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h-STABILITY OF PERTURBED VOLTERRA DIFFERENCE SYSTEMS

Sung Kyu Choi, Nam Jip Koo, and Yoon Hoe Goo

ABSTRACT. We discuss the h-stability of perturbed Volterra difference systems by means of the resolvent matrix and discrete inequalities.

1. Introduction

Medina and Pinto [8-10] extended the notion of exponential stability to a variety of reasonable difference systems called h-systems. The new concept of stability (called h-stability) permits to obtain a uniform treatment for the concept of stability in difference systems. Also, it allows to get asymptotic formulae for weakly stable difference systems.

The stability problem for Volterra difference systems was studied by Elaydi [5], Elaydi and Murakami [6], Raffoul [11], Sheng and Agarwal [12], Zouyousefain and Leela [13], Choi and Koo [3], and others.

Let $A(n)$ and $K(n,s)$ be $m \times m$ nonsingular matrices whose elements are real functions defined on $\mathbb{N}(n_0) = \{n_0, n_0+1, \cdots, n_0+k, \cdots\}$ and $\mathbb{N}(n_0) \times \mathbb{N}(n_0)$, respectively. Here n_0 is a fixed nonnegative integer and k is a positive integer.

We consider the linear Volterra difference system

$$(1)\qquad \begin{aligned} \triangle x(n) &\equiv x(n+1) - x(n) \\ &= A(n)x(n) + \sum_{s=n_0}^{n-1} K(n,s)x(s),\ x(n_0) = x_0, \end{aligned}$$

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and its perturbation

$$(2) \qquad \triangle y(n) = A(n)y(n) + \sum_{s=n_0}^{n-1} K(n,s)y(s) + F(n), \; y(n_0) = y_0,$$

where $x(n)$ and $y(n)$ are vectors in the m-dimensional real Euclidean space $\mathbb{R}^m$, and $F(n)$ is a vector function defined on $\mathbb{N}(n_0)$. Let $x(n) = x(n, n_0, x_0)$ and $y(n) = y(n, n_0, y_0)$ be the unique solutions of (1) and (2) satisfying the initial conditions $x(n_0) = x_0$ and $y(n_0) = y_0$, respectively. The symbol $|\cdot|$ will be used to denote any convenient vector norm.

In this paper we discuss the h-stability of perturbed Volterra difference systems :

$$\triangle y(n) = A(n)y(n) + \sum_{s=n_0}^{n-1} K(n,s)y(s) + g(n, y(n))$$

and

$$\triangle y(n) = A(n)y(n) + \sum_{s=n_0}^{n-1} \big[K(n,s)y(s) + G(n,s,y(s))\big] + F(n, y(n)),$$

where $g : \mathbb{N}(n_0) \times \mathbb{R}^m \to \mathbb{R}^m$ is a vector function satisfying $g(n,0) = 0, \; n \geq n_0$ and $F(n,x) = \int_0^1 [f_x(n,\theta x) - f_x(n,0)]d\theta \cdot x, \; G(n,s,x) = \int_0^1 [g_x(n,s,x\theta) - g_x(n,s,0)]d\theta \cdot x$.

For this purpose, we use the resolvent matrix and discrete inequalities. We recall definitions of stability notions in [7, 10].

Definition 1.1. The zero solution of (1) is said to be

(S) *stable* if given $\varepsilon > 0$ and $n_0 \geq 0$, there exists $\delta = \delta(\varepsilon, n_0) > 0$ such that $|x_0| < \delta$ implies $|x(n, n_0, x_0)| < \varepsilon$ for all $n \geq n_0$.

(US) *uniformly stable* if it is stable and δ can be chosen independent of n_0.

(ULS) *uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that

$$|x(n, n_0, x_0)| \leq M|x_0| \;\text{ whenever } |x_0| \leq \delta \;\text{ and } n \geq n_0 \geq 0.$$

(ES) *exponentially stable* if there exist $\delta > 0, a > 0$ and $\eta \in (0,1)$ such that $|x_0| < \delta$ implies $|x(n, n_0, x_0)| \leq a|x_0|\eta^{n-n_0}$.

