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STABILITY OF ISOMETRIES ON HILBERT SPACES

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Abstract. Let X and Y be real Banach spaces and $\epsilon, p \geq 0$. A mapping T between X and Y is called an (ϵ, p)-isometry if $|\|T(x)-T(y)\| - \|x-y\|| \leq \epsilon\|x-y\|^p$ for $x, y \in X$. Let H be a real Hilbert space and $T : H \rightarrow H$ an (ϵ, p)-isometry with $T(0) = 0$. If $p \neq 1$ is a nonnegative number, then there exists a unique isometry $I : H \rightarrow H$ such that $\|T(x) - I(x)\| \leq C(\epsilon)(\|x\|^{(1+p)/2} + \|x\|^p)$ for all $x \in H$, where $C(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.

1. Introduction

Throughout this paper, X and Y denote real Banach spaces and H denote real Hilbert space and $\epsilon, p \geq 0$. It is a well-known classical result of Mazur and Ulam [7] that an isometry T from X onto Y for which $T(0) = 0$ is automatically linear. A mapping $T : X \rightarrow Y$ is called an (ϵ, p)-isometry if

$$|\|T(x) - T(y)\| - \|x - y\|| \leq \epsilon\|x - y\|^p$$

for $x, y \in X$.

In 1983, J. Gevirtz [2] showed that if $T : X \rightarrow Y$ is a surjective $(\epsilon, 0)$-isometry, then there exists a unique isometry $I : X \rightarrow Y$ such that

$$\|T(x) - I(x)\| \leq 20(\sqrt{2} - 1)^{-1}((\epsilon\|x\|)^{1/2} + 40\epsilon)$$

for all $x \in X$ and, by using a result of P. M. Gruber [3], there exists a unique surjective isometry $I : X \rightarrow Y$ for which $\|T(x) - I(x)\| \leq 5\epsilon$ for all $x \in X$.

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In 1995, M. Omladič and P. Šermrl [8] showed that for any an $(\epsilon, 0)$-isometry $T : X \to Y$ there exists a unique surjective linear isometry $I : X \to Y$ such that $\|T(x) - I(x)\| \leq 2\epsilon$ for all $x \in X$.

In 2000, G. Dolinar [1] showed that if $T : X \to Y$ is a surjective (ϵ, p)-isometry with $T(0) = 0$ and $0 \leq p < 1$ then there exists a constant $N(p)$ and a surjective isometry $I : X \to Y$ such that

$$\|T(x) - I(x)\| \leq \epsilon N(p)\|x\|^p \text{ for all } x \in X.$$

Also he showed that every for (ϵ, p)-isometry $T : X \to H$ with $T(0) = 0$, and $0 < p < 1$, there exists a linear isometry $I : X \to H$ such that

$$\|T(x) - I(x)\| \leq C(\epsilon, p) \max\{\|x\|^p, \|x\|^{(1+p)/2}\}$$

for all $x \in X$, where $C(\epsilon, p) \to 0$ as $\epsilon \to 0$.

The authors [4] showed that if $T : X \to Y$ is an $(\epsilon, 1)$-isometry with $T(0) = 0$, then

$$\|T(x + y) - T(x) - T(y)\| \leq C(\epsilon)(\|x\| + \|y\|)$$

for all $x, y \in X$, where $C(\epsilon) \to 0$ as $\epsilon \to 0$.

In this paper, let p be a nonnegative number with $p \neq 1$. If $T : H \to H$ is an (ϵ, p)-isometry with $T(0) = 0$, then we show that there exists a mapping $\phi_{(\epsilon,p)}\ (x, y) : H \times H \to [0, \infty)$ such that

$$\|T(x + y) - T(x) - T(y)\| \leq \phi_{(\epsilon,p)}(x, y)$$

for all $x, y \in H$, where $\sum_{k=0}^{\infty} a^{-k}\phi_{(\epsilon,p)}(a^k x, a^k y) < \infty$ for some positive rational number a and $\phi_{(\epsilon,p)}(x, y) \to 0$ as $\epsilon \to 0$. Hence we obtain a unique isometry $I : H \to H$ such that

$$\|T(x) - I(x)\| \leq C(\epsilon)(\|x\|^{(1+p)/2} + \|x\|^p)$$

for all $x \in H$, where $C(\epsilon) \to 0$ as $\epsilon \to 0$.

