

FUZZY r -REGULAR OPEN SETS AND FUZZY ALMOST r -CONTINUOUS MAPS

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ABSTRACT. We introduce the concepts of fuzzy r -regular open sets and fuzzy almost r -continuous maps in the fuzzy topology of Chattopadhyay. Also we investigate the equivalent conditions of the fuzzy almost r -continuity.

1. Introduction

Chang [2] introduced fuzzy topological spaces and other authors continued the investigation of such spaces. Azad [1] introduced the concepts of fuzzy regular open set and fuzzy almost continuous maps in Chang's fuzzy topology. Chattopadhyay et al. [4] introduced another definition of fuzzy topology as a generalization of Chang's fuzzy topology. By generalizing the definitions of Azad, we introduce the concepts of fuzzy r -regular open sets and fuzzy almost r -continuous maps in the fuzzy topology of Chattopadhyay. Then the concepts introduced by Azad become special cases of our definition. Also we investigate the equivalent conditions of the fuzzy almost r -continuity.

2. Preliminaries

In this paper, we denote by I the unit interval $[0, 1]$ of the real line and $I_0 = (0, 1]$. A member μ of I^X is called a fuzzy set in X . For any $\mu \in I^X$, μ^c denotes the complement $1 - \mu$. By $\tilde{0}$ and $\tilde{1}$ we denote constant maps on X with value 0 and 1, respectively. All other notations are standard notations of fuzzy set theory.

Received July 24, 2001.

2000 Mathematics Subject Classification: 54A40.

Key words and phrases: fuzzy r -regular open, fuzzy almost r -continuous.

This work was supported partly by the Basic Science Research Institute (BSRI-01-4) of Chungbuk National University.

A *Chang's fuzzy topology* on X is a family T of fuzzy sets in X which satisfies the following three properties:

- (1) $\tilde{0}, \tilde{1} \in T$.
- (2) If $\mu_1, \mu_2 \in T$ then $\mu_1 \wedge \mu_2 \in T$.
- (3) If $\mu_i \in T$ for each i , then $\bigvee \mu_i \in T$.

The pair (X, T) is called a *Chang's fuzzy topological space*.

A *fuzzy topology* on X is a map $\mathcal{T} : I^X \rightarrow I$ which satisfies the following properties:

- (1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$,
- (2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$,
- (3) $\mathcal{T}(\bigvee \mu_i) \geq \bigwedge \mathcal{T}(\mu_i)$.

The pair (X, \mathcal{T}) is called a *fuzzy topological space*.

For each $\alpha \in (0, 1]$, a *fuzzy point* x_α in X is a fuzzy set in X defined by

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

In this case, x and α are called the *support* and the *value* of x_α , respectively. A fuzzy point x_α is said to *belong* to a fuzzy set μ in X , denoted by $x_\alpha \in \mu$, if $\alpha \leq \mu(x)$. A fuzzy point x_α in X is said to be *quasi-coincident* with μ , denoted by $x_\alpha q\mu$, if $\alpha + \mu(x) > 1$. A fuzzy set ρ in X is said to be *quasi-coincident* with a fuzzy set μ in X , denoted by $\rho q\mu$, if there is an $x \in X$ such that $\rho(x) + \mu(x) > 1$.

DEFINITION 2.1. ([5]) Let μ be a fuzzy set in a fuzzy topological space (X, \mathcal{T}) and $r \in I_0$. Then μ is called

- (1) a *fuzzy r -open set* in X if $\mathcal{T}(\mu) \geq r$,
- (2) a *fuzzy r -closed set* in X if $\mathcal{T}(\mu^c) \geq r$.

DEFINITION 2.2. ([3]) Let (X, \mathcal{T}) be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy r -closure* is defined by

$$\text{cl}(\mu, r) = \bigwedge \{ \rho \in I^X : \mu \leq \rho, \mathcal{T}(\rho^c) \geq r \}.$$

DEFINITION 2.3. ([5]) Let (X, \mathcal{T}) be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy r -interior* is defined by

$$\text{int}(\mu, r) = \bigvee \{ \rho \in I^X : \mu \geq \rho, \mathcal{T}(\rho) \geq r \}.$$

THEOREM 2.4. ([5]) For a fuzzy set μ in a fuzzy topological space (X, \mathcal{T}) and $r \in I_0$, we have:

- (1) $\text{int}(\mu, r)^c = \text{cl}(\mu^c, r)$.
- (2) $\text{cl}(\mu, r)^c = \text{int}(\mu^c, r)$.