DEFINITION 1.2. System (1) is called an *h-system* around the null solution, or more briefly an *h-system*, if there exist a positive function $h : \mathbb{N}(n_0) \to \mathbb{R}$ and a constant $c \geq 1$ such that

$$|x(n, n_0, x_0)| \leq c|x_0|h(n)h^{-1}(n_0), \quad n \geq n_0$$

for $|x_0|$ small enough (here $h^{-1}(n) = \frac{1}{h(n)}$).

If h is a bounded function, then an h-system permits the following types of stability:

The zero solution of system (1), or more briefly system (1), is said to be

(hS) *h-stable* if $c \geq 1$, δ exist as well as a positive bounded function $h : \mathbb{N}(n_0) \to \mathbb{R}$ such that

$$|x(n, n_0, x_0)| \leq c|x_0|h(n)h^{-1}(n_0), \quad n \geq n_0$$

for $|x_0| \leq \delta$,

(GhS) *globally h-stable* if system (1) is hS for every $x_0 \in D$, where $D \subset \mathbb{R}^m$ is a region which includes the origin.

The various notions about hS given by Definition 1.2 include several types of known stability properties as uniform stability, uniform Lipschitz stability and exponential stability. See [2-4, 8-10]. Also, some examples about hS for difference systems are presented in [2, 10].

DEFINITION 1.3. A function $w : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be of the *class* $\hat{F}$ if

(i) $w(u)$ is nondecreasing and continuous for $u \geq 0$ and positive for $u > 0$,

(ii) there exists a nonnegative function r (multiplier function) defined on $(0, \infty)$ such that

$$w(\alpha u) \leq r(\alpha)w(u) \text{ for } \alpha > 0, \; u \geq 0,$$

(iii) $\lim_{\alpha \to 0+} \frac{r(\alpha)}{\alpha}$ exists.

2. Main results

For our discussion we need the following lemmas. The first lemma is a result corresponding to Fubini's theorem, which can be proved by induction.

LEMMA 2.1 ([13, Lemma 2.1]). *Let $L(n,s)$, $K(n,s)$ be $m\times m$ matrices defined for $s,n \ge n_0$ such that L and K are zero matrices for $s,n < n_0$. Then the relation*

$$\sum_{s=n_0}^{n-1} L(n,s+1)\sum_{\sigma=n_0}^{s-1} K(s,\sigma)x(\sigma) = \sum_{s=n_0}^{n-1}\sum_{\sigma=s+1}^{n-1} L(n,\sigma+1)K(\sigma,s)x(s)$$

holds, where $x : \mathbb{N}(n_0) \to \mathbb{R}^m$ is a vector function defined on $\mathbb{N}(n_0)$.

The *resolvent matrix* $L(n,s)$ of (2) can be defined as the unique solution of the matrix difference system

$$\begin{aligned} &L(n,s+1) - L(n,s) + L(n,s+1)A(s) \\ &+ \sum_{\sigma=s+1}^{n-1} L(n,\sigma+1)K(\sigma,s) = 0, \quad n > s, \end{aligned} \tag{3}$$

with $L(n,n) = I$, the identity matrix, (see Theorem 2.1 of [9]). We can obtain the representation of the solution $y(n,n_0,y_0)$ of (2) with respect to the resolvent matrix in the following :

LEMMA 2.2 ([13, Theorem 2.2]). *The unique solution $y(n,n_0,y_0)$ of (2) with $y(n_0,n_0,0) = y_0$ is given by*

$$y(n,n_0,y_0) = L(n,n_0)y_0 + \sum_{s=n_0}^{n-1} L(n,s+1)F(s). \tag{4}$$

Proof. Let $y(n) = y(n,n_0,y_0)$. Setting $p(s) = L(n,s)y(s)$, we have

$$\begin{aligned} \triangle p(s) &= [L(n,s+1) - L(n,s)]y(s) + L(n,s+1)[y(s+1) - y(s)] \\ &= [L(n,s+1) - L(n,s) + L(n,s+1)A(s)]y(s) \\ &\quad + L(n,s+1)\Big(\sum_{\sigma=n_0}^{s-1} K(s,\sigma)y(\sigma) + F(s)\Big). \end{aligned} \tag{5}$$

Summing both sides of (5) from n_0 to $n-1$, we obtain

$$\begin{aligned} &L(n,n)y(n) - L(n,n_0)y_0 \\ &= \sum_{s=n_0}^{n-1} \big[L(n,s+1) - L(n,s) + L(n,s+1)A(s) \\ &\quad + \sum_{\sigma=s+1}^{n-1} L(n,\sigma+1)K(\sigma,s)\big]y(s) + \sum_{s=n_0}^{n-1} L(n,s+1)F(s), \end{aligned}$$

by Lemma 2.1 and system (3). Thus we have the formula (4). □

The following lemma obtained by Medina in [9, Theorem 7] shows one condition under which a linear Volterra difference system (1) is an h-system.

