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RINGS WHOSE MAXIMAL ONE-SIDED IDEALS ARE TWO-SIDED

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ABSTRACT. In this note we are concerned with relationships between one-sided ideals and two-sided ideals, and study the properties of polynomial rings whose maximal one-sided ideals are two-sided, in the viewpoint of the Nullstellensatz on noncommutative rings. Let R be a ring and $R[x]$ be the polynomial ring over R with x the indeterminate. We show that eRe is right quasi-duo for $0 \neq e^2 = e \in R$ if R is right quasi-duo; $R/J(R)$ is commutative with $J(R)$ the Jacobson radical of R if $R[x]$ is right quasi-duo, from which we may characterize polynomial rings whose maximal one-sided ideals are two-sided; if $R[x]$ is right quasi-duo then the Jacobson radical of $R[x]$ is $N(R)[x]$ and so the Köthe's conjecture (i.e., the upper nilradical contains every nil left ideal) holds, where $N(R)$ is the set of all nilpotent elements in R. Next we prove that if the polynomial ring $R[X]$, over a reduced ring R with $|X| \geq 2$, is right quasi-duo, then R is commutative. Several counterexamples are included for the situations that occur naturally in the process of this note.

Throughout this paper, all rings are associative with identity. Given a ring R, the Jacobson radical of R, the polynomial ring over R and the formal power series ring over R are denoted by $J(R)$, $R[x]$ and $R[[x]]$, respectively. A ring R is *right (left) duo* if every right (left) ideal of R is two-sided. A ring R is called *weakly right (left) duo* if for each a in R there exists a positive integer $n = n(a)$, depending on a, such that $a^n R(Ra^n)$ is two-sided. A ring R is called *right (left) quasi-duo* if every

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maximal right (left) ideal of R is two-sided. Commutative rings and direct sums of division rings are clearly right and left duo, and right duo rings are obviously weakly right duo. A ring is called *abelian* if every idempotent is central. Weakly right duo rings are both abelian and right quasi-duo by [12, Lemma 4] and [13, Proposition 2.2]. But the converses do not hold in general by the following examples.

EXAMPLE 1. (1) Let R be the subring $\left\{\begin{pmatrix} a & a_{12} & a_{13} \\ 0 & a & a_{23} \\ 0 & 0 & a \end{pmatrix} \mid a, a_{ij} \in D\right\}$ of the 3 by 3 full matrix ring over a division ring D. Then R is weakly right duo because every element is either invertible or nilpotent; but R is not right duo by the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. In fact, letting $a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, the right ideal $aR = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & D \\ 0 & 0 & 0 \end{pmatrix}$ does not contain

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

(2) The 2 by 2 upper triangular matrix ring over a division ring is right quasi-duo by [13, Proposition 2.1]; but it is not weakly right duo by the noncentral idempotent $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ since weakly right duo rings are abelian by [12, Lemma 4]. $\square$

In this note we also continue the study of quasi-duo rings that was initiated by Yu in [13], related to the Bass' conjecture in [2]. We first study the properties of right quasi-duo rings as follows. According to Nicholson [10], given a ring R a right R-module M is called *very semisimple* if every principal submodule mR, with $0 \neq m \in M$, is simple. Every such modules are obviously semisimple, but the converse is not true in general by the example in [10]. Recall that a module is called *homogeneous* if any two simple submodules are isomorphic.

PROPOSITION 1. *For a ring R, the following conditions are equivalent*:
(i) *R is right quasi-duo*;
(ii) *$R/J(R)$ is right quasi-duo*;
(iii) *Every maximal right ideal of R is a maximal ideal of R*;

(iv) *Every right primitive factor ring of R is a division ring;*

(v) *Every homogeneous semisimple right R-module is very semisimple.*

Proof. (i)⇔(ii)⇔(iii): By the definition.

(i)⇒(iv): Let P be a right primitive ideal of R. Then P is the maximal ideal in a maximal right ideal I of R. Hence $P = I$ and R/P is a division ring if R is right quasi-duo.

