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NOTE ON GOOD IDEALS IN GORENSTEIN LOCAL RINGS

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ABSTRACT. Let I be an ideal in a Gorenstein local ring A with the maximal ideal $\mathfrak{m}$ and $d = \dim A$. Then we say that I is a *good ideal* in A, if I contains a reduction $Q = (a_1, a_2, \cdots, a_d)$ generated by d elements in A and $\mathrm{G}(I) = \oplus_{n \geq 0} I^n/I^{n+1}$ of I is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(I)) = 1 - d$, where $\mathrm{a}(\mathrm{G}(I))$ denotes the a-invariant of $\mathrm{G}(I)$. Let $S = A[Q/a_1]$ and $P = \mathfrak{m}S$. In this paper, we show that the following conditions are equivalent.

(1) $I^2 = QI$ and $I = Q : I$.
(2) $I^2 S = a_1 IS$ and $IS = a_1 S :_S IS$.
(3) $I^2 S_P = a_1 I S_P$ and $I S_P = a_1 S_P :_{S_P} I S_P$.

We denote by $\mathcal{X}_A(Q)$ the set of good ideals I in $\mathcal{X}_A$ such that I contains Q as a reduction. As a Corollary of this result, we show that

$$I \in \mathcal{X}_A(Q) \Longleftrightarrow IS_P \in \mathcal{X}_{S_P}(Q_P).$$

1. Introduction

Let A be a Gorenstein local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A$. Let I denote an $\mathfrak{m}$-primary ideal in A. Then we say that I is a *good* ideal in A if I contains a parameter ideal $(c_1, c_2, \cdots, c_d)$ in A as a reduction and the associated graded ring $\mathrm{G}(I) = \oplus_{n \geq 0} I^n/I^{n+1}$ of I is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(I)) = 1 - d$ ([3]), where $\mathrm{a}(\