In case $p = 1$, from the result of J. Lindenstrauss and A. Szankowski [6] we obtain an $(\epsilon, 1)$-isometry T from ℓ_2 onto itself which satisfies for every linear operator L on ℓ_2 there is an $x \in \ell_2$ so that $\|T(x) - L(x)\| \geq \|x\|$. Thus we cannot obtain an isometry $I : \ell_2 \to \ell_2$ such that

$$\|T(x) - I(x)\| \leq C(\epsilon)\|x\|$$

for all $x \in \ell_2$ where $(\epsilon) \to 0$ as $\epsilon \to 0$.

2. Results

We denote by G a vector space. Let a be a fixed positive rational number and let q, r be natural numbers with $a = r/q$. Assume $\phi : G \times G \to [0, \infty)$ is a mapping such that

$$\Phi^{(a)}(x, y) := \sum_{k=0}^{\infty} a^{-k} \phi(a^k x, a^k y) < \infty$$

for all $x, y \in G$. We also assume that $\sum_{i=1}^{n-1}(\cdot) = 0$ if $n = 1$.

The following theorem is a generalization of some results of Y.-H. Lee and K.-W. Jun [5].

THEOREM 1. *Let $T : G \to X$ be a mapping such that*

(1) $$\|T(x + y) - T(x) - T(y)\| \le \phi(x, y) \text{ for all } x, y \in G.$$

Then there exists a unique additive mapping $I : G \to X$ such that

$$\|T(x) - I(x)\| \le \sum_{i=0}^{q-1} \Phi^{(a)}\Big(\frac{1}{q}x, \frac{i}{q}x\Big) + \sum_{i=0}^{r-1} a^{-1}\Phi^{(a)}\Big(\frac{1}{q}x, \frac{i}{q}x\Big)$$

for all $x \in G$.

Proof. From (1), we easily get the following formula by induction

(2) $$\|T(kx) - kT(x)\| \le \sum_{i=0}^{k-1} \phi(x, ix)$$

for all $x \in G$ and $k \in N$.

Replacing x by $\frac{1}{q}x$ and k by q, we have

$$\Big\|T(x) - qT\Big(\frac{1}{q}x\Big)\Big\| \le \sum_{i=0}^{q-1} \phi\Big(\frac{1}{q}x, \frac{i}{q}x\Big).$$

Thus

(3) $$\Big\|\frac{1}{q}T(x) - T\Big(\frac{1}{q}x\Big)\Big\| \le \sum_{i=0}^{q-1} q^{-1}\phi\Big(\frac{1}{q}x, \frac{i}{q}x\Big)$$

for all $x \in G$. Replacing x by $\frac{1}{q}x$ and k by r in (2), it follows that

$$\Big\| T\Big(\frac{r}{q}x\Big) - rT\Big(\frac{1}{q}x\Big)\Big\| \le \sum_{i=0}^{r-1} \phi\Big(\frac{1}{q}x, \frac{i}{q}x\Big),$$

and so we have

$$\Big\| \frac{1}{r}T\Big(\frac{r}{q}x\Big) - T\Big(\frac{1}{q}x\Big)\Big\| \le \sum_{i=0}^{r-1} r^{-1}\phi\Big(\frac{1}{q}x, \frac{i}{q}x\Big) \tag{4}$$

for all $x \in G$. (3) and (4) imply that

$$\begin{aligned}\Big\| \frac{1}{q}T(x) - \frac{1}{r}T\Big(\frac{r}{q}x\Big)\Big\| \le & \sum_{i=0}^{q-1} q^{-1}\phi\Big(\frac{1}{q}x, \frac{i}{q}x\Big) \\ & + \sum_{i=0}^{r-1} r^{-1}\phi\Big(\frac{1}{q}x, \frac{i}{q}x\Big).\end{aligned}$$