(iv)⇒(i): Let M be a maximal right ideal of R. Then there exists a right primitive ideal P of R such that $P \subseteq M$; hence $P = M$ by the condition. Thus R is right quasi-duo.

(i)⇔(v): By the right version of [10, Corollary 1]. □

We may obtain the same result for left quasi-duo rings by replacing right by left in the preceding proposition and its proof. A ring is called *reduced* if it has no nonzero nilpotent elements. As an elementary fact, a semiprimitive commutative ring is a subdirect product of fields; we may see a similar result for semiprimitive right (left) quasi-duo rings in the following.

COROLLARY 2. (i) *A semiprimitive right (left) quasi-duo ring is a subdirect product of division rings.*

(ii)[13, Lemma 2.3] *If R is a right quasi-duo ring then $R/J(R)$ is reduced.*

(iii) *A right primitive right quasi-duo ring is a division ring.*

(iv) *If a ring R is right (left) quasi-duo then $J(R)$ coincides with the Brown-McCoy radical of it.*

(v) *Every homomorphic image of a right quasi-duo ring is also right quasi-duo.*

(vi) *Let R be a right quasi-duo ring and M be a simple right R-module. Then $r_R(M)$ is a maximal ideal of R with $R/r_R(M)$ is a division ring, where $r_R(M)$ is the right annihilator of M over R. So M is isomorphic to $R/r_R(M)$.*

Proof. (i), (iii) and (vi): By Proposition 1. (ii): By the result (i). (iv): By the definition.

(v) Let R be a right quasi-duo ring and H be a homomorphic image of R. Then every right primitive ideal of H is an isomorphic image of some right primitive ideal of R which contains the kernel of the homomorphism. Hence we obtain the proof with the help of Proposition 1. □

The converses of Corollary 2(ii) and (iv) do not hold in general by the first Weyl algebra over a field of characteristic zero. Subrings of right quasi-duo rings need not be right quasi-duo, considering the first Weyl algebra over the field of rationals and the right quotient division ring of it. But we have a kind of subring that may be right quasi-duo as in the following that is one of our main results in this paper.

THEOREM 3. *Let R be a ring and $0 \neq e^2 = e \in R$. Then we have the following assertions:*

(i) *If R is right duo then so is eRe.*

(ii) *If R is weakly right duo then so is eRe.*

(iii) *If R is right quasi-duo then so is eRe.*

Proof. (i): Let $a, b \in eRe$. Since R is right duo, there is $r \in R$ with $ba = ar$ and $ba = aer = aere$; hence eRe is right duo.

(ii): Similar to the proof of (i).

(iii): Let M be a maximal right ideal of eRe. Then clearly $MR = MeR = eMR$ is a right ideal of R contained in eR. Assume that $MR = eR$. Then $M = MeRe = MRe = eRe$, a contradiction. So $MR \subsetneqq eR$. Since eR is finitely generated as a right R-module, there exists a maximal submodule N of eR containing MR by the right version of [1, Theorem 2.8]. Note that $Ne = eNe$ is a right ideal of eRe such that $M = MRe \subseteq Ne$. Assume that $Ne = eRe$. Then $e \in eRe = Ne \subseteq N$ and so $eR \subseteq N$ because N is right ideal of R, a contradiction. Thus Ne is a proper right ideal of eRe containing M; hence $Ne = M$ by the maximality of M. Now we claim that $N + (1-e)R$ is a maximal right ideal of R. To see this, let $J = N + (1-e)R$ and note that $J = N \oplus (1-e)R \subsetneqq eR \oplus (1-e)R$. If $x \in R$ and $x \notin J$ then $ex \notin N$ and so $N + exR = eR$ by the maximality of N in eR; hence $J + exR = N + (1-e)R + exR = (N + exR) + (1-e)R = eR + (1-e)R = R$. Since $xR + (1-e)R = exR + (1-e)R$ it follows that $J + xR = N + (1-e)R + xR = N + (1-e)R + exR = J + exR = R$. Thus J is a maximal right ideal of R. Now since R is right quasi-duo by hypothesis, J is a maximal ideal of R such that $eJe = eNe = Ne = M$. Let $x \in R$ with $exe \notin eJe$. Then $exe \notin J$. Since J is maximal, $J + R(exe)R = R$ and so $eJe + (eRe)(exe)(eRe) = eRe$; hence eJe is a maximal ideal of eRe. Therefore eRe is right quasi-duo. □