DEFINITION 2.5. ([5]) Let μ be a fuzzy set in a fuzzy topological space (X, \mathcal{T}) and $r \in I_0$. Then μ is said to be

- (1) *fuzzy r -semiopen* if there is a fuzzy r -open set ρ in X such that $\rho \leq \mu \leq \text{cl}(\rho, r)$,
- (2) *fuzzy r -semiclosed* if there is a fuzzy r -closed set ρ in X such that $\text{int}(\rho, r) \leq \mu \leq \rho$.

DEFINITION 2.6. ([5]) Let x_α be a fuzzy point in a fuzzy topological space (X, \mathcal{T}) and $r \in I_0$. Then a fuzzy set μ in X is called

- (1) a *fuzzy r -neighborhood* of x_α if there is a fuzzy r -open set ρ in X such that $x_\alpha \in \rho \leq \mu$,
- (2) a *fuzzy r -quasi-neighborhood* of x_α if there is a fuzzy r -open set ρ in X such that $x_\alpha q \rho \leq \mu$.

DEFINITION 2.7. ([5]) Let $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is called

- (1) a *fuzzy r -continuous* map if $f^{-1}(\mu)$ is a fuzzy r -open set of X for each fuzzy r -open set μ in Y ,
- (2) a *fuzzy r -semicontinuous* map if $f^{-1}(\mu)$ is a fuzzy r -semiopen set of X for each fuzzy r -open set μ in Y ,
- (3) a *fuzzy r -irresolute* map if $f^{-1}(\mu)$ is a fuzzy r -semiopen set of X for each fuzzy r -semiopen set μ in Y .

All the other nonstandard definitions and notations can be found in [5] and [6].

3. Fuzzy r -regular open sets

We define the notions of fuzzy r -regular open sets and fuzzy r -regular closed sets, and investigate some of their properties.

DEFINITION 3.1. Let μ be a fuzzy set in a fuzzy topological space (X, \mathcal{T}) and $r \in I_0$. Then μ is said to be

- (1) *fuzzy r -regular open* if $\text{int}(\text{cl}(\mu, r), r) = \mu$,
- (2) *fuzzy r -regular closed* if $\text{cl}(\text{int}(\mu, r), r) = \mu$.

THEOREM 3.2. *Let μ be a fuzzy set in a fuzzy topological space (X, \mathcal{T}) and $r \in I_0$. Then μ is fuzzy r -regular open if and only if μ^c is fuzzy r -regular closed.*

Proof. It follows from Theorem 2.4. □

REMARK 3.3. Clearly, every fuzzy r -regular open (r -regular closed) set is fuzzy r -open (r -closed). That the converse need not be true is shown by the following example. The example also shows that the union (intersection) of any two fuzzy r -regular open (r -regular closed) sets need not be fuzzy r -regular open (r -regular closed).

EXAMPLE 3.4. Let $X = I$ and μ_1, μ_2 and μ_3 be fuzzy sets in X defined by

$$\mu_1(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1; \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{4}, \\ -4x + 2 & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1; \end{cases}$$

and

$$\mu_3(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{4}, \\ \frac{1}{3}(4x - 1) & \text{if } \frac{1}{4} \leq x \leq 1. \end{cases}$$

Define $\mathcal{T} : I^X \rightarrow I$ by

$$\mathcal{T}(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \mu_2, \mu_1 \vee \mu_2, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly \mathcal{T} is a fuzzy topology on X .

(1) Clearly, $\mu_1 \vee \mu_2$ is fuzzy $\frac{1}{2}$ -open. Since $\text{int}(\text{cl}(\mu_1 \vee \mu_2, \frac{1}{2}), \frac{1}{2}) = \tilde{1} \neq \mu_1 \vee \mu_2$, $\mu_1 \vee \mu_2$ is not a fuzzy $\frac{1}{2}$ -regular open set.

