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A Note to the Stability of Fuzzy Closed-Loop Control Systems

DUG HUN HONG [1]

Abstract

Chen and Chen(FSS, 1993, 159-168) presented a reasonable analytical model of fuzzy closed-loop systems and proposed a method to analyze the stability of fuzzy control by the relational matrix of fuzzy system. Chen, Lu and Chen(IEEE Trans. Syst. Man Cybern., 1995, 881-888) formulated the sufficient and necessary conditions on stability of fuzzy closed-loop control systems. Gang and Chen(FSS, 1996, 27-34) deduced a linguistic relation model of a fuzzy closed loop control system from the linguistic models of the fuzzy controller and the controlled process and discussed the linguistic stability of fuzzy closed loop system by a linguistic relation matrix. In this paper, we study more on their models. Indeed, we prove the existence and uniqueness of equilibrium state X_e in which fuzzy system is stable and give closed form of X_e. The same examples in Chen and Chen and Gang and Chen are treated to analyze the stability of fuzzy control systems.

Key Words and Phrases: Fuzzy control; fuzzy closed-loop systems; stability.

1. Introduction

Many authors [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] have studied the stability of fuzzy systems with the concepts of fuzzy sets. Gupta et al.[5], Kiszka et al.[7], and Pedricz[8] discussed the controllability of fuzzy open loop systems, Tong[4, 5] introduced the model of fuzzy closed loop systems by which it is very difficult to analyze the stability of fuzzy systems. Recently Chen and Chen[1], and Chen et al[2, 3] proposed an analytical model of fuzzy closed loop systems and relational matrix of fuzzy system and Gang and Chen[4] deduced a linguistic relation model of a fuzzy closed loop control system from the linguistic models of the fuzzy controller and the controlled

[1]Associate Professor, School of Mechanical and Automotive Engineering, Catholic University of Taegu, Kyungbuk, 712 - 702, Korea

process and discussed the linguistic stability of fuzzy closed loop system by a linguistic relation matrix. In this note, we study more on the stability of fuzzy closed loop control systems.

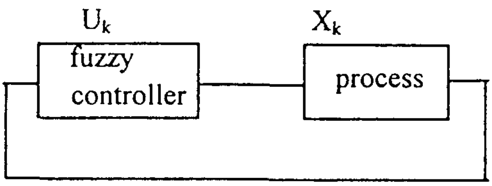

Fig. 1. Fuzzy closed-loop control system.

Indeed we prove the existence and the uniqueness of equilibrium state X_e in which fuzzy system is stable and give closed form of X_e. We also treat the same examples in Chen and Chen[1] and Gang and Chen[4] to analyze the stability of fuzzy control systems.

2. Fuzzy model of closed-loop systems.

In this section, we consider an analytical model of fuzzy closed loop systems proposed by Chen and Chen[1]. We consider the fuzzy control system shown in Fig. 1. where $X_k \in F(\bar{X})$ denotes the state variable of the system at time k, $U_k \in F(\bar{U})$ is the control variable and both $\bar{X}$ and $\bar{U}$ represent the universes of discourse which consist of finite elements. Assume that the behavior of the process is described by the following fuzzy rules:

$$\begin{aligned} R_1 &: \text{ if } X_k \text{ is } A_1 \text{ and } U_k \text{ is } B_1, \text{ then } X_{k+1} \text{ is } C_1 \text{ else} \\ R_2 &: \text{ if } X_k \text{ is } A_2 \text{ and } U_k \text{ is } B_2, \text{ then } X_{k+1} \text{ is } C_2 \text{ else} \\ &\vdots \\ R_n &: \text{ if } X_k \text{ is } A_n \text{ and } U_k \text{ is } B_n, \text{ then } X_{k+1} \text{ is } C_n \text{ else} \end{aligned} \tag{1}$$

Input : X_k is A' and U_k is B'

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Output : X_{k+1} is C'

Combining all the relations corresponding to the individual implications $R_i\ :\ (A_i, B_i) \rightarrow$