In spite of Theorem 3, the conditions are not Morita invariant properties by the n by n full matrix rings over division rings which are neither right nor left quasi-duo, where n is any positive integer ≥ 2. The converses of Theorem 3 do not hold in general by the preceding rings. We next observe some properties of polynomial rings and formal power series rings over such kinds of rings.

PROPOSITION 4. *For a ring R, the following conditions are equivalent:*
(i) *R is right quasi-duo;*
(ii) *$R[[x;\theta]]$ is right quasi-duo for every endomorphism $\theta : R \to R$;*
(iii) *$R[[x;\theta]]$ is right quasi-duo for some endomorphism $\theta : R \to R$;*
(iv) *$R[[x]]$ is right quasi-duo,*
where $R[[x;\theta]]$ is the skew power series ring over R by θ, every element of which is of the form $\sum_{n=1}^{\infty} a_n x^n$, only subject to $xa = \theta(a)x$ for each $a \in R$.

Proof. (i)⇒(ii): Let J be a proper right ideal of $R[[x;\theta]]$, then the set of constant terms of elements of J forms a proper right ideal of R. So every maximal right ideal of $R[[x;\theta]]$ is of the form $I + R[[x;\theta]]x$ with I a maximal right ideal of R. Since R is right quasi-duo, I is a maximal ideal of R and hence $I + R[[x;\theta]]x$ is also a maximal ideal of $R[[x;\theta]]$.

(ii)⇒(iii): Straightforward.

(iii)⇒(i): Let I be a maximal right ideal of R. Then $I + R[[x;\theta]]x$ is a maximal right ideal of $R[[x;\theta]]$. Since $R[[x;\theta]]$ is right quasi-duo, $I + R[[x;\theta]]x$ is a maximal ideal of $R[[x;\theta]]$ and thus I is also a maximal ideal of R.

(i)⇒(iv) and (iv)⇒(i) are similar to the proofs of (i)⇒(ii) and (iii)⇒(i), respectively. □

The argument, in Proposition 4, for right duo rings and weakly right duo rings does not hold in general by [5, Example 4] and [6, Example 5], respectively. But if a ring R is a right self-injective von Neumann regular ring then we have the following equivalences by [6, Theorem 6]: R is right (left) duo if and only if R is weakly right (left) duo ⇔ R is right (left) quasi-duo ⇔ R is a reduced ring ⇔ $R[[x]]$ is right (left) duo if and only if $R[[x]]$ is weakly right (left) duo if and only if $R[[x]]$ is right (left) quasi-duo if and only if $R[[x]]$ is a reduced ring. Next by a similar method to the proof of Proposition 4, we obtain the following result.

LEMMA 5. *Given a ring R if $R[x]$ is right quasi-duo then R is right quasi-duo.*

Based on Proposition 4 and Lemma 5, we may ask whether $R[x]$ is right quasi-duo if R is right quasi-duo. However the answer is negative by the following Example 2. To see that, we first need the following lemma.

LEMMA 6. *Let R be a right primitive ring. Then the following conditions are equivalent:*

(i) *$R[x]$ is a right duo ring;*

(ii) *$R[x]$ is a weakly right duo ring;*

(iii) *$R[x]$ is a right quasi-duo ring;*

(iv) *R is a field.*

Proof. (i)$\Rightarrow$(ii)$\Rightarrow$(iii) and (iv)$\Rightarrow$(i) are straightforward.

