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ON POSITIVE-NORMAL OPERATORS

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Abstract. In this paper we study the properties of positive-normal operators and show that Weyl's theorem holds for some totally positive-normal operators.

1. Introduction

Let H be an infinite dimensional complex Hilbert space, let $\mathcal{L}(H)$ the algebra of all bounded linear operators on H. An operator $T \in \mathcal{L}(H)$ is called a *positive-normal* (or *posinormal*) operator if there exists a positive operator $P \in \mathcal{L}(H)$, called the interrupter, such that $TT^* = T^*PT$. This class of operators was introduced and studied in [12]. Recall ([13]) that an operator $T \in \mathcal{L}(H)$ is said to be *dominant* if for each $\lambda \in \mathbb{C}$ there exists a positive number M_λ such that

$$(T-\lambda)(T-\lambda)^* \leq M_\lambda (T-\lambda)^*(T-\lambda).$$

If the constants M_λ are bounded by a positive number M, then T is said to be *M-hyponormal.* Also, we may note that if T is 1-hyponormal, then T is *hyponormal.* In [12] it is well known that the class of dominant operators is a proper subset of the class of positive-normal operators. So

$$\begin{aligned} \text{hyponormal} \ &\Rightarrow \ M-\text{hyponormal} \\ &\Rightarrow \ \text{dominant} \\ &\Rightarrow \ \text{positive-normal.} \end{aligned} \tag{1}$$

In view of (1), it is very natural to study the properties of positive-normal operators.

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An operator $T \in \mathcal{L}(H)$ is called *Fredholm* if the range of T, denoted by $\operatorname{Ran}(T)$, is closed, and the kernel of T, denoted by $\operatorname{Ker}(T)$, and $H/R(T)$ are both finite dimensional. If T is Fredholm, then the *index* of T is defined by

$$\operatorname{ind}(T) = \dim \operatorname{Ker}(T) - \dim H/R(T)$$

and a Fredholm operator with index zero is called *Weyl* [7]. We shall denote $\sigma(T)$, $\sigma_p(T)$, $\pi_0(T)$, and $\operatorname{iso}\sigma(T)$ by the spectrum of T, the set of all eigenvalues of T, the set of all eigenvalues of finite multiplicity of T, and the set of all isolated points of $\sigma(T)$, respectively. We write

$$\pi_{00}(T) = \pi_0(T) \cap \operatorname{iso}\sigma(T)$$

for the set of all isolated eigenvalues of finite multiplicity of T. The *Weyl spectrum* of T, denoted by $w(T)$, is defined by

$$w(T) = \{\lambda \in \mathbb{C} : \ T - \lambda \text{ is not Weyl}\}.$$

Following Coburn [4] we say that *Weyl's theorem holds for T* if

$$\sigma(T) \setminus w(T) = \pi_{00}(T).$$

In a vast literature, there exist several classes of operators for which Weyl's theorem holds. In particular, Coburn [4] showed that Weyl's theorem holds for hyponormal operators, which was extended to M-hyponormal operators by Arora and Kumer [1]. Also, in [3] it was shown that Weyl's theorem holds for p-hyponormal operators. On the other hand, using results of Oberai [11], Lee and Lee [10] showed that the spectral mapping theorem holds for $w(T)$ and Weyl's theorem holds for $f(T)$ when T is hyponormal and f is a function analytic on a neighborhood of $\sigma(T)$. Recently, this was also improved by Hou and Zhang [8] to show that the spectral mapping theorem holds for Weyl spectrum of a dominant operator T and that Weyl's theorem holds for $f(T)$ when T is M-hyponormal.

In this paper we study important properties of positive-normal operators and show that Weyl's theorem holds for some totally positive-normal operators.

2. Main results

The following result suggests that the definition of a positive-normal operator can be weaken.