LEMMA 2.3. *The linear Volterra difference system* (1) *is an h-system, if and only if, there exist a constant $c \geq 1$ and a positive function $h : \mathbb{N}(n_0) \to \mathbb{R}$ such that*

$$|L(n, n_0)| \leq ch(n)h^{-1}(n_0), \; n \geq n_0,$$

where $L(n, n_0)$ is the resolvent matrix of system (2).

REMARK 1. For the trivial solution $x = 0$ of (1) we obtain the equivalence of stability properties

$$\text{US} \iff \text{ULS} \iff \text{GhS} \iff \text{hS}$$

by Lemma 2.3.

Now, we consider the perturbed system

$$\Delta y(n) = A(n)y(n) + \sum_{s=n_0}^{n-1} K(n,s)y(s) + g(n, y(n)) \tag{6}$$

of (1), here $g : \mathbb{N}(n_0) \times \mathbb{R}^m \to \mathbb{R}^m$ is a vector function satisfying $g(n, 0) = 0$ for all $n \geq n_0$.

THEOREM 2.4. *Assume that*

(i) *system* (1) *is an h-system,*
(ii) *the perturbing term g satisfies that*

$$|g(n, y)| \leq \lambda(n)w(|y|),$$

where w is of the class $\hat{F}$ with the corresponding multiplier function r, and $\lambda : \mathbb{N}(n_0) \to \mathbb{R}^+$ satisfies $\lambda(n)\ r(h(n)h^{-1}(n_0))\ h^{-1}(n+1) \in l_1(\mathbb{N}(n_0))$.

Then every solution $y(n) = y(n, n_0, y_0)$ of (6) satisfies

$$|y(n, n_0, y_0)| \le Kh(n)h^{-1}(n_0), \ n \ge n_0,$$

where K is a positive constant.

Proof. By Lemma 2.2, the solution $y(n)$ of (6) is given by

$$y(n) = L(n, n_0)y_0 + \sum_{s=n_0}^{n-1} L(n, s+1)g(s, y(s)),$$

where $L(n, s)$ is the resolvent matrix of system (2). Then, in view of the assumptions and Lemma 2.3, we have the following inequality :

$$\begin{aligned} |y(n)| &\le |L(n, n_0)||y_0| + \sum_{s=n_0}^{n-1} |L(n, s+1)||g(s, y(s))| \\ &\le ch(n)h^{-1}(n_0)|y_0| + \sum_{s=n_0}^{n-1} ch(n)h^{-1}(s+1)\lambda(s)w(|y(s)|). \end{aligned}$$

Thus we obtain

$$\frac{|y(n)|h(n_0)}{h(n)} \le c|y_0| + c\sum_{s=n_0}^{n-1} \lambda(s)\frac{r(\frac{h(s)}{h(n_0)})}{h(s+1)} w\Big(\frac{|y(s)|h(n_0)}{h(s)}\Big).$$

Put $u(n) = \frac{|y(n)|h(n_0)}{h(n)}$. Then, by the discrete Bihari inequality in [7], we have

$$|y(n)| \le h(n)h^{-1}(n_0)W^{-1}\Big[W(c|y_0|) + c\sum_{s=n_0}^{n-1} \hat{\lambda}(s)\Big], \ n \ge n_0,$$

where $\hat{\lambda}(n) = \frac{\lambda(n)r(h(n)h^{-1}(n_0))}{h(n+1)}$. Hence we obtain

$$|y(n)| \le Kh(n)h^{-1}(n_0),$$

where $K = W^{-1}\big[W(c|y_0|) + c\sum_{s=n_0}^{\infty} \hat{\lambda}(s)\big]$ is a positive constant. □

Remark 2 ([8]). In Theorem 2.4, we assume that $|y_0|$ is small enough and

$$W(0+) = \infty,$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $u > 0$. Then the inverse function $W^{-1}(\nu)$ is defined for $\nu \in (0, \delta_0)$, where δ_0 is small enough. For instance, if we let $w(u) = u^\gamma, \gamma \geq 1$ and $\phi(s) = W^{-1}[W(s) + \alpha], s \geq 0, \alpha \geq 0$, then $\phi(0+) = 0$ follows from $\int_0^1 \frac{ds}{w(s)} = +\infty$. Thus, if s is small enough, then there exists a positive constant $M \geq 1$ such that $\phi(s) \leq Ms$. This observation leads the following corollary.