C_i, $i = 1, 2, \cdots, n$, by taking union $R = \bigvee_{i=1}^{n} R_i$, we have for fuzzy reasoning from (1)

$$\begin{aligned} C' &= (A', B') \circ \bigvee_{i=1}^{n} (A_i \text{ and } B_i \to C) \\ &= (A', B') \circ \bigvee_{i=1}^{n} R_i \\ &= (A', B') \circ R. \end{aligned}$$

Generally $\to$ is the Mamdani implication and $\circ$ is max - min composition rule. Namely, we have

$$M_{c'}(c) = \bigvee_{(a,b)\in \bar{X}\times\bar{X}} [(A'(a) \wedge B'(b))] \wedge \bigvee_{i=1}^{n} (A_i(a) \wedge B_i(b) \wedge C_i(c)).$$

We now briefly summarize the model of fuzzy control system suggested by Chen and Chen. Let the fuzzy controller in Figure 1 be given by

$$U_k = X_k \circ R_c \tag{2}$$

where $R_c \in F(\bar{X} \times \bar{U})$ is fuzzy relational matrix.

When $X_k = A'$ and $U_k = B'$, (2) gives $B' = A' \circ R_c$. Assuming that $R_c \circ G_i = \bar{B}_i, C_i = A' \circ [(A_i \circ \bar{B}_i) \to C_i]$, then we have $C' = A' \circ \bigvee_{i=1}^{n}(\bar{A}_i \wedge C_i) = A \circ R$, where $\bar{A}_i = A_i \wedge \bar{B}_i = A_i \wedge (R_c \circ B_i), \bar{A}_i \in F(X)$ and $R = \bigvee_{i=1}^{n}(\bar{A}_i \wedge C_i), R \in F(\bar{X} \times \bar{X})$. Substituting fuzzy variables X_{k+1} and X_k for fuzzy variables C' and A' in the above equality gives

$$X_{k+1} = X_k \circ R. \tag{3}$$

Equation (3) is the model of the closed-loop systems suggested by Chen and Chen[1]. Chen et al[2, 3] treated similar fuzzy closed-loop control systems.

Recently, Gang and Chen[2] also deduced a linguistic relation model of fuzzy closed loop control system given by $X_{k+1} = X_k \circ R$, where the element of R take their values from $\{0, 1\}$, from the linguistic models of the fuzzy controller and the controlled process.

3. Stability of fuzzy control systems

There are several papers [1, 2, 4, 5, 6, 7, 9, 10] which discussed the stability of fuzzy systems given by $X_{k+1} = X_k \circ R$. The stability degree between arbitrary states X_k and the equilibrium state X_e defined in [9] can be used to indicate the degree of X_k approaching X_e.

Definition 1. If a state X_e satisfies $X_e = X_e \circ R$ for the system $X_{k+1} = X_k \circ R$, then X_e is called the equilibrium state of the fuzzy system.

Definition 2[9]. Let X_e be a normal fuzzy set and an equilibrium state of fuzzy systems denoted by $X_{k+1} = X_k \circ R$. Then the degree of stability is given by

$$\alpha(X_n, X_e) = 1 - X_n \circ X_e$$

where the smaller $\alpha(X_n, X_e)$, the nearer is the equilibrium X_e to the state X_n.

Chen and Chen[1] and Chen et al[2] introduced the following theorem:

Theorem Any fuzzy system given by $X_{k+1} = X_k \circ R$ must be stable and will approach the equilibrium state X_e for arbitrary initial state X_k which is normal if $R^n \circ X_e = [1, 1, \cdots, 1]^t$ for any number n after $n \geq N$, where N is defined as a step-number after which the fuzzy system will be near X_e, i.e., when $n < N$, $R^n \circ X_e < [1, 1, \cdots, 1]^t$ and when $n \geq N$, $R^n \circ X_e = [1, 1, \cdots, 1]^t$.

From this result, we don't have any information about the existence of X_e. In the following, we prove the existence and the uniqueness of equilibrium state X_e in which fuzzy system is stable.

We first consider the following lemmas.

Lemma 1. Let $X_1, X_2 \in F(\bar{X})$ and $R \in F(\bar{X} \times \bar{X})$. Then $(X_1 \vee X_2) \circ R = (X_1 \circ R) \vee (X_2 \circ R)$.