THEOREM 1. *An operator $T \in \mathcal{L}(H)$ is positive-normal if and only if there exists a positive operator $P \in \mathcal{L}(H)$ such that $TT^* \leq T^*PT$.*

Proof. It suffices to show that if there exists a positive operator $P \in \mathcal{L}(H)$ such that $TT^* \leq T^*PT$, then T is positive-normal. For any $x \in H$

$$\begin{aligned} \|T^*x\|^2 &= \langle TT^*x, x\rangle \\ &\leq \langle T^*PTx, x\rangle \\ &= \langle \sqrt{P}Tx, \sqrt{P}Tx\rangle \\ &= \|\sqrt{P}Tx\|^2 \\ &\leq \|\sqrt{P}\|^2\|Tx\|^2. \end{aligned}$$

Hence for any $x \in H$

$$\|T^*x\| \leq \|\sqrt{P}\|\|Tx\|.$$

Set $\lambda = \|\sqrt{P}\|$. Then for any $x \in H$

$$\|T^*x\| \leq \lambda\|Tx\|.$$

Thus

$$TT^* \leq \lambda^2 T^*T \quad \text{for some } \lambda \geq 0.$$

By [5], there exists $B \in \mathcal{L}(H)$ such that $T = T^*B$. Therefore,

$$TT^* = (T^*B)(B^*T) = T^*(BB^*)T.$$

So T is a positive-normal operator with an interrupter BB^*. □

If we take $P = I$ in Theorem 1, we get the following corollary.

COROLLARY 2. *Every hyponormal operator is positive-normal.*

PROPOSITION 3. *If $T \in \mathcal{L}(H)$ is positive-normal, then $Ker(T) = Ker(T^2)$.*

Proof. It suffices to show that $\mathrm{Ker}(T^2) \subset \mathrm{Ker}(T)$. If $x \in \mathrm{Ker}(T^2)$, then $T^2x = 0$. Hence $Tx \in \mathrm{Ker}(T)$. Since $\mathrm{Ker}(T) \subset \mathrm{Ker}(T^*)$, $Tx \in \mathrm{Ker}(T^*)$. Hence $T^*Tx = 0$. Now,

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leq \|T^*Tx\|\|x\| = 0$$

Thus $Tx = 0$, and so we have $x \in \mathrm{Ker}(T)$. □

We remark that the unilateral backward shift U^* is not positive-normal since $\mathrm{Ker}(U^*) \neq \mathrm{Ker}(U^{*2})$.

PROPOSITION 4. *If $T \in \mathcal{L}(H)$ is positive-normal with interrupter P and $\mathcal{M} \in Lat(T)$, then $T|_{\mathcal{M}}$ is also positive-normal.*

Proof. If $\mathcal{M} \in \mathrm{Lat}(T)$, let $Q : H \longrightarrow \mathcal{M}$ be the orthogonal projection of H onto $\mathcal{M}$. Then for all $m \in \mathcal{M}$

$$\begin{aligned}\langle (T|_{\mathcal{M}})^* m, m\rangle &= \langle m, T|_{\mathcal{M}} m\rangle \\ &= \langle m, Tm\rangle \\ &= \langle T^* m, m\rangle \\ &= \langle T^* Q m, m\rangle.\end{aligned}$$

Hence $(T|_{\mathcal{M}})^* = T^*Q$ on $\mathcal{M}$. For all $m \in \mathcal{M}$,

$$\|(T|_{\mathcal{M}})^* m\| = \|T^* Q m\| = \|(\sqrt{P}T)|_{\mathcal{M}} m\|.$$

Hence $T|_{\mathcal{M}}$ is positive-normal. □

Halmos showed in [6, # 204] that a partial isometry is subnormal if and only if it is the direct sum of an isometry and zero. We generalize this theorem to the case of a positive-normal operator.

THEOREM 5. *A partial isometry T is quasinormal (i.e. $(T^*T)T = T(T^*T)$) if and only if T is positive-normal.*

Proof. Assume T is a partial isometry and positive-normal operator. Since $\mathrm{Ker}(T)$ is a reducing subspace for T from [12, Corollary 2.3],

$$T = \begin{pmatrix} T|_{\mathrm{Ker}(T)} & 0 \\ 0 & T|_{(\mathrm{Ker}(T))^{\perp}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix},$$

where $A = T|_{(\mathrm{Ker}(T))^{\perp}}$ is isometry. Hence

$$T^*T = \begin{pmatrix} 0 & 0 \\ 0 & A^*A \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

So $(T^*T)T = T(T^*T)$. Thus T is quasinormal. Since the converse implication is trivial, we complete the proof. □

It is well known that if T is hyponormal and compact then T is normal. But it is not true in the case of positive-normal operators.