(iii)$\Rightarrow$(iv): Since $R[x]$ is right quasi-duo it follows that R is a right primitive right quasi-duo ring by Lemma 5, and so Proposition 1 implies that R is a division ring. Let $a, b \in R$. Notice that $(a+x)R[x]$ is a maximal right ideal of $R[x]$ so it is two-sided. Then $b(a+x) \in (a+x)R[x]$, say $b(a+x) = (a+x)c$ for some $c \in R$; hence by a simple computation $ba = ab$. The converse is obvious. □

We may compare Lemma 6 with [8, Corollary 17], and obtain the following equivalences: Let D be a division ring with F its center, and suppose that F is algebraically closed. Then $D[x]$ is right quasi-duo if and only if every maximal right ideal contains a maximal ideal in $D[x]$ if and only if D is a field.

Recall that a ring R is called a *PI-ring* if R satisfies a polynomial identity with coefficients in the ring of integers. PI-rings include commutative rings, so we may conjecture that the answer of the preceding question with PI-condition is affirmative. However the following examples may be counterexamples to that.

EXAMPLE 2. (i) Consider the field $F = \{0, 1, u, 1+u\}$ with $u^2 = 1+u$ and the automorphism $\theta : F \to F$ given by $\theta(\alpha) = \alpha^2$ for each $\alpha \in F$. Let $R = F[[t;\theta]]$ be the skew power series ring over R by θ with t its indeterminate, subject to $t\alpha = \theta(\alpha)t$ for each $\alpha \in F$. By Proposition 4, R is right quasi-duo. Since t^2 is central and $\alpha^2 + \alpha \in \mathbb{Z}_2$ for all $\alpha \in F$, R satisfies the polynomial identity $x_1(x_2x_3 - x_3x_2)^2 - (x_2x_3 - x_3x_2)^2x_1$. Let $I = (1+tx)R[x]$ then since I is a proper right ideal of $R[x]$ there exists a maximal right ideal M of R such that $I \subseteq M$. Now assume that $R[x]$ is right quasi-duo, then M is two-sided and so $1 = u + u^2 = u(1+tx) + (1+tx)u^2 \in M$, a contradiction. Thus $R[x]$ cannot be right quasi-duo.

(ii) Let R be the Hamilton quaternion over the field of real numbers. Then clearly R is right quasi-duo and right primitive; but by Lemma 6, $R[x]$ cannot be right quasi-duo. In fact $(1+ix)R[x]$ is a maximal right ideal but not a maximal ideal in $R[x]$ since

$$(k(1+ix)+(1+ix)k)(2k)^{-1}=1.$$

□

Note that $R[x]$ is an infinite centralizing extension of a ring R, so one may suspect that finite centralizing extensions of duo related rings are also duo related. However the following example erases the possibility: Let R be any right quasi-duo ring and S be the n by n full matrix ring over R with $n \geq 2$. Then S is a finite centralizing extension of R but S is not right quasi-duo obviously.

REMARK. Let R be a right primitive ring. If R is right quasi-duo then it is reduced by Corollary 2. The converse is not true in general by the first Weyl algebra over a field of characteristic zero, say W. Note that W is simple but not a PI-ring. So we suppose that R is a primitive PI-ring. Then R is a division ring if it is reduced, by the well-known Kaplansky's Theorem; consequently we have that R is right quasi-duo $\Leftrightarrow$ R is reduced $\Leftrightarrow$ R is a division ring. □

A ring R is commutative if and only if $R[x]$ is right duo by [5, Lemma 3]. Since right duo rings are right quasi-duo, we may conjecture that a ring R is commutative if $R[x]$ is right quasi-duo, considering Lemma 6. But it does not hold in general as in the following.

EXAMPLE 3. Let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then R is right quasi-duo and so is $R[x] = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}[x] \cong \begin{pmatrix} F[x] & F[x] \\ 0 & F[x] \end{pmatrix}$. But R is noncommutative obviously. □

We now find a condition under which polynomial rings may be duo related. For doing it, we first need to show the following that is one of our main results of this paper.