Proof. For $x \in \bar{X}$,

$$\begin{aligned}
(X_1 \vee X_2) \circ R(x) &= \max_{y \in \bar{X}} (X_1(y) \vee X_2(y)) \wedge R(y, x) \\
&= \max_{y \in \bar{X}} [X_1(y) \wedge R(y, x)] \vee [(X_2(y) \wedge R(y, x)] \\
&= \max_{y \in \bar{X}} [(X_1(y) \wedge R(y, x)] \vee \max_{y \in \bar{X}} [X_2(y) \wedge R(y, x)] \\
&= (X_1 \circ R(x)) \vee (X_2 \circ R(x)),
\end{aligned}$$

which proves the lemma.

The following lemma is immediate from Lemma 1.

Lemma 2. Let $X_i, i = 1, \cdots, m$, be equilibrium states of fuzzy system denoted by $X_{k+1} = X_k \circ R$, then $\bigvee_{i=1}^{n} X_i$ is also an equilibrium state for the fuzzy system $X_{k+1} = X_k \circ R$.

Lemma 3. Let $\lim_{n\to\infty} R^n = R^*$, then for any $X \in F(\bar{X})$, $X \circ R^*$ is an equilibrium state for the fuzzy system $X_{k+1} = X_k \circ R$.

Proof. From the condition that $\lim_{n\to\infty} R^n = R^*$, we see that $R^* \circ R = R^* = R \circ R^*$. Now, for $X \in F(\bar{X})$, $(X \circ R^*) \circ R = X \circ (R^* \circ R) = X \circ R^*$, which completes this lemma.

We now prove our main theorem. We denote $I_A \in F(\bar{X}), A \subset \bar{X}$, by

$$I_A(x) = \begin{cases} 1 & x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1(Existence). Let $\lim_{n\to\infty} R^n = R^*$ and let any fuzzy system denoted by $X_{k+1} = X_k \circ R$ be given. Then $I_{\{x\}} \circ R^*$ is normal fuzzy set for all $x \in \bar{X}$, if and only if there exists an equilibrium state X_e such that , for any initial state X_k which is normal,

$$\alpha(X_n, X_e) \to 0 \text{ as n} \to \infty.$$

Proof. We first note that $I_{\{x\}} \circ R^*$, $x \in \bar{X}$ and $I_{\bar{X}} \circ R^*$ are equilibrium state by Lemma 3 and also note that

$$\bigvee_{x\in\bar{X}} (I_{\{x\}} \circ R^*) = I_{\bar{X}} \circ R^*.$$

Suppose that $I_{\{x\}} \circ R^*$ is a normal fuzzy set for all $x \in \bar{X}$. Then for arbitrary given a normal fuzzy set X_k, there exists $x \in \bar{X}$ such that $X_k(x) = 1$. Now

$$X_{k+n} \circ R = X_k \circ R^n \to X_k \circ R^* \geq I_{\{x\}} \circ R^*$$

Let $X_e = I_{\bar{X}} \circ R^*$ then, as $n \to \infty$. Since $I_{\{x\}} \circ R^*$ is normal by assumption,

$$\begin{aligned} (X_k \circ R^*) \circ X_e &\geq (X_k \circ R^*) \circ (I_{\{x\}} \circ R^*) \\ &\geq (I_{\{x\}} \circ R^*) \circ (I_{\{x\}} \circ R^*) = 1, \end{aligned}$$

which means $\alpha(X_n, X_e) \to 0$ as $n \to \infty$.

Conversely, we assume that there exists a $x \in \bar{X}$ such that $I_{\{x\}} \circ R^*$ is not normal fuzzy set. Let $X_k = I_{\{x\}}$ then

$$X_{k+n} \circ R = X_k \circ R^n \to I_{\{x\}} \circ R^*$$

and

$$(I_{\{x\}} \circ R^*) \circ X_e \leq \max_{y \in \bar{X}} (I_{\{x\}} \circ R^*)(y) < 1.$$

Hence $\alpha(X_n, X_e) \not\to 0$ as $n \to \infty$, which proves the theorem.

