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TRACE-CLASS INTERPOLATION FOR VECTORS IN TRIDIAGONAL ALGEBRAS

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ABSTRACT. Given vectors x and y in a Hilbert space, an interpolating operator is a bounded operator T such that $Tx = y$. An interpolating operator for n vectors satisfies the equation $Tx_i = y_i$, for $i = 1, 2, \cdots, n$. In this article, we obtained the following : Let $x = (x_i)$ and $y = (y_i)$ be two vectors in $\mathcal{H}$ such that $x_i \neq 0$ for all $i = 1, 2, \cdots$. Then the following statements are equivalent.

(1) There exists an operator A in Alg$\mathcal{L}$ such that $Ax = y$, A is a trace-class operator and every E in $\mathcal{L}$ reduces A.

(2) $\sup\left\{\frac{\|\sum_{k=1}^{l}\alpha_k E_k y\|}{\|\sum_{k=1}^{l}\alpha_k E_k x\|} : l \in N, \alpha_k \in \mathbb{C} \text{ and } E_k \in \mathcal{L}\right\} < \infty$ and $\sum_{n=1}^{\infty} |y_n||x_n|^{-1} < \infty$.

1. Introduction

Let $\mathcal{C}$ be a collection of operators acting on a Hilbert space $\mathcal{H}$ and let x and y be vectors on $\mathcal{H}$. An *interpolation question* for $\mathcal{C}$ asks for which x and y is there a bounded operator $T \in \mathcal{C}$ such that $Tx = y$. A variation, the 'n-vector interpolation problem', asks for an operator T such that $Tx_i = y_i$ for fixed finite collections $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$. The n-vector interpolation problem was considered for a C^*-algebra $\mathcal{U}$ by Kadison [9]. In case $\mathcal{U}$ is a nest algebra, the (one-vector) interpolation problem was solved by Lance [10]: his result was extended by Hopenwasser [4] to the case that $\mathcal{U}$ is a CSL-algebra.

In this article, we investigate Trace-class interpolation problems for vectors in tridiagonal algebra : Given vectors x and y in a Hilbert space,

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when does there exist a Trace-class operator A in a tridiagonal algebra such that $Ax = y$?

We establish some notations and conventions. A commutative subspace lattice $\mathcal{L}$, or CSL $\mathcal{L}$ is a strongly closed lattice of pairwise-commuting projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If $\mathcal{L}$ is CSL, Alg$\mathcal{L}$ is called a CSL-algebra. The symbol Alg$\mathcal{L}$ is the algebra of all bounded linear operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. Let x and y be two vectors in some Hilbert space. Then $\langle x, y\rangle$ means the inner product of the vectors x and y. Let N be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers.

2. Trace-class interpolation for vectors in tridiagonal algebra

Let $\mathcal{H}$ be a separable complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, \cdots\}$. Let x_1, x_2, $\cdots$, x_n be vectors in $\mathcal{H}$. Then $[x_1, x_2, \cdots, x_n]$ means the closed subspace generated by the vectors x_1, x_2, $\cdots$, x_n. Let M be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of M and $\overline{M}^{\perp}$ means the orthogonal complement of M. Let $\mathcal{L}$ be the subspace lattice of orthogonal projections generated by the subspaces $[e_{2k-1}]$, $[e_{2k-1}, e_{2k}, e_{2k+1}]$ $(k = 1, 2, \cdots)$. Then the algebra Alg$\mathcal{L}$ is called a tridiagonal algebra which was introduced by F. Gilfeather and D. Larson [3]. These algebras have been found to be useful counterexample to a number of plausible conjectures. Recently, such algebras have been found to be use in physics, in electrical engineering and in general system theory.

Let $\mathcal{A}$ be the algebra consisting of all bounded operators acting on $\mathcal{H}$ of the form

$$\begin{pmatrix} * & * & & & \\ & * & & & \\ & * & * & * & \\ & & & * & \\ & & & * & \ddots \end{pmatrix}$$

with respect to the orthonormal basis $\{e_1, e_2, \cdots\}$, where all non-starred entries are zero. It is easy to see that Alg$\mathcal{L}$=$\mathcal{A}$. Let

$$D = \{A : \ A \text{ is a diagonal operator in } \mathcal{B}(\mathcal{H})\}.$$

Then D is a masa (maximal abelian subalgebra) of Agl$\mathcal{L}$ and $\mathcal{D}$=(Alg$\mathcal{L}$)$\cap$ (Alg$\mathcal{L}$)*, where (Alg$\mathcal{L}$)$^* = \{A^* : A \in$Alg$\mathcal{L}\}$.