Corollary 2.5. *The zero solution $y = 0$ of (6) is hS if all conditions of Theorem 2.5 and $\int_{0+}^{1} \frac{ds}{w(s)} = \infty$ hold.*

Example. To illustrate Theorem 2.4, we consider the Volterra difference equation

$$\text{(V)} \qquad \Delta x(n) = 2x(n) + \sum_{s=n_0}^{n-1} 2^{n-s} x(s), \quad x(n_0) = x_0$$

and its perturbation with the same initial value

$$\text{(P)} \qquad \Delta y(n) = 2y(n) + \sum_{s=n_0}^{n-1} 2^{n-s} y(s) + g(n, y(n)),$$

where g satisfies

$$|g(n, y)| \leq \lambda(n) w(|y|).$$

If $w(u) = u^\gamma$, $\gamma \geq 1$ and $\lambda(n) h^\gamma(n) h^{-1}(n+1) \in l_1(\mathbb{N}(n_0))$, then system (P) is an h-system.

Proof. Any solution $x(n, n_0, x_0)$ of (V) through the initial point $x(n_0, n_0, x_0) = x_0$ is given by

$$\begin{aligned} x(n, n_0, x_0) &= \frac{x_0}{3}[1 + 2 \cdot 4^{n-n_0}] \\ &= L(n, n_0) x_0, \end{aligned}$$

where $L(n, n_0) = \frac{1+2\cdot 4^{n-n_0}}{3}$, $n \geq n_0$. Then we obtain

$$|x(n, n_0, x_0)| \leq 4^{n-n_0} |x_0|,$$

where $h(n) = 4^n$. Thus (V) is an h-system. In the proof of Theorem 2.4, if we let $u(n) = \frac{|y(n)| h(n_0)}{h(n)}$, then we have

$$|y(n)| \leq h(n) h^{-1}(n_0) W^{-1}[W(c|y_0|) + c \sum_{s=n_0}^{\infty} \hat{\lambda}(s)],$$

where $\hat{\lambda}(n) = \frac{ch(n_0)}{h(n+1)}\lambda(n)\big(\frac{h(n)}{h(n_0)}\big)^{\gamma}$. Therefore, by the discrete Bihari inequality in [7], we obtain

$$|y(n)| \leq cMh(n)h^{-1}(n_0)|y_0|, \quad n \geq n_0,$$

since $W^{-1}[W(c|y_0|)+c\sum_{s=n_0}^{\infty}\hat{\lambda}(s)] \leq Mc|y_0|$ for some positive constant $M \geq 1$. □

Now, we consider the nonlinear Volterra difference system

$$\Delta x(n) = f(n, x(n)) + \sum_{s=n_0}^{n-1} g(n, s, x(s)), \quad x(n_0) = x_0, \tag{7}$$

where $f : \mathbb{N}(n_0) \times S(\rho) \to \mathbb{R}^m$ and $g : \mathbb{N}(n_0) \times \mathbb{N}(n_0) \times S(\rho) \to \mathbb{R}^m$ with $f(n, 0) = 0$, $g(n, s, 0) = 0$, are continuous in $x \in S(\rho)$, $S(\rho) = \{x \in \mathbb{R}^m : |x| < \rho, \rho > 0\}$. Moreover, we assume that f_x, g_x exist and are continuous in x.