Theorem 2(Uniqueness). Let $lim_{n\to\infty} R^n = R^*$ and let any fuzzy system denoted by $X_{k+1} = X_k \circ R$ be given. Suppose that for any $x \in \bar{X}$, there exists $x' \in \bar{X}$ such that $I_{\{x\}} \circ R = I_{\{x'\}}$. Then $X_e = I_{\bar{X}} \circ R^*$ is the unique equilibrium state in which the fuzzy system is stable for any initial state X_k.

Proof. By Theorem 1, $X_e = I_{\bar{X}} \circ R$ is an equilibrium state in which the fuzzy system is stable for any initial state X_k. So it remains to prove the uniqueness. Suppose that $X_{e'}$ is another equilibrium state in which the fuzzy system is stable for any initial state X_k. Then, since $X_{e'} = X_{e'}^n \circ R = X_{e'} \circ R^n \leq I_{\bar{X}} \circ R^n$ for all n, $X_{e'} \leq X_e$. On the other hand, noting that for any $x \in \bar{X}$, there exists $x'' \in \bar{X}$ such that $I_{\{x\}} \circ R^n \to I_{\{x\}} \circ R^* = I_{\{x''\}}$ as $n \to \infty$ by the assumption, $\alpha(X_{e'}, I_{\{x''\}}) = 1$, which means that $X_{e'} \circ I_{\{x''\}} = 1$. Then $X_{e'} \geq I_{\{x\}} \circ R^* = I_{\{x''\}}$ for all $x \in \bar{X}$. Hence

$$X_{e'} \geq \max_{x \in \bar{X}} (I_{\{x\}} \circ R^*) = I_{\bar{X}} \circ R^* = X_e,$$

which proves the theorem.

4. Examples.

In this section, we consider the same examples in Chen and Chen[1] and Gang and Chen[4] and evaluate the stability of fuzzy systems.

Example 1[1]. Let the universe of discourse $\bar{X}$ consist of five element -2, -1, 0, 1, 2. For a fuzzy system given by $X_{k+1} = X_k \circ R$, consider the following fuzzy relations of the closed loop control systems:

$$R_1 = \begin{pmatrix} 0 & 0.5 & 1 & 0.5 & 0 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0 & 0.5 & 1 & 0.5 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 1 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 1 & 0 \end{pmatrix}$$

and

$$R_3 = \begin{pmatrix} 0 & 1 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 & 1 \end{pmatrix}.$$

Then

$$R_1^* = R_2^* = \begin{pmatrix} 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \end{pmatrix}, \quad R_3^* = \begin{pmatrix} 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 & 1 \end{pmatrix}$$

and, by Theorem 1, we can easily find an equilibrium state X_{e_i} for the fuzzy system $X_{k+1} = X_k \circ R_i, \quad i = 1, 2, 3$ such that, for any initial state X_k which is normal, $\alpha(X_{k+n}, X_{e_i}) \to 0$ as $n \to \infty$ as follows:

$$X_{e_1} = X_{e_2} = [1, \ 1, \ 1, \ 1, \ 1] \begin{pmatrix} 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \end{pmatrix} = [0.5, \ 0.5, \ 1, \ 0.5, \ 0.5]$$

and

$$X_{e_3} = [1, \ 1, \ 1, \ 1, \ 1] \begin{pmatrix} 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 & 1 \end{pmatrix} = [0.5, \ 0.5, \ 1, \ 0.5, \ 1].$$

Example 2[4]. Let the universe of discourse consists of seven elements and let two linguistic relation matrixes be given as follow:

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$R_1^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_2^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and by Theorem 2, the following X_{e_1} and X_{e_2} are the unique equilibrium state such that for any initial state X_k which is normal, the fuzzy system $X_{k+1} = X_k \circ R_i$, $i = 1, 2$ is stable, respectively :

$$\begin{aligned} X_{e_1} &= [1,1,1,1,1,1,1] \circ R_1^* = [1,1,0,0,0,0,0] \\ X_{e_2} &= [1,1,1,1,1,1,1] \circ R_2^* = [0,1,0,0,0,0,0]. \end{aligned}$$

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