In this paper, we use the convention $\frac{0}{0} = 0$, when necessary.

DEFINITION. Let $\mathcal{H}$ be a Hilbert space and let A be an operator acting on $\mathcal{H}$. A is called *positive* if $\langle Ax, x\rangle \geq 0$ for all x in $\mathcal{H}$.

DEFINITION. Let $\mathcal{H}$ be a Hilbert space, $\{e_n\}_{n=1}^{\infty}$ an orthonormal basis. Then for any positive operator A acting on $\mathcal{H}$, we define $\mathrm{tr}A = \sum_{n=1}^{\infty}\langle e_n, Ae_n\rangle$. The number $\mathrm{tr}A$ is called *the trace of A.*

DEFINITION. Let $\mathcal{H}$ be a Hilbert space and let A be an operator acting on $\mathcal{H}$. A is called *a Hilbert-Schmidt operator* if $\mathrm{tr}A^*A < \infty$.

DEFINITION. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the set of all bounded operators acting on $\mathcal{H}$. Let $\mathcal{B}_2(\mathcal{H})$ be the set of all Hilbert-Schmidt operators acting on $\mathcal{H}$. Let $\mathcal{B}_1(\mathcal{H}) = \{AB|A, B \in \mathcal{B}_2(\mathcal{H})\}$. Operators belonging to $\mathcal{B}_1(\mathcal{H})$ are called *trace-class operators.*

The following theorem is well-known.

THEOREM 1. *Let $\mathcal{H}$ be a Hilbert space and let A be an operator in $\mathcal{B}(\mathcal{H})$. Then the following are equivalent.*

(1) $A \in \mathcal{B}_1(\mathcal{H})$.

(2) $|A| = (A^*A)^{\frac{1}{2}} \in \mathcal{B}_1(\mathcal{H})$.

(3) $|A|^{\frac{1}{2}} \in \mathcal{B}_2(\mathcal{H})$.

(4) $tr(|A|) < \infty$.

From Theorem 1, we can get the following theorem.

THEOREM 2. *Let A be a diagonal operator in $\mathcal{B}(\mathcal{H})$ with diagonal $\{a_n\}$. A is a trace-class operator if and only if $\sum_{n=1}^{\infty}|a_n| < \infty$.*

THEOREM 3. *Let $x = (x_i)$ and $y = (y_i)$ be two vectors in $\mathcal{H}$ such that $x_i \neq 0$ for all $i = 1, 2, \cdots$. If*

$$\sup\left\{\frac{\|\sum_{k=1}^{l}\alpha_k E_k y\|}{\|\sum_{k=1}^{l}\alpha_k E_k x\|} : l \in N, \alpha_k \in \mathbb{C} \text{ and } E_k \in \mathcal{L}\right\} < \infty$$

and

$$\sum_{n=1}^{\infty}|y_n||x_n|^{-1} < \infty,$$

then there is an operator A in Alg$\mathcal{L}$ such that $Ax = y$, every E in $\mathcal{L}$ reduces A and A is a trace-class operator.

Proof. If $\sup\left\{\frac{\|\sum_{k=1}^{l}\alpha_k E_k y\|}{\|\sum_{k=1}^{l}\alpha_k E_k x\|} : l \in N, \alpha_k \in \mathbb{C} \text{ and } E_k \in \mathcal{L}\right\} < \infty$, then, there is an operator A in Alg$\mathcal{L}$ such that $Ax = y$ and every E in $\mathcal{L}$ reduces A by Theorem 1 [8]. Since every E in $\mathcal{L}$ reduces A, A is diagonal. Let $A = (a_{ii})$. Since $y = Ax$, $y_i = a_{ii}x_i$ for all $i = 1, 2, \cdots$. Since $\sum_{n=1}^{\infty}|y_n||x_n|^{-1} < \infty$, A is a trace-class operator. □