(2) Since $\text{int}(\text{cl}(\mu_1, \frac{1}{2}), \frac{1}{2}) = \text{int}(\mu_1^c, \frac{1}{2}) = \mu_1$ and $\text{int}(\text{cl}(\mu_2, \frac{1}{2}), \frac{1}{2}) = \text{int}(\mu_2^c, \frac{1}{2}) = \mu_2$, μ_1 and μ_2 are fuzzy $\frac{1}{2}$ -regular open sets. But $\mu_1 \vee \mu_2$ is not a fuzzy $\frac{1}{2}$ -regular open set.

(3) In view of Theorem 3.2, μ_1^c and μ_2^c are fuzzy $\frac{1}{2}$ -regular closed sets but $\mu_1^c \wedge \mu_2^c = (\mu_1 \vee \mu_2)^c$ is not a fuzzy $\frac{1}{2}$ -regular closed set.

THEOREM 3.5. (1) *The intersection of two fuzzy r -regular open sets is fuzzy r -regular open.*

(2) *The union of two fuzzy r -regular closed sets is fuzzy r -regular closed.*

Proof. (1) Let μ and ρ be any two fuzzy r -regular open sets in a fuzzy topological space X . Then μ and ρ are fuzzy r -open sets and hence $\mathcal{T}(\mu \wedge \rho) \geq \mathcal{T}(\mu) \wedge \mathcal{T}(\rho) \geq r$. Thus $\mu \wedge \rho$ is a fuzzy r -open set. Since $\mu \wedge \rho \leq \text{cl}(\mu \wedge \rho, r)$,

$$\text{int}(\text{cl}(\mu \wedge \rho, r), r) \geq \text{int}(\mu \wedge \rho, r) = \mu \wedge \rho.$$

Now, $\mu \wedge \rho \leq \mu$ and $\mu \wedge \rho \leq \rho$ implies

$$\text{int}(\text{cl}(\mu \wedge \rho, r), r) \leq \text{int}(\text{cl}(\mu, r), r) = \mu$$

and

$$\text{int}(\text{cl}(\mu \wedge \rho, r), r) \leq \text{int}(\text{cl}(\rho, r), r) = \rho.$$

Hence $\text{int}(\text{cl}(\mu \wedge \rho, r), r) \leq \mu \wedge \rho$. Therefore $\mu \wedge \rho$ is fuzzy r -regular open.

(2) It follows from (1) and Theorem 3.2. □

THEOREM 3.6. (1) *The fuzzy r -closure of a fuzzy r -open set is fuzzy r -regular closed.*

(2) *The fuzzy r -interior of a fuzzy r -closed set is fuzzy r -regular open.*

Proof. (1) Let μ be a fuzzy r -open set in a fuzzy topological space X . Then clearly $\text{int}(\text{cl}(\mu, r), r) \leq \text{cl}(\mu, r)$ implies that

$$\text{cl}(\text{int}(\text{cl}(\mu, r), r), r) \leq \text{cl}(\text{cl}(\mu, r), r) = \text{cl}(\mu, r).$$

Since μ is fuzzy r -open, $\mu = \text{int}(\mu, r)$. Also since $\mu \leq \text{cl}(\mu, r)$, $\mu = \text{int}(\mu, r) \leq \text{int}(\text{cl}(\mu, r), r)$. Thus $\text{cl}(\mu, r) \leq \text{cl}(\text{int}(\text{cl}(\mu, r), r), r)$. Hence $\text{cl}(\mu, r)$ is a fuzzy r -regular closed set.

(2) Similar to (1). □

Let (X, \mathcal{T}) be a fuzzy topological space. For an r -cut $\mathcal{T}_r = \{\mu \in I^X \mid \mathcal{T}(\mu) \geq r\}$, it is obvious that (X, \mathcal{T}_r) is a Chang's fuzzy topological space for all $r \in I_0$.