THEOREM 7. *Given a ring R if $R[x]$ is right quasi-duo then $R/J(R)$ is a commutative reduced ring.*

Proof. Assume to the contrary that there exist $a, b \in R$ such that $ab - ba \notin J(R)$, then there exists a maximal right ideal I of R such that $ab - ba \notin I$. By hypothesis and Lemma 5, R is right quasi-duo so I is a maximal ideal of R. Notice that $R[x]/I[x] \cong (R/I)[x]$ and that

R/I is a division ring by Proposition 1. By hypothesis and Corollary 2, $R[x]/I[x]$ is also right quasi-duo and so Lemma 6 implies that R/I is commutative; hence $ab - ba \in I$, which is a contradiction. Therefore $R/J(R)$ is a commutative reduced ring with the help of Corollary 2 and Lemma 5. □

As the converse of Theorem 7, one may conjecture that for a right quasi-duo ring R if $R/J(R)$ is commutative then $R[x]$ is right quasi-duo. However it fails in general by the following example.

EXAMPLE 4. Let $\mathbb{C}$ be the field of complex numbers. Define a field isomorphism $\sigma : \mathbb{C} \longrightarrow \mathbb{C}$ by $\sigma(a+bi) = a-bi$ where a, b are real numbers. Next consider the skew power series ring $R = \mathbb{C}[[t;\sigma]]$, every element is of the form $\sum_{n=1}^{\infty} a_n t^n$ over $\mathbb{C}$, only subject to $t(a+bi) = (\sigma(a+bi))t$, where t is the indeterminate of R. Note that R is a right quasi-duo local domain and $R/J(R) \cong \mathbb{C}$; hence $R/J(R)$ is commutative. Next consider a maximal right ideal M of the polynomial ring $R[x]$ over R such that $1 + (it)x \in M$. Assume that $R[x]$ is right quasi-duo. Then M is 2-sided and so $2i = i(1 + (it)x) + (1 + (it)x)i \in M$, which is a contradiction. Thus $R[x]$ is not right quasi-duo. □

In the following result we have a condition under which ground rings may be commutative.

COROLLARY 8. *Let R be a semiprimitive ring. Then the following conditions are equivalent:*

(i) *$R[x]$ is a right duo ring;*
(ii) *$R[x]$ is a weakly right duo ring;*
(iii) *$R[x]$ is a right quasi-duo ring;*
(iv) *R is a subdirect product of fields;*
(v) *R is a commutative ring.*

Proof. (i)⇒(ii)⇒(iii) and (iv)⇒(v)⇒(i) are straightforward.

(iii)⇒(iv): Note that R is a subdirect product of division rings by Proposition 1 and Lemma 5. Let R/P be any right primitive factor ring of R, then since $R[x]$ is right quasi-duo it follows that $(R/P)[x] \cong R[x]/P[x]$ is right quasi-duo by Corollary 2. So R/P is a field by Lemma 6 and R is a subdirect product of fields. □

In Corollary 8, the ring R is semiprimitive right quasi-duo, so it is reduced by Corollary 2; hence as a generalization of Corollary 8 we may ask whether R is commutative if R is reduced and $R[x]$ is right quasi-duo. But the following proposition shows that this question is equivalent to the corresponding one in the case where R is a domain.