The nonlinear system (7) becomes

$$\begin{aligned}\Delta x(n) = A(n)x(n) + \sum_{s=n_0}^{n-1} \big[K(n, s)x(s) + G(n, s, x(s))\big] \\ + F(n, x(n)), \quad x(n_0) = x_0,\end{aligned} \tag{8}$$

where

$$F(n, x) = \int_0^1 [f_x(n, \theta x) - f_x(n, 0)]d\theta \cdot x$$
$$G(n, s, x) = \int_0^1 [g_x(n, s, x\theta) - g_x(n, s, 0)]d\theta \cdot x$$

if we put $f_x(n, 0) = A(n)$ and $g_x(n, 0) = K(n, s)$, and use the mean value theorem. Then we can get the following result :

THEOREM 2.6. *Suppose that*

(i) *the zero solution $v = 0$ of the linear Volterra difference system*

$$\Delta v(n) = A(n)v(n) + \sum_{s=n_0}^{n-1} K(n, s)v(s), \ v(n_0) = v_0 \tag{9}$$

is hS,

(ii) *for each* $(n,x) \in \mathbb{N}(n_0) \times S(\rho)$ *we have* $|F(n,x)| \le a(n)|x|$ *and* $|G(n,s,x)| \le b(n,s)|x|$, *where* $a : \mathbb{N}(n_0) \to \mathbb{R}^+$ *and* $b : \mathbb{N}(n_0) \times \mathbb{N}(n_0) \to \mathbb{R}^+$,

(iii) $\lambda(n) = h(n)[h^{-1}(n+1)a(n)+K] \in l_1(\mathbb{N}(n_0))$ *with* $\sum_{\sigma=s+1}^{n-1} h^{-1}(\sigma+1)b(\sigma,s) \le K$, *where* K *is a constant.*

Then the zero solution $x = 0$ *of* (8) *is hS.*

Proof. By Lemmas 2.1 and 2.2, it follows that any solution $x(n)$ of (8) is given by

$$\begin{aligned} x(n) &= L(n,n_0)x_0 + \sum_{s=n_0}^{n-1} L(n,s+1)\big[F(s,x(s)) + \sum_{\sigma=n_0}^{s-1} G(s,\sigma,x(\sigma))\big] \\ &= L(n,n_0)x_0 + \sum_{s=n_0}^{n-1} L(n,s+1)F(s,x(s)) \\ &\quad + \sum_{s=n_0}^{n-1}\sum_{\sigma=s+1}^{n-1} L(n,\sigma+1)G(\sigma,s,x(s)), \end{aligned}$$

where $L(n,s)$ is the resolvent matrix of system (2). Thus, by the assumptions and Lemma 2.3, we have

$$\begin{aligned} |x(n)| &\le ch(n)h^{-1}(n_0)|x_0| + \sum_{s=n_0}^{n-1} ch(n)h^{-1}(s+1)a(s)|x(s)| \\ &\quad + \sum_{s=n_0}^{n-1}\sum_{\sigma=s+1}^{n-1} ch(n)h^{-1}(\sigma+1)b(\sigma,s)|x(s)| \\ &\le ch(n)h^{-1}(n_0)|x_0| + ch(n)\sum_{s=n_0}^{n-1}\big[h^{-1}(s+1)a(s)+K\big]|x(s)|. \end{aligned}$$

Letting $u(n) = \frac{|x(n)|}{h(n)}$, we obtain

$$u(n) \le cu(n_0) + c\sum_{s=n_0}^{n-1} h(s)\big(h^{-1}(s+1)a(s)+K\big)u(s).$$

It follows from the discrete Gronwall inequality in [1, 7] that

$$\begin{aligned} |x(n)| &\le ch(n)h^{-1}(n_0)|x_0| \exp\Big(c\sum_{s=n_0}^{n-1}\lambda(s)\Big) \\ &\le c_1 h(n)h^{-1}(n_0)|x_0|, \end{aligned}$$

where $c_1 = c\exp(c\sum_{s=n_0}^{\infty}\lambda(s))$ is a positive constant. Hence the zero solution $x = 0$ of (8) is hS. This completes the proof. □

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Sung Kyu Choi, Department of Mathematics, Chungnam National University, Taejon 305–764, Korea
E-mail: skchoi@math.cnu.ac.kr

Nam Jip Koo, Department of Mathematics, Chungnam National University, Taejon 305–764, Korea
E-mail: njkoo@math.cnu.ac.kr

Yoon Hoe Goo, Department of Mathematics, Hanseo University, Seosan 356–820, Korea
E-mail: yhgoo@gaya.hanseo.ac.kr