Let (X, \mathcal{T}) be a Chang's fuzzy topological space and $r \in I_0$. Recall [4] that a fuzzy topology $T^r : I^X \rightarrow I$ is defined by

$$T^r(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ r & \text{if } \mu \in \mathcal{T} - \{\tilde{0}, \tilde{1}\}, \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 3.7. *Let μ be a fuzzy set in a fuzzy topological space (X, \mathcal{T}) and $r \in I_0$. Then μ is fuzzy r -regular open (r -regular closed) in (X, \mathcal{T}) if and only if μ is fuzzy regular open (regular closed) in (X, \mathcal{T}_r) .*

Proof. Straightforward. \square

THEOREM 3.8. *Let μ be a fuzzy set of a Chang's fuzzy topological space (X, \mathcal{T}) and $r \in I_0$. Then μ is a fuzzy regular open (regular closed) in (X, \mathcal{T}) if and only if μ is fuzzy r -regular open (r -regular closed) in (X, \mathcal{T}^r) .*

Proof. Straightforward. \square

4. Fuzzy almost r -continuous maps

We are going to introduce the notions of fuzzy almost r -continuous maps and investigate some of their properties. Also, we describe the relations among fuzzy almost r -continuous maps, fuzzy r -continuous maps and fuzzy r -semicontinuous maps.

DEFINITION 4.1. Let $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is called

- (1) a *fuzzy almost r -continuous* map if $f^{-1}(\mu)$ is a fuzzy r -open set of X for each fuzzy r -regular open set μ in Y , or equivalently, $f^{-1}(\mu)$ is a fuzzy r -closed set in X for each fuzzy r -regular closed set μ in Y ,
- (2) a *fuzzy almost r -open* map if $f(\rho)$ is a fuzzy r -open set in Y for each fuzzy r -regular open set ρ in X ,
- (3) a *fuzzy almost r -closed* map if $f(\rho)$ is a fuzzy r -closed set in Y for each fuzzy r -regular closed set ρ in X .

THEOREM 4.2. *Let $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent:*

- (1) f is a fuzzy almost r -continuous map.
- (2) $f^{-1}(\mu) \leq \text{int}(f^{-1}(\text{int}(\text{cl}(\mu, r), r)), r)$ for each fuzzy r -open set μ in Y .
- (3) $\text{cl}(f^{-1}(\text{cl}(\text{int}(\mu, r), r)), r) \leq f^{-1}(\mu)$ for each fuzzy r -closed set μ in Y .

Proof. (1) \Rightarrow (2) Let f be fuzzy almost r -continuous and μ any fuzzy r -open set in Y . Then

$$\mu = \text{int}(\mu, r) \leq \text{int}(\text{cl}(\mu, r), r).$$

By Theorem 3.6(2), $\text{int}(\text{cl}(\mu, r), r)$ is a fuzzy r -regular open set in Y . Since f is fuzzy almost r -continuous, $f^{-1}(\text{int}(\text{cl}(\mu, r), r))$ is a fuzzy r -open set in X . Hence

$$f^{-1}(\mu) \leq f^{-1}(\text{int}(\text{cl}(\mu, r), r)) = \text{int}(f^{-1}(\text{int}(\text{cl}(\mu, r), r)), r).$$

(2) \Rightarrow (3) Let μ be a fuzzy r -closed set of Y . Then μ^c is a fuzzy r -open set in Y . By (2),

$$f^{-1}(\mu^c) \leq \text{int}(f^{-1}(\text{int}(\text{cl}(\mu^c, r), r)), r).$$

Hence

$$\begin{aligned} f^{-1}(\mu) &= f^{-1}(\mu^c)^c \geq \text{int}(f^{-1}(\text{int}(\text{cl}(\mu^c, r), r)), r)^c \\ &= \text{cl}(f^{-1}(\text{cl}(\text{int}(\mu, r), r)), r). \end{aligned}$$

(3) \Rightarrow (1) Let μ be a fuzzy r -regular closed set in Y . Then μ is a fuzzy r -closed set in Y and hence

$$f^{-1}(\mu) \leq \text{int}(f^{-1}(\text{int}(\text{cl}(\mu, r), r)), r) = \text{int}(f^{-1}(\mu), r).$$

Thus $f^{-1}(\mu) = \text{cl}(f^{-1}(\mu), r)$ and hence $f^{-1}(\mu)$ is a fuzzy r -closed set in X . Therefore, f is a fuzzy almost r -continuous map. \square

THEOREM 4.3. Let $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U})$ be a map and $r \in I_0$. Then f is fuzzy almost r -open if and only if $f(\text{int}(\rho, r)) \leq \text{int}(f(\rho), r)$ for each fuzzy r -semiclosed set ρ in X .