PROPOSITION 9. *For a ring R the following conditions are equivalent:*
(i) *If R is reduced and $R[x]$ is right quasi-duo then R is commutative;*
(ii) *If R is a domain and $R[x]$ is right quasi-duo then R is commutative.*

Proof. (i)$\Rightarrow$(ii): Straightforward.
(ii)$\Rightarrow$(i): Assume that R is noncommutative, then there exist $a, b \in R$ such that $ab - ba \neq 0$. Since R is reduced, $R[x]$ is semiprimitive; hence there exists a maximal right ideal M of $R[x]$ such that $ab - ba \notin M$. Note that M is two-sided since $R[x]$ is right quasi-duo. But we have $R/(R \cap M) \cong (R + M)/M \subseteq R[x]/M$. Note that $R[x]/M$ is a division ring by Proposition 1 because $R[x]$ is right quasi-duo, whence $(R{+}M)/M$ is a domain. Also the fact that $R[x]$ is right quasi-duo implies that $R[x]/(R \cap M)[x]$ is right quasi-duo by Corollary 2. Then $R[x]/(R \cap M)[x] \cong (R/(R \cap M))[x]$ gives $((R + M)/M)[x]$ is right quasi-duo. Now, by the condition, $(R{+}M)/M$ is commutative and it follows that $ab-ba \in M$, which is a contradiction. Therefore R is commutative. □

For the preceding question, we have a modified answer as in the following.

PROPOSITION 10. *Let R be a reduced ring. If the polynomial ring $R[X]$ is right quasi-duo with $|X| \geq 2$, then R is commutative, where X is a set of commuting indeterminates (possibly infinite) over R.*

Proof. Let $x_{i_1}, x_{i_2} \in X$. Then since R is reduced, $R[X \backslash \{x_{i_1}, x_{i_2}\}]$ is clearly a reduced ring and so $J(R[X \backslash \{x_{i_1}, x_{i_2}\}][x_{i_1}]) = 0$ by [11, Reproof of Amitsur's Theorem (2.5.23) after Lemma 2.5.41]. By hypothesis, $R[X] = R[X \backslash \{x_{i_1}, x_{i_2}\}][x_{i_1}][x_{i_2}]$ is right quasi-duo; hence Corollary 8 implies that $R[X \backslash \{x_{i_1}, x_{i_2}\}][x_{i_1}]$ is commutative. It follows that R is commutative. □

Given a ring R we denote the prime radical of R and the set of all nilpotent elements of R by $P(R)$ and $N(R)$, respectively. In the following we obtain some informations about $J(R[x])$ when $R[x]$ is right quasi-duo, related to $N(R)$ and $P(R)$. In fact, the following results are concerned with the Köthe's conjecture (i.e., the upper nilradical contains every nil left ideal).

PROPOSITION 11. *Let R be a ring and suppose that $R[x]$ is right* *quasi-duo. Then we have the following assertions:*

(i) *$N(R)$ is an ideal of R and $J(R[x]) = N(R)[x]$. So the Köthe's conjecture holds.*

(ii) *If $a \in R$ is nilpotent then aR and Ra are nil.*

Proof. (i): By [11, Reproof of Amitsur's Theorem (2.5.23) after Lemma 2.5.41], $J(R[x]) = I[x]$ for some nil ideal I of R. Take $a \in N(R)$, then $a \in J(R[x])$ by Corollary 2 since $R[x]$ is right quasi-duo. Hence $N(R) \subseteq I$; but clearly $I \subseteq N(R)$ and so $N(R) = I$ (consequently $N(R)$ is the upper nilradical of R). Thus $J(R[x]) = N(R)[x]$ and so the Köthe's conjecture holds by [11, Theorem 2.6.35].

(ii) comes immediately from (i). □

REMARK. Let R be a ring such that $R[x]$ is right quasi-duo. Then we have $N(R) = P(R)$ in the following situations:

(i) Let R be a ring with right Krull dimension (in the sense of Gabriel and Rentschler, see [4] for details.) Then $N(R)$ is a nilpotent ideal of R by Proposition 11 and [9], hence $N(R) = P(R)$ and Proposition 11 implies $J(R[x]) = P(R)[x]$.

(ii) Let R be a ring which is right Goldie or satisfies ascending chain condition on both right and left annihilators. Then $N(R)$ is a nilpotent ideal of R by Proposition 11, [3, Theorem 1.34] and [7], hence $N(R) = P(R)$ and Proposition 11 implies $J(R[x]) = P(R)[x]$.

