + MG\|u_1 - u_2\|_Y]\, e^{aML(1+aK)}. \tag{18}$$

and, therefore, (11) holds.

Finally, inequality (12) is a consequence of inequality (11). Hence the proof is complete. □

REMARK. If $G = 0$ then inequality (11) is reduced to the classical inequality

$$\|u_1 - u_2\|_Y \leq Me^{aML(1+Ka)} [\|\phi_1 - \phi_2\|_X]$$

which is characteristic for the continuous dependence of the semilinear functional-differential equation with the classical initial condition.

5. Application

As an application of the Theorem 3.1, we shall consider the system (1) with control parameter

$$\frac{du(t)}{dt} + Au(t) = Bv(t) + f(t, u_t, \int_0^t k(t, \tau, u_\tau)d\tau), \qquad t \in [0, a] \tag{19}$$

$$u(s) + [g(u_{t_1}, .., u_{t_p})](s) = \phi(s), \qquad s \in [-r, 0], \tag{20}$$

where B is a bounded linear operator from V, a Banach space, to E and $v \in L^2(I : V)$. Then the mild solution is given by

$$\begin{aligned} u(t) &= T(t)\phi(0) - T(t)[g(u_{t_1}, .., u_{t_p})](0) + \int_0^t T(t-\tau)Bv(\tau)d\tau \\ &+ \int_0^t T(t-\tau)f(\tau, u_\tau, \int_0^\tau k(\tau, \theta, u_\theta)d\theta)d\tau, \quad t \in I, \\ u(s) &+ [g(u_{t_1}, .., u_{t_p})](s) = \phi(s), \quad s \in I_0. \end{aligned}$$

We say that the system (19) is *controllable to the origin* if for any given initial function $\phi \in X$ there exists a control $v \in L^2(I : V)$ such that the mild solution $u(t)$ of (19) satisfies $u(a) = 0$.

For the controllability of nonlinear delay systems one can refer the papers [8, 9, 11]. To establish the result we need the following additional hypotheses:

(A_6) The linear operator W from V into E, defined by

$$Wv = \int_0^a T(a-s)Bv(s)ds$$

has an inverse operator W^{-1} defined on $L^2(I; V) / kerW$, such that the operator BW^{-1} is bounded.

(A_7) $MG + M\|BW^{-1}\|a[MG + MLa + MLKa^2 + ML + MLKa) < 1.$

THEOREM 5.1. *If the hypotheses* (A_1)-(A_4) *and* (A_6)-(A_7) *are satisfied, then the system* (19) *with* (20) *is controllable.*

Proof. Using the hypothesis (A_6), for an arbitrary function $x(.)$ define the control

$$\begin{aligned} v(t) &= -W^{-1}[T(a)\phi(0) - T(a)g(u_{t_1},..,u_{t_p})(0)) \\ &+ \int_0^a T(a-s)f(s,u_s,\int_0^s k(s,\tau,u_\tau)d\tau)ds](t). \end{aligned}$$

Now we shall show that, when using this control, the operator defined by

$$(\Phi u)(t) = \begin{cases} T(t)\phi(0) - T(t)[g(u_{t_1},..,u_{t_p})](0) + \int_0^t T(t-\tau)Bv(\tau)d\tau \\ \\ +\int_0^t T(t-\tau)f(\tau,u_\tau,\int_0^\tau k(\tau,\theta,u_\theta)d\theta)d\tau, \quad t \in I, \\ \\ \phi(s) - [g(u_{t_1},..,u_{t_p})](s), \quad s \in I_0 \end{cases}$$

has a fixed point. This fixed point is then a solution of equation (19). Substituting $v(t)$ in the above equation, we get

$$(\Phi u)(t) = \begin{cases} T(t)\phi(0) - T(t)[g(u_{t_1},..,u_{t_p})](0) \\ \qquad -\int_0^t T(t-\tau)BW^{-1}\Big[T(a)\phi(0) - T(a)g(u_{t_1},..,u_{t_p})(0)) \\ \qquad\qquad +\int_0^a T(a-s)f(s,u_s,\int_0^s k(s,\theta,u_\theta)d\theta)ds\Big](\tau)d\tau \\ \qquad +\int_0^t T(t-\tau)f(\tau,u_\tau,\int_0^\tau k(\tau,\theta,u_\theta)d\theta)d\tau, \quad t \in I, \\ \phi(s) - [g(u_{t_1},..,u_{t_p})](s), \quad s \in I_0. \end{cases}$$

Clearly, $(\Phi u)(a) = 0$, which means that the control v steers the semilinear integrodifferential system from the given initial function ϕ to the origin in time a, provided we can obtain a fixed point of the nonlinear operator Φ. The remaining part of the proof is similar to Theorem 3.1 and hence it is omitted. □

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