Proof. Let f be fuzzy almost r -open and ρ a fuzzy r -semiclosed set in X . Then $\text{int}(\rho, r) \leq \text{int}(\text{cl}(\rho, r), r) \leq \rho$. Note that $\text{cl}(\rho, r)$ is a fuzzy r -closed set of X . By Theorem 3.6(2), $\text{int}(\text{cl}(\rho, r), r)$ is a fuzzy r -regular open set in X . Since f is fuzzy almost r -open, $f(\text{int}(\text{cl}(\rho, r), r))$ is a fuzzy r -open set in X . Thus we have

$$f(\text{int}(\rho, r)) \leq f(\text{int}(\text{cl}(\rho, r), r)) = \text{int}(f(\text{int}(\text{cl}(\rho, r), r)), r) \leq \text{int}(f(\rho), r).$$

Conversely, let ρ be a fuzzy r -regular open set of X . Then ρ is fuzzy r -open and hence $\text{int}(\rho, r) = \rho$. Since $\text{int}(\text{cl}(\rho, r), r) = \rho$, ρ is fuzzy r -semiclosed. So

$$f(\rho) = f(\text{int}(\rho, r)) \leq \text{int}(f(\rho), r) \leq f(\rho).$$

Thus $f(\rho) = \text{int}(f(\rho), r)$ and hence $f(\rho)$ is a fuzzy r -open set in Y . \square

The global property of fuzzy almost r -continuity can be rephrased to the local property in terms of neighborhood and quasi-neighborhood, respectively, in the following two theorems.

THEOREM 4.4. *Let $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U})$ be a map and $r \in I_0$. Then f is fuzzy almost r -continuous if and only if for every fuzzy point x_α in X and every fuzzy r -neighborhood μ of $f(x_\alpha)$, there is a fuzzy r -neighborhood ρ of x_α such that $x_\alpha \in \rho$ and $f(\rho) \leq \text{int}(\text{cl}(\mu, r), r)$.*

Proof. Let x_α be a fuzzy point in X and μ a fuzzy r -neighborhood of $f(x_\alpha)$. Then there is a fuzzy r -open set λ in Y such that $f(x_\alpha) \in \lambda \leq \mu$. So $x_\alpha \in f^{-1}(\lambda) \leq f^{-1}(\mu)$. Since f is fuzzy almost r -continuous,

$$f^{-1}(\lambda) \leq \text{int}(f^{-1}(\text{int}(\text{cl}(\lambda, r)), r) \leq \text{int}(f^{-1}(\text{int}(\text{cl}(\mu, r)), r)).$$

Put $\rho = f^{-1}(\text{int}(\text{cl}(\mu, r), r))$. Then $x_\alpha \in f^{-1}(\lambda) \leq \text{int}(\rho, r) \leq \rho$. By Theorem 3.6(2), $\text{int}(\text{cl}(\mu, r), r)$ is fuzzy r -regular open. Since f is fuzzy almost r -continuous, $\rho = f^{-1}(\text{int}(\text{cl}(\mu, r), r))$ is fuzzy r -open. Thus ρ is a fuzzy r -neighborhood of x_α and

$$f(\rho) = f f^{-1}(\text{int}(\text{cl}(\mu, r), r)) \leq \text{int}(\text{cl}(\mu, r), r).$$

Conversely, let μ be a fuzzy r -regular open set in Y and $x_\alpha \in f^{-1}(\mu)$. Then μ is fuzzy r -open and hence μ is a fuzzy r -neighborhood of $f(x_\alpha)$. By hypothesis, there is a fuzzy r -neighborhood ρ_{x_α} of x_α such that $x_\alpha \in \rho_{x_\alpha}$ and $f(\rho_{x_\alpha}) \leq \text{int}(\text{cl}(\mu, r), r) = \mu$. Since ρ_{x_α} is a fuzzy r -neighborhood of x_α , there is a fuzzy r -open set λ_{x_α} in X such that

$$x_\alpha \in \lambda_{x_\alpha} \leq \rho_{x_\alpha} \leq f^{-1} f(\rho_{x_\alpha}) \leq f^{-1}(\mu).$$