THEOREM 4. *Let $x = (x_i)$ and $y = (y_i)$ be two vectors in $\mathcal{H}$ such that $x_i \neq 0$ for all $i = 1, 2, \cdots$. If there exists an operator A in Alg$\mathcal{L}$ such that $Ax = y$, every E in $\mathcal{L}$ reduces A and A is a trace-class operator, then*

$$\sup\left\{\frac{\|\sum_{k=1}^{l}\alpha_k E_k y\|}{\|\sum_{k=1}^{l}\alpha_k E_k x\|} : l \in N, \alpha_k \in \mathbb{C} \text{ and } E_k \in \mathcal{L}\right\} < \infty$$

and

$$\sum_{n=1}^{\infty}|y_n||x_n|^{-1} < \infty.$$

Proof. Since $Ax = y$ and every E in $\mathcal{L}$ reduces A, $AEx = Ey$ for every E in $\mathcal{L}$. So $A(\sum_{k=1}^{l}\alpha_k E_k x) = \sum_{k=1}^{l}\alpha_k E_k y$ for every $l \in N$, every $\alpha_k \in \mathbb{C}$ and every $E_k \in \mathcal{L}$. Thus $\|\sum_{k=1}^{l}\alpha_k E_k y\| \leq \|A\|\|\sum_{k=1}^{l}\alpha_k E_k x\|$. If $\|\sum_{k=1}^{l}\alpha_k E_k x\| \neq 0$, then

$$\frac{\|\sum_{k=1}^{l}\alpha_k E_k y\|}{\|\sum_{k=1}^{l}\alpha_k E_k x\|} \leq \|A\|.$$

Hence

$$\sup\left\{\frac{\|\sum_{k=1}^{l}\alpha_k E_k y\|}{\|\sum_{k=1}^{l}\alpha_k E_k x\|} : l \in N, \alpha_k \in \mathbb{C} \text{ and } E_k \in \mathcal{L}\right\} < \infty.$$

Since every E in $\mathcal{L}$ reduces A, A is diagonal. Let $A = (a_{ii})$. Since $Ax = y$, $y_i = a_{ii}x_i$ and hence $a_{ii} = y_i x_i^{-1}$ for all $i = 1, 2, \cdots$. Since A is a trace-class operator, $\sum_{n=1}^{\infty}|y_n||x_n|^{-1} < \infty$. □

If we summarize Theorems 3 and 4, then we can get the following theorem.

THEOREM 5. *Let $x = (x_i)$ and $y = (y_i)$ be two vectors in $\mathcal{H}$ such that $x_i \neq 0$ for all $i = 1, 2, \cdots$. Then the following statements are equivalent.*

(1) *There exists an operator A in Alg$\mathcal{L}$ such that $Ax = y$, A is a trace-class operator and every E in $\mathcal{L}$ reduces A.*

(2) $\sup\left\{\dfrac{\|\sum_{k=1}^{l}\alpha_k E_k y\|}{\|\sum_{k=1}^{l}\alpha_k E_k x\|} : l \in N, \alpha_k \in \mathbb{C} \text{ and } E_k \in \mathcal{L}\right\} < \infty$ *and* $\displaystyle\sum_{n=1}^{\infty}|y_n||x_n|^{-1} < \infty.$

THEOREM 6. *Let $x_p = (x_{p,i})$ and $y_p = (y_{p,i})$ be vectors in $\mathcal{H}$ such that $x_{q,i} \neq 0$ $(p = 1, 2, \cdots, n)$ for some fixed q and all $i = 1, 2, \cdots$. If there is an operator A in Alg$\mathcal{L}$ such that $Ax_p = y_p$ $(p = 1, 2, \cdots, n)$, every E in $\mathcal{L}$ reduces A and A is a trace-class operator, then*

$$\sup\left\{\frac{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}y_p\|}{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}x_p\|} : m_p \in N, l \leq n, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C}\right\}$$

$< \infty$ *and* $\displaystyle\sum_{n=1}^{\infty}|y_{q,n}||x_{q,n}|^{-1} < \infty.$

Proof. Since $Ax_p = y_p$ and every E in $\mathcal{L}$ reduces A, $AEx_p = Ey_p$. So $A(\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}x_p) = \sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}y_p$, $m_p \in N$, $l \leq n$, $E_{k,p} \in \mathcal{L}$ and $\alpha_{k,p} \in \mathbb{C}$. Thus $\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}y_p\| \leq \|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}x_p\|\,\|A\|$. If $\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}x_p\| \neq 0$, then

$$\frac{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k.p}E_{k,p}y_p\|}{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}x_p\|} \leq \|A\|.$$