This implies that

$$\|aT(x) - T(ax)\| \le \sum_{i=0}^{q-1} a\phi\Big(\frac{1}{q}x, \frac{i}{q}x\Big) + \sum_{i=0}^{r-1} \phi\Big(\frac{1}{q}x, \frac{i}{q}x\Big) \tag{5}$$

for all $x \in G$. Replacing x by $a^{k-1}x$ in (5) we have

$$\begin{aligned}\|aT(a^{k-1}x) - T(a^k x)\| \le & \sum_{i=0}^{q-1} a\phi\Big(\frac{1}{q}a^{k-1}x, \frac{i}{q}a^{k-1}x\Big) \\ & + \sum_{i=0}^{r-1} \phi\Big(\frac{1}{q}a^{k-1}x, \frac{i}{q}a^{k-1}x\Big).\end{aligned}$$

Thus we obtain

$$\begin{aligned}\|a^{-k+1}T(a^{k-1}x) - a^{-k}T(a^k x)\| \le & \sum_{i=0}^{q-1} a^{-k+1}\phi\Big(\frac{1}{q}a^{k-1}x, \frac{i}{q}a^{k-1}x\Big) \\ & + a^{-k}\sum_{i=0}^{r-1} \phi\Big(\frac{1}{q}a^{k-1}x, \frac{i}{q}a^{k-1}x\Big)\end{aligned} \tag{6}$$

for all $x \in G$. For $n > m$ (6) implies that

$$\|a^{-n}T(a^n x) - a^{-m}T(a^m x)\| \le \sum_{k=m+1}^{n} \Big[\sum_{i=0}^{q-1} a^{-k+1}\phi\Big(\frac{1}{q}a^{k-1}x, \frac{i}{q}a^{k-1}x\Big) + a^{-k}\sum_{i=0}^{r-1}\phi\Big(\frac{1}{q}a^{k-1}x, \frac{i}{q}a^{k-1}x\Big)\Big]$$

$$= \sum_{i=0}^{q-1}\sum_{k=m+1}^{n} a^{-k+1}\phi\Big(\frac{1}{q}a^{k-1}x, \frac{i}{q}a^{k-1}x\Big) + \sum_{i=0}^{r-1}\sum_{k=m+1}^{n} a^{-k}\phi\Big(\frac{1}{q}a^{k-1}x, \frac{i}{q}a^{k-1}x\Big) \tag{7}$$

for all $x \in G$. Thus $\{a^{-k}T(a^n x)\}$ is a Cauchy sequence and converges for all $x \in G$. So we define $I : G \to X$ by

$$I(x) = \lim_{n\to\infty} a^{-n}T(a^n x)$$

for all $x \in G$. From (1) I is an additive mapping. From (7) we have

$$\Big\|T(x) - I(x)\Big\| \le \sum_{i=0}^{q-1}\Phi^{(a)}\Big(\frac{1}{q}x, \frac{i}{q}x\Big) + \sum_{i=0}^{r-1} a^{-1}\Phi^{(a)}\Big(\frac{1}{q}x, \frac{i}{q}x\Big) \tag{8}$$

for all $x \in G$. It remains to show that I is uniquely defined. Let $I' : G \to X$ be another additive mapping satisfying (8). Then

$$\begin{aligned}\|I(x) - I'(x)\| &= \|a^{-n}I(a^n x) - a^{-n}I'(a^n x)\| \\ &\le \|a^{-n}I(a^n x) - a^{-n}T(a^n x)\| \\ &\quad + \|a^{-n}T(a^n x) - a^{-n}I'(a^n x)\| \\ &\le 2\Big[\sum_{i=0}^{q-1}\sum_{k=0}^{\infty} a^{-n-k}\phi\Big(\frac{1}{q}a^{n+k}x, \frac{i}{q}a^{n+k}x\Big) \\ &\quad + \sum_{i=0}^{r-1}\sum_{k=0}^{\infty} a^{-n-k-1}\phi\Big(\frac{1}{q}a^{n+k}x, \frac{i}{q}a^{n+k}x\Big)\Big] \\ &\le 2\Big[\sum_{i=0}^{q-1}\sum_{k=n}^{\infty} a^{-k}\phi\Big(\frac{1}{q}a^{k}x, \frac{i}{q}a^{k}x\Big) \\ &\quad + \sum_{i=0}^{r-1}\sum_{k=n}^{\infty} a^{-k-1}\phi\Big(\frac{1}{q}a^{k}x, \frac{i}{q}a^{k}x\Big)\Big]\end{aligned} \tag{9}$$

for all $x \in G$. Taking the limit of (9) as $n \to \infty$ we have

$$I(x) = I'(x)$$

for all $x \in G$. This completes the proof of the theorem. □

The following corollary is due to Th. M. Rassias [8].