EXAMPLE 6. Let H be a 2-dimensional Hilbert space and let T be defined on H as

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then $T^*T \neq TT^*$. Hence T is not normal. On the other hand, consider a positive operator

$$P = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Then $TT^* = T^*PT$. Hence T is positive-normal.

From Example 6 we conclude that positive-operators on a finite dimensional Hilbert space are not necessary normal.

It is trivial that the positive-normality is invariant under unitary equivalence. But the similarity does not preserve the positive-normality.

EXAMPLE 7. Let H be a 3-dimensional Hilbert space and let A and T be defined on H as

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

Then A is positive-normal, and if we take

$$X = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

then $T = XAX^{-1}$. Hence T is similar to A. Now

$$\mathrm{Ker}(T) = \{x = (x_1, x_2, x_3) \in H : x_1 = x_2 + x_3\}$$

and

$$\mathrm{Ker}(T^*) = \{x = (x_1, x_2, x_3) \in H : x_2 = -x_3\}.$$

Hence

$$\mathrm{Ker}(T) \not\subseteq \mathrm{Ker}(T^*).$$

Thus T is not positive-normal.

If $T \in \mathcal{L}(H)$ is hyponormal and quasinilpotent (i.e. $\sigma(T) = \{0\}$), then T is a zero operator. But it is not true for the case of positive-normal operators.

EXAMPLE 8. If T is a unilateral weighted shift with positive weights w_n such that $w_n \longrightarrow 0$, then T is positive-normal by [12, Proposition 1.1] and $\sigma(T) = \{0\}$ by [6, # 96].

Let $\mathcal{Q}$ be the class of operators on H such that if $\sigma(T) = \{0\}$ then $T = 0$. For example, the class of hyponormal operators is included in $\mathcal{Q}$.

We say that an operator T is *totally positive-normal* (or *totally posinormal*) if the translates $T - \lambda$ are positive-normal for all $\lambda \in \mathbb{C}$. Rhaly gave an example of a positive-normal operator whose translate is not a positive-normal operator ([12]).

It is well known ([12, Proposition 3.5]) that T is totally positive-normal if and only if T is dominant.

THEOREM 9. *If $T \in \mathcal{L}(H)$ is totally positive-normal and $T|_{\mathcal{M}} \in \mathcal{Q}$ for every $\mathcal{M} \in Lat(T)$, then T is isoloid* (i.e. $\text{iso}\sigma(T) \subset \sigma_p(T)$).

Proof. Since $T - \lambda$ is positive-normal for all $\lambda \in \mathbb{C}$, it suffices to show that if $0 \in \text{iso}\sigma(T)$ then $0 \in \sigma_p(T)$. Choose $\rho > 0$ sufficiently small that 0 is the only point of $\sigma(T)$ contained in the circle $|\lambda| = \rho$. Define

$$E = \int_{|\lambda|=\rho} (\lambda I - T)^{-1} d\lambda.$$

Then E is the Riesz projection corresponding to 0. So $\mathcal{M} := EH$ is an invariant subspace for T. Moreover, $\mathcal{M} \neq \{0\}$ and $\sigma(T|_{\mathcal{M}}) \in \mathcal{Q}$, and so $T|_{\mathcal{M}} = 0$. Thus $\mathcal{M} \subseteq \text{Ker}(T)$. Since $\text{Ker}(T) \subseteq \mathcal{M}$ is clear, we have that $\mathcal{M} = \text{Ker}(T) \neq \{0\}$. Therefore, T is not injective, i.e. $0 \in \sigma_p(T)$. □