Hence

$$\sup\left\{\frac{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}y_p\|}{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}x_p\|} : m_p \in N, l \leq n, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C}\right\}$$

$< \infty$. Since every E in $\mathcal{L}$ reduces A, A is diagonal. Let $A = (a_{ii})$. Since $Ax_p = y_p$, $y_{p,i} = a_{ii}x_{p,i}$ $(p = 1, 2, \cdots, n$ and $i = 1, 2, \cdots)$. Since $x_{q,i} \neq 0$, $a_{ii} = y_{q,i}x_{q,i}^{-1}$ $(i = 1, 2, \cdots)$. Since A is a trace-class operator, $\sum_{n=1}^{\infty}|y_{q,n}||x_{q,n}|^{-1} < \infty$. □

THEOREM 7. *Let $x_p = (x_{p,i})$ and $y_p = (y_{p,i})$ be vectors in $\mathcal{H}$ such that $x_{q,i} \neq 0$ $(p = 1, 2, \cdots, n)$ for some fixed q and all $i = 1, 2, \cdots$.*

If $\sup\left\{\frac{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}y_p\|}{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}x_p\|} : m_p \in N,\ l \leq n,\ E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C}\right\} < \infty$ *and* $\sum_{n=1}^{\infty}|y_{q,n}||x_{q,n}|^{-1} < \infty$, *then there is an operator* A *in Alg*$\mathcal{L}$ *such that* $Ax_p = y_p$ $(p = 1, 2, \cdots, n)$, *every* E *in* $\mathcal{L}$ *reduces* A *and* A *is a trace-class operator.*

Proof. If $\sup\left\{\frac{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}y_p\|}{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}x_p\|} : m_p \in N, l \leq n, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C}\right\} < \infty$, then there is an operator A in Alg$\mathcal{L}$ such that $Ax = y$ and every E in $\mathcal{L}$ reduces A. So A is a diagonal operator. Let $A = (a_{ii})$. Since $y_p = Ax_p$, $y_{p,i} = a_{ii}x_{p,i}$ $(p = 1, 2, \cdots, n$ and $i = 1, 2, \cdots)$. Since $\sum_{n=1}^{\infty}|y_{q,n}||x_{q,n}|^{-1} < \infty$, A is a trace-class operator. □

If we summarize Theorems 6 and 7, then we can get the following theorem.

THEOREM 8. *Let* $x_p = (x_{p,i})$ *and* $y_p = (y_{p,i})$ *be vectors in* $\mathcal{H}$ *such that* $x_{q,i} \neq 0$ *for some fixed* q *and all* $i = 1, 2, \cdots$. *Then the following statements are equivalent.*

(1) *There exists an operator* A *in Alg*$\mathcal{L}$ *such that* $Ax_p = y_p$ $(p = 1, \cdots, n)$, *every* E *in* $\mathcal{L}$ *reduces* A *and* A *is a trace-class operator.*

(2) $\sup\left\{\frac{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}y_p\|}{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}x_p\|} : m_p \in N,\ l \leq n,\ E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C}\right\} < \infty$ and $\sum_{n=1}^{\infty}|y_{q,n}||x_{q,n}|^{-1} < \infty$.

If we modify the proof of Theorems 6 and 7, then we can get the following theorem that is considered for infinite vectors.

THEOREM 9. *Let* $x_p = (x_{p,i})$ *and* $y_p = (y_{p,i})$ *be vectors in* $\mathcal{H}(p = 1, 2, \cdots)$ *such that* $x_{q,i} \neq 0$ *for all* i *and for some fixed* q. *Then the following statements are equivalent.*

(1) *There exists an operator* A *in Alg*$\mathcal{L}$ *such that* $Ax_p = y_p$ $(p = 1, 2, \cdots)$, *every* E *in* $\mathcal{L}$ *reduces* A *and* A *is a trace-class operator.*

(2) $\sup\left\{\frac{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}y_p\|}{\|\sum_{k=1}^{m_p}\sum_{p=1}^{l}\alpha_{k,p}E_{k,p}x_p\|} : m_p,\ l \in N,\ E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C}\right\} < \infty$ and $\sum_{n=1}^{\infty} |y_{q,n}|\ |x_{q,n}|^{-1} < \infty$.

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