COROLLARY 2. *Given $\epsilon > 0$ and nonnegative number $p \neq 1$, let $T : X \to Y$ be a mapping such that*

$$\|T(x+y) - T(x) - T(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then there exists a unique additive mapping $I : X \to Y$ such that

$$\|T(x) - I(x)\| \leq \frac{2\epsilon}{|2-2^p|}\|x\|^p$$

for all $x \in X$.

Proof. Apply Theorem 1. □

Let H be a real Hilbert space with inner product $\langle , \rangle$ and we define $\phi_{(\epsilon,p)}(x,y) : H \times H \to [0,\infty)$ by

$$\begin{aligned}\phi_{(\epsilon,p)}(x,y) = {} & [2\epsilon(\|x+y\|^{1+p} + 2\|x\|^{1+p} + 2\|y\|^{1+p} + \|x-y\|^{1+p}) \\ & + 2\epsilon^2(\|x+y\|^{2p} + 2\|x\|^{2p} + 2\|y\|^{2p} + \|x-y\|^{2p})]^{1/2}\end{aligned}$$

for all $x, y \in H$. Let $\Phi^{(a)}_{(\epsilon,p)}(x,y) = \sum_{k=0}^{\infty} a^{-k}\phi_{(\epsilon,p)}(a^k x, a^k y)$. If $p > 1$ then $\Phi^{(a)}_{(\epsilon,p)}(x,y) < \infty$ for $0 < a < 1$ and if $0 \leq p < 1$ then $\Phi^{(a)}_{(\epsilon,p)}(x,y) < \infty$ for $a > 1$.

THEOREM 3. *Let H be a real Hilbert space and let $T : H \to H$ be an (ϵ, p)-isometry with $T(0) = 0$. Then*

$$\|T(x+y) - T(x) - T(y)\| \leq \phi_{(\epsilon,p)}(x,y)$$

for all $x, y \in H$.

Proof. Since T is an (ϵ, p)-isometry, we have

$$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + 2\epsilon\|x - y\|^{1+p} + \epsilon^2\|x - y\|^{2p},$$

and so

$$\begin{aligned} -2\langle T(x), T(y)\rangle \leq\ & - \|T(x)\|^2 - \|T(y)\|^2 + \|x - y\|^2 \\ & + 2\epsilon\|x - y\|^{1+p} + \epsilon^2\|x - y\|^{2p} \end{aligned}$$

for all $x, y \in H$. This implies

$$\begin{aligned} & \|T(x + y) - T(x) - T(y)\|^2 \\ \leq\ & - \|T(x + y)\|^2 + \|x\|^2 + \|y\|^2 \\ & + 2\epsilon(\|x\|^{1+p} + \|y\|^{1+p}) + \epsilon^2(\|x\|^{2p} + \|y\|^{2p}) \\ & + 2\langle T(x), T(y)\rangle \end{aligned} \tag{10}$$

for all $x, y \in H$. Assume $x, y \in H$. Then we divide the proof into four cases: □

Case 1. $\|x - y\| > \epsilon\|x - y\|^p$ and $\|x + y\| > \epsilon\|x + y\|^p$.
It follows from $\|x - y\| > \epsilon\|x - y\|^p$ that

$$\begin{aligned} 2\langle T(x), T(y)\rangle \leq\ & \|T(x)\|^2 + \|T(y)\|^2 - \|x - y\|^2 \\ & + 2\epsilon\|x - y\|^{1+p} - \epsilon^2\|x - y\|^{2p} \\ \leq\ & \|x\|^2 + \|y\|^2 - \|x - y\|^2 \\ & + 2\epsilon(\|x\|^{1+p} + \|y\|^{1+p} + \|x - y\|^{1+p}) \\ & + \epsilon^2(\|x\|^{2p} + \|y\|^{2p} - \|x - y\|^{2p}). \end{aligned} \tag{11}$$

Since $\|x + y\| > \epsilon\|x + y\|^p$, we obtain

$$\|T(x + y)\|^2 \geq \|x + y\|^2 - 2\epsilon\|x + y\|^{1+p} + \epsilon^2\|x + y\|^{2p}. \tag{12}$$