However the converses of Proposition 11 and Remark do not hold in general by the ring R in Example 4. R is a domain so $J(R[x]) = P(R)[x]$ with $P(R) = N(R) = 0$; but $R[x]$ is not right quasi-duo.

Recall that for a ring R and an endomorphism $\sigma : R \to R$, the *Ore* extension $R[x;\sigma]$ of R is the ring obtained by giving $R[x]$ the new multiplication $rx = x\sigma(r)$ for all $r \in R$. This is also called a *skew* polynomial ring over R by σ. So it is natural to ask whether skew polynomial rings over right quasi-duo rings may be right quasi-duo. But the answer is negative by the following.

EXAMPLE 5. Let R be the quotient field of the polynomial ring $F[t]$ over a field F with t its indeterminate and define a ring homomorphism $\sigma : R \to R$ by $\sigma(\frac{f(t)}{g(t)}) = \frac{f(t^2)}{g(t^2)}$. Next let $S = R[x;\sigma]$. Then since σ is not onto, S is not right quasi-duo by [8, Proposition 27]. Actually $(1+x)S$ is a maximal right ideal but not a maximal ideal in S because $xS = SxS$ is the unique maximal ideal of S by the argument in the proof of [8, Proposition 27]. □

Artinian rings need not be right or left quasi-duo by the n by n full matrix ring over a division ring with $n \geq 2$, but we have the following relation between them.

PROPOSITION 12. *Let R be a right or left Artinian ring. Then the following conditions are equivalent:*

(i) *R is a right quasi-duo ring;*

(ii) *$R/J(R)$ is a finite direct product of division rings.*

Proof. (i)⇒(ii): Since R is a right or left Artinian ring, there are only a finite number of distinct right primitive ideals in R and $R/J(R)$ is a finite direct product of simple Artinian rings, by [11, Theorem 2.3.9]. But since R is right quasi-duo, $R/J(R)$ is a finite direct product of division rings by Proposition 1 and Corollary 2.

(ii)⇒(i): Note that every right primitive factor ring of $R/J(R)$ (hence, of R) is a division ring by the condition. So R is right quasi-duo by Proposition 1.

The preceding argument does not hold in general for Noetherian rings, considering the polynomial ring over a field.

References

[1] F. W. Anderson and K.R. Fuller, Springer-Verlag New York, Inc. (1992).

[2] H. Bass, *Finitistic dimension and a generalization of semiprimary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488.

[3] A. W. Chatters and C. R. Hajarnavis, Pitman Advanced Publishing Program (1980).

[4] R. Gordon and J. C. Robson, *Krull dimension*, Memoirs Amer. Math. Soc. **133** (1973).

[5] Y. Hirano, C. Y. Hong, J. Y. Kim and J. K. Park, *On strongly bounded rings and duo rings*, Comm. Algebra **23** (1995), no. 6, 2199–2214.

[6] C. O. Kim, H. K. Kim and S. H. Jang, *A study on quasi-duo rings*, Bull. Kor. Math. Soc. **36** (1999), no. 3, 579–588.

[7] C. Lanski, *Nil subrings of Goldie rings are nilpotent*, Canad. J. Math. **21** (1969), 904–907.

[8] Y. Lee and C. Huh, *On rings in which every maximal one-sided ideal contains a maximal ideal*, Comm. Algebra **27** (1999), no. 8, 3969–3978.

[9] T. H. Lenagan, *Nil ideals in rings with finite Krull dimension*, J. Algebra **29** (1974), 77–87.

[10] W. K. Nicholson, *Very semisimple modules*, American Math. Monthly **104** (1997), no. 2, 159–162.

[11] L. H. Rowen, Academic Press, Inc. (1991).

[12] X. Yao, *Weakly right duo rings*, Pure Appl. Math. Sci. **21** (1985), 19–24.

[13] H.-P. Yu, *On quasi-duo rings*, Glasgow Math. J. **37** (1995), 21–31.

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