So we have

$$\begin{aligned} f^{-1}(\mu) &= \bigvee \{x_\alpha : x_\alpha \in f^{-1}(\mu)\} \\ &\leq \bigvee \{\lambda_{x_\alpha} : x_\alpha \in f^{-1}(\mu)\} \\ &\leq f^{-1}(\mu). \end{aligned}$$

Thus $f^{-1}(\mu) = \bigvee \{\lambda_{x_\alpha} : x_\alpha \in f^{-1}(\mu)\}$ is fuzzy r -open in X and hence f is almost r -continuous. \square

THEOREM 4.5. *Let $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U})$ be a map and $r \in I_0$. Then f is a fuzzy almost r -continuous map if and only if for every fuzzy point x_α in X and every fuzzy r -quasi-neighborhood μ of $f(x_\alpha)$, there is a fuzzy r -quasi-neighborhood ρ of x_α such that $x_\alpha \mathbf{q} \rho$ and $f(\rho) \leq \text{int}(\text{cl}(\mu, r), r)$.*

Proof. Let x_α be a fuzzy point in X and μ a fuzzy r -quasi-neighborhood of $f(x_\alpha)$. Then there is a fuzzy r -open set λ in Y such that $f(x_\alpha) \mathbf{q} \lambda \leq \mu$. So $x_\alpha \mathbf{q} f^{-1}(\lambda)$. Since f is fuzzy almost r -continuous,

$$f^{-1}(\lambda) \leq \text{int}(f^{-1}(\text{int}(\text{cl}(\lambda, r)), r) \leq \text{int}(f^{-1}(\text{int}(\text{cl}(\mu, r)), r).$$

Put $\rho = f^{-1}(\text{int}(\text{cl}(\mu, r), r))$. Then $x_\alpha \mathbf{q} f^{-1}(\lambda) \leq \text{int}(\rho, r) \leq \rho$. So $x_\alpha \mathbf{q} \rho$. Since $\text{int}(\text{cl}(\mu, r), r)$ is fuzzy r -regular open and f is fuzzy almost r -continuous, $\rho = f^{-1}(\text{int}(\text{cl}(\mu, r), r))$ is fuzzy r -open. Thus ρ is a fuzzy r -quasi-neighborhood of x_α and

$$f(\rho) = f f^{-1}(\text{int}(\text{cl}(\mu, r), r)) \leq \text{int}(\text{cl}(\mu, r), r).$$

Conversely, let μ be a fuzzy r -regular open set in Y . If $f^{-1}(\mu) = \tilde{0}$, then it is obvious. Suppose x_α is a fuzzy point in $f^{-1}(\mu)$ such that $\alpha < f^{-1}(\mu)(x)$. Then $\alpha < \mu(f(x))$ and hence $f(x)_{1-\alpha} \mathbf{q} \mu$. So μ is a fuzzy r -quasi-neighborhood of $f(x)_{1-\alpha} = f(x_{1-\alpha})$. By hypothesis, there is a fuzzy r -quasi-neighborhood ρ_{x_α} of $x_{1-\alpha}$ such that $x_{1-\alpha} \mathbf{q} \rho_{x_\alpha}$ and $f(\rho_{x_\alpha}) \leq \text{int}(\text{cl}(\mu, r), r) = \mu$. Since ρ_{x_α} is a fuzzy r -quasi-neighborhood of $x_{1-\alpha}$, there is a fuzzy r -open set λ_{x_α} in X such that

$$x_{1-\alpha} \mathbf{q} \lambda_{x_\alpha} \leq \rho_{x_\alpha} \leq f^{-1} f(\rho_{x_\alpha}) \leq f^{-1}(\mu).$$

Then $\alpha < \lambda_{x_\alpha}(x)$ and hence $x_\alpha \in \lambda_{x_\alpha}$. So

$$\begin{aligned} f^{-1}(\mu) &= \bigvee \{x_\alpha : x_\alpha \text{ is a fuzzy point in } f^{-1}(\mu) \text{ such that} \\ &\quad \alpha < f^{-1}(\mu)(x)\} \\ &\leq \bigvee \{\lambda_{x_\alpha} : x_\alpha \text{ is a fuzzy point in } f^{-1}(\mu) \text{ such that} \\ &\quad \alpha < f^{-1}(\mu)(x)\} \\ &\leq f^{-1}(\mu) \end{aligned}$$

and hence

$$\begin{aligned} f^{-1}(\mu) &= \bigvee \{\lambda_{x_\alpha} : x_\alpha \text{ is a fuzzy point in } f^{-1}(\mu) \text{ such that} \\ &\quad \alpha < f^{-1}(\mu)(x)\}. \end{aligned}$$