From (10), (11), (12) and the parallelogram identity we get

$$\begin{aligned} & \|T(x + y) - T(x) - T(y)\|^2 \\ \leq\ & 2\epsilon(\|x + y\|^{1+p} + 2\|x\|^{1+p} + 2\|y\|^{1+p} + \|x - y\|^{1+p}) \\ & + 2\epsilon^2(-\|x + y\|^{2p} + 2\|x\|^{2p} + 2\|y\|^{2p} - \|x - y\|^{2p}) \\ \leq\ & \phi^2_{(\epsilon,p)}(x, y). \end{aligned}$$

□

Case 2. $\|x-y\| > \epsilon\|x-y\|^p$ and $\|x+y\| \le \epsilon\|x+y\|^p$.
(10), (11) and the parallelogram identity imply

$$\begin{aligned}
&\|T(x+y)-T(x)-T(y)\|^2 \\
\le\ &\|x+y\|^2 \\
&+ 2\epsilon(2\|x\|^{1+p} + 2\|y\|^{1+p} + \|x-y\|^{1+p}) \\
&+ 2\epsilon^2(2\|x\|^{2p} + 2\|y\|^{2p} - \|x-y\|^{2p}).
\end{aligned}$$

Since $\|x+y\| \le \epsilon\|x+y\|^p$, we have

$$\begin{aligned}
&\|T(x+y)-T(x)-T(y)\|^2 \\
\le\ &2\epsilon(2\|x\|^{1+p} + 2\|y\|^{1+p} + \|x-y\|^{1+p}) \\
&+ 2\epsilon^2(2\|x\|^{2p} + 2\|y\|^{2p} + \|x+y\|^{2p} - \|x-y\|^{2p}) \\
\le\ &\phi^2_{(\epsilon,p)}(x,y). \qquad \square
\end{aligned}$$

Case 3. $\|x-y\| \le \epsilon\|x-y\|^p$ and $\|x+y\| > \epsilon\|x+y\|^p$.
We obtain easily

$$2\langle T(x), T(y)\rangle \le \|T(x)\|^2 + \|T(y)\|^2. \tag{13}$$

From (10), (12), (13) and the parallelogram identity we have

$$\begin{aligned}
\|T(x+y)-T(x)-T(y)\|^2 \le\ &2\epsilon(2\|x\|^{1+p} + 2\|y\|^{1+p} + \|x+y\|^{1+p}) \\
&+ \epsilon^2(2\|x\|^{2p} + 2\|y\|^{2p}) + \|x-y\|^2.
\end{aligned}$$

Since $\|x-y\| \le \epsilon\|x-y\|^p$, we obtain

$$\begin{aligned}
\|T(x+y)-T(x)-T(y)\|^2 \le\ &2\epsilon(2\|x\|^{1+p} + 2\|y\|^{1+p} + \|x+y\|^{1+p}) \\
&+ \epsilon^2(2\|x\|^{2p} + 2\|y\|^{2p} + \|x-y\|^{2p}) \\
\le\ &\phi^2_{(\epsilon,p)}(x,y). \qquad \square
\end{aligned}$$

Case 4. $\|x-y\| \le \epsilon\|x-y\|^p$ and $\|x+y\| \le \epsilon\|x+y\|^p$.
(10), (13) and the parallelogram identity imply

$$\begin{aligned}
\|T(x+y)-T(x)-T(y)\|^2 \le\ &\|x+y\|^2 + \|x-y\|^2 \\
&+ 2\epsilon(2\|x\|^{1+p} + 2\|y\|^{1+p}) \\
&+ \epsilon^2(2\|x\|^{2p} + 2\|y\|^{2p}).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\|T(x+y)-T(x)-T(y)\|^2 \\
&\le 2\epsilon(2\|x\|^{1+p}+2\|y\|^{1+p}) \\
&\quad + \epsilon^2(2\|x\|^{2p}+2\|y\|^{2p}+\|x-y\|^{2p}+\|x+y\|^{2p}) \\
&\le \phi^2_{(\epsilon,p)}(x,y).
\end{aligned}$$