COROLLARY 10. *Let $T \in \mathcal{L}(H)$ be totally positive-normal and $T|_{\mathcal{M}} \in \mathcal{Q}$ for every $\mathcal{M} \in Lat(T)$. If f is analytic on a neighborhood of $\sigma(T)$, then*

$$f(\sigma(T) \setminus \pi_{00}(T)) = \sigma(f(T)) \setminus \pi_{00}(f(T)).$$

Proof. The proof follows from Theorem 9 and [10]. $\square$

LEMMA 11. *Let $T \in \mathcal{L}(H)$ be totally positive-normal and let $\lambda, \mu \in \sigma_p(T)$, where $\lambda \neq \mu$. If x and y are eigenvectors of λ and μ, respectively, then $\langle x, y\rangle = 0$.*

Proof. Since $\mathrm{Ker}(T-\mu) \subset \mathrm{Ker}(T^* - \overline{\mu})$ we can see that

$$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Tx, y\rangle = \langle x, T^*y\rangle = \langle x, \overline{\mu}y\rangle = \mu\langle x, y\rangle.$$

Hence we have that $\langle x, y\rangle = 0$. $\square$

LEMMA 12 ([2, Lemma 3]). *Let $T \in \mathcal{L}(H)$. Suppose that T satisfies the following condition C:*

C. If $\{\lambda_n\}$ is an infinite sequence of distinct points of the set of eigenvalues of finite multiplicity of T and $\{x_n\}$ is any sequence of corresponding normalized eigenvectors, then the sequence $\{x_n\}$ does not converge.

Then

$$\sigma(T) \setminus \pi_{00}(T) \subset w(T).$$

THEOREM 13. *If $T \in \mathcal{L}(H)$ is totally positive-normal and $T|_{\mathcal{M}} \in \mathcal{Q}$ for every $\mathcal{M} \in Lat(T)$, then T satisfies Weyl's theorem.*

Proof. If T is totally positive-normal, then by Lemma 11 T satisfies the condition C. From Lemma 12 we have

$$\sigma(T) \setminus \pi_{00}(T) \subset w(T).$$

Conversely, without loss of generality it may be assumed that $0 \in \pi_{00}(T)$. Since $0 \in \mathrm{iso}(T)$, as in the proof of Theorem 9, consider Riesz projection E corresponding to 0. Then we also see that

$$EH = \mathrm{Ker}(T) \text{ and } S := T|_{EH^\perp} \text{ is invertible.}$$

Therefore,

$$T = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \quad \text{on } H = EH \oplus EH^\perp.$$

Let

$$K = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{on } H = EH \oplus EH^\perp.$$

Since by the assumption $\dim(EH) < \infty$, K is a finite rank operator and

$$T + K = \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}$$

is invertible. Thus by [7, Theorem 6.5.5] T is Weyl, i.e. $0 \notin w(T)$. Hence we have that $\sigma(T) \setminus \pi_{00}(T) \supset w(T)$. □

COROLLARY 14. *Let $T \in \mathcal{L}(H)$ be totally positive-normal and $T|_{\mathcal{M}} \in \mathcal{Q}$ for every $\mathcal{M} \in Lat(T)$. If N is a nilpotent commuting with T, then $T + N$ satisfies Weyl's theorem.*

Proof. The proof follows from Theorem 13 and [11, Theorem 3]. □

COROLLARY 15. *Let $T \in \mathcal{L}(H)$ be totally positive-normal and $T|_{\mathcal{M}} \in \mathcal{Q}$ for every $\mathcal{M} \in Lat(T)$. If F is a finite rank commuting with T, then $T + F$ satisfies Weyl's theorem.*

Proof. The proof follows from Theorem 9, Theorem 13 and [8, Theorem 3.3]. □

THEOREM 16. *Let $T \in \mathcal{L}(H)$ be totally positive-normal and $T|_{\mathcal{M}} \in \mathcal{Q}$ for every $\mathcal{M} \in Lat(T)$. If f is analytic on a neighborhood of $\sigma(T)$, then Weyl's theorem holds for $f(T)$.*

Proof. The proof follows from Theorem 13, Corollary 10, and [8, Theorem 2.2]. □

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