Thus $f^{-1}(\mu)$ is fuzzy r -open in X . Therefore f is fuzzy almost r -continuous. □

THEOREM 4.6. *Let $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U})$ be fuzzy r -semicontinuous and fuzzy almost r -open. Then f is fuzzy r -irresolute.*

Proof. Let μ be fuzzy r -semiclosed in Y . Then $\text{int}(\text{cl}(\mu, r), r) \leq \mu$. Since f is fuzzy r -semicontinuous,

$$\text{int}(\text{cl}(f^{-1}(\mu), r), r) \leq f^{-1}(\text{cl}(\mu, r)).$$

Thus we have

$$\text{int}(\text{cl}(f^{-1}(\mu), r), r) = \text{int}(\text{int}(\text{cl}(f^{-1}(\mu), r), r), r) \leq \text{int}(f^{-1}(\text{cl}(\mu, r)), r).$$

Since f is fuzzy r -semicontinuous and $\text{cl}(\mu, r)$ is fuzzy r -closed, $f^{-1}(\text{cl}(\mu, r))$ is a fuzzy r -semiclosed set in X . Since f is fuzzy almost r -open,

$$f(\text{int}(f^{-1}(\text{cl}(\mu, r)), r) \leq \text{int}(ff^{-1}(\text{cl}(\mu, r)), r) \leq \text{int}(\text{cl}(\mu, r), r) \leq \mu.$$

Hence we have

$$\begin{aligned} \text{int}(\text{cl}(f^{-1}(\mu), r), r) &\leq f^{-1}f(\text{int}(\text{cl}(f^{-1}(\mu), r), r)) \\ &\leq f^{-1}f(\text{int}(f^{-1}(\text{cl}(\mu, r)), r)) \\ &\leq f^{-1}(\mu). \end{aligned}$$

Thus $f^{-1}(\mu)$ is fuzzy r -semiclosed in X and hence f is fuzzy r -irresolute. \square

REMARK 4.7. Clearly a fuzzy r -continuous map is a fuzzy almost r -continuous map. That the converse need not be true is shown by the following example. Also, the example shows that a fuzzy almost r -continuous map need not be a fuzzy r -semicontinuous map.

EXAMPLE 4.8. Let $X = I$ and μ_1, μ_2 and μ_3 be fuzzy sets in X defined by

$$\begin{aligned} \mu_1(x) &= x; \\ \mu_2(x) &= 1 - x; \end{aligned}$$

and

$$\mu_3(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Define $\mathcal{T}_1 : I^X \rightarrow I$ and $\mathcal{T}_2 : I^X \rightarrow I$ by

$$\mathcal{T}_1(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1} \\ \frac{1}{2} & \text{if } \mu = \mu_1, \mu_2, \mu_1 \vee \mu_2, \mu_1 \wedge \mu_2 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathcal{T}_2(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1} \\ \frac{1}{2} & \text{if } \mu = \mu_1, \mu_2, \mu_3, \mu_1 \vee \mu_2, \mu_1 \wedge \mu_2 \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $\mathcal{T}_1, \mathcal{T}_2$ are fuzzy topologies on X . Consider the identity map $1_X : (X, \mathcal{T}_1) \rightarrow (X, \mathcal{T}_2)$. It is clear that $\mu_1, \mu_2, \mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ are fuzzy $\frac{1}{2}$ -regular open in (X, \mathcal{T}_2) while μ_3 is not. Noting that $\mathcal{T}_1(\mu_3) = 0$, it is obvious that 1_X is a fuzzy $\frac{1}{2}$ -almost continuous map which is not a fuzzy $\frac{1}{2}$ -continuous map. Also, because $\tilde{0}$ is the only fuzzy $\frac{1}{2}$ -open set contained in μ_3 , $\mu_3 = 1_X^{-1}(\mu_3)$ is not a fuzzy $\frac{1}{2}$ -semiopen set in (X, \mathcal{T}_1) and hence 1_X is not a fuzzy $\frac{1}{2}$ -semicontinuous map.

EXAMPLE 4.9. A fuzzy r -semicontinuous map need not be a fuzzy almost r -continuous map.

Let (X, \mathcal{T}) be a fuzzy topological space as described in Example 3.4 and let $f : (X, \mathcal{T}) \rightarrow (X, \mathcal{T})$ be defined by $f(x) = \frac{x}{2}$. Simple computations give $f^{-1}(\tilde{0}) = \tilde{0}, f^{-1}(\tilde{1}) = \tilde{1}, f^{-1}(\mu_1) = \tilde{0}$ and $f^{-1}(\mu_2) = \mu_1^c = f^{-1}(\mu_1 \vee \mu_2)$. Since $\text{cl}(\mu_2, \frac{1}{2}) = \mu_1^c$, μ_1^c is a fuzzy $\frac{1}{2}$ -semiopen set and hence f is a fuzzy $\frac{1}{2}$ -semicontinuous map. But $f^{-1}(\mu_2) = \mu_1^c$ and

$$\begin{aligned} \text{int}(f^{-1}(\text{int}(\text{cl}(\mu_2, \frac{1}{2}), \frac{1}{2})), \frac{1}{2}) &= \text{int}(f^{-1}(\text{int}(\mu_1^c, \frac{1}{2})), \frac{1}{2}) \\ &= \text{int}(f^{-1}(\mu_2), \frac{1}{2}) \\ &= \text{int}(\mu_1^c, \frac{1}{2}) = \mu_2. \end{aligned}$$

Thus $f^{-1}(\mu_2) \not\subseteq \text{int}(f^{-1}(\text{int}(\text{cl}(\mu_2, \frac{1}{2}), \frac{1}{2})), \frac{1}{2})$ and hence f is not a fuzzy almost $\frac{1}{2}$ -continuous map.

From Example 4.8 and 4.9 we have the following result.

THEOREM 4.10. *Fuzzy r -semicontinuity and fuzzy almost r -continuity are independent notions.*

DEFINITION 4.11. Let (X, \mathcal{T}) be a fuzzy topological space and $r \in I_0$. Then (X, \mathcal{T}) is called a *fuzzy r -semiregular space* if each fuzzy r -open set in X is a union of fuzzy r -regular open sets.

THEOREM 4.12. Let $r \in I_0$ and $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U})$ be a map from a fuzzy topological space X to a fuzzy r -semiregular space Y . Then f is fuzzy almost r -continuous if and only if f is fuzzy r -continuous.

Proof. Due to Remark 4.7, it suffices to show that if f is fuzzy almost r -continuous then it is fuzzy r -continuous. Let μ be a fuzzy r -open set in Y . Since (Y, \mathcal{U}) is a r -semiregular space, $\mu = \bigvee \mu_i$, where μ_i 's are fuzzy r -regular open sets in Y . Then since f is a fuzzy almost r -continuous map, $f^{-1}(\mu_i)$ is a fuzzy r -open set for each i . So

$$\mathcal{T}(f^{-1}(\mu)) = \mathcal{T}(f^{-1}(\bigvee \mu_i)) = \mathcal{T}(\bigvee f^{-1}(\mu_i)) \geq \bigwedge \mathcal{T}(f^{-1}(\mu_i)) \geq r.$$

Thus $f^{-1}(\mu)$ is fuzzy r -open in X and hence f is a fuzzy r -continuous map. \square

THEOREM 4.13. Let $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is fuzzy almost r -continuous (r -open, r -closed) if and only if $f : (X, \mathcal{T}_r) \rightarrow (Y, \mathcal{U}_r)$ is fuzzy almost continuous (open, closed).

Proof. Straightforward. \square

THEOREM 4.14. Let $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U})$ be a map from a Chang's fuzzy topological space X to another Chang's fuzzy topological space Y and $r \in I_0$. Then f is fuzzy almost continuous (open, closed) if and only if $f : (X, \mathcal{T}^r) \rightarrow (Y, \mathcal{U}^r)$ is fuzzy almost r -continuous (r -open, r -closed).

Proof. Straightforward. \square

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