Hence we arrive at

$$\|T(x+y)-T(x)-T(y)\| \le \phi_{(\epsilon,p)}(x,y)$$

for all $x, y \in H$. This completes the proof of the theorem. □

THEOREM 4. *Let H be a real Hilbert space and let $T : H \to H$ be an (ϵ, p)-isometry with $T(0) = 0$. If $p \neq 1$ is a nonnegative number then there exists a unique isometry $I : H \to H$ such that*

$$\begin{aligned}
\|T(x)-I(x)\| \le &\sum_{i=0}^{q-1} \Phi^{(a)}_{(\epsilon,p)}\Big(\frac{1}{q}x, \frac{i}{q}x\Big) \\
&+ \sum_{i=0}^{r-1} a^{-1}\Phi^{(a)}_{(\epsilon,p)}\Big(\frac{1}{q}x, \frac{i}{q}x\Big)
\end{aligned}$$

for all $x \in H$, and for $a > 1$ if $0 \le p < 1$ or $0 < a < 1$ if $p > 1$, where $a = \frac{r}{q}$ (q, r : positive integers).

Proof. It is sufficient to prove the theorem for $a > 1$, $0 \le p < 1$, $a = \frac{r}{q}$ (q, r: positive integers). By Theorem 1 and Theorem 3, there exists a unique additive mapping $I : H \to H$ such that

$$\begin{aligned}
\|T(x)-I(x)\| \le &\sum_{i=0}^{q-1} \Phi^{(a)}_{(\epsilon,p)}\Big(\frac{1}{q}x, \frac{i}{q}x\Big) \\
&+ \sum_{i=0}^{r-1} a^{-1}\Phi^{(a)}_{(\epsilon,p)}\Big(\frac{1}{q}x, \frac{i}{q}x\Big)
\end{aligned}$$

for all $x \in H$. Since $I(x) = \lim_{n\to\infty} a^{-n}T(a^n x)$ in the proof of Theorem 1, I is an isometry. This completes the proof of the theorem. □

REMARKS. In Theorem 4, if $0 < p < 1$, then we put $a = 2$ and so $q = 1, r = 2$. Thus we get

$$\begin{aligned}\Phi^{(2)}_{(\epsilon,p)}(x,x) &= \sum_{k=0}^{\infty} 2^{-k}\phi_{(\epsilon,p)}(2^k x, 2^k x)\\ &\le \frac{(2+2^p)^{1/2}}{1-2^{(p-1)/2}} 2\sqrt{\epsilon}\|x\|^{(1+p)/2}\\ &\quad + \frac{(2^{2p}+4)^{1/2}}{1-2^{p-1}}\sqrt{2}\epsilon\|x\|^p.\end{aligned}$$

By Theorem 4, there exists a unique isometry $I : H \to H$ such that

$$\begin{aligned}\|T(x) - I(x)\| &\le \frac{(2+2^p)^{1/2}}{1-2^{(p-1)/2}}\sqrt{\epsilon}\|x\|^{(1+p)/2}\\ &\quad + \frac{(2^{2p}+4)^{1/2}}{2-2^p}\sqrt{2}\epsilon\|x\|^p\end{aligned}$$

for all $x \in H$. If $p > 1$ and $a = 1/2$, we put $q = 2, r = 1$. So we have

$$\begin{aligned}\Phi^{(1/2)}_{(\epsilon,p)}\left(\frac{1}{2}x, \frac{1}{2}x\right) &= \sum_{k=0}^{\infty} 2^{k}\phi_{(\epsilon,p)}(2^{-k-1} x, 2^{-k-1} x)\\ &\le \frac{(2+2^{2-p})^{1/2}}{1-2^{(1-p)/2}}\sqrt{\epsilon}\|x\|^{(1+p)/2}\\ &\quad + \frac{(2+2^{-2p+2})^{1/2}}{1-2^{1-p}}\epsilon\|x\|^{2p}\end{aligned}$$

for all $x \in H$. Thus there exists a unique isometry $I : H \to H$ such that

$$\begin{aligned}\|T(x) - I(x)\| &\le \frac{(2+2^{2-p})^{1/2}}{1-2^{(1-p)/2}}\sqrt{\epsilon}\|x\|^{(1+p)/2}\\ &\quad + \frac{(2+2^{-2p+2})^{1/2}}{1-2^{1-p}}\epsilon\|x\|^{2p}\end{aligned}$$

for all $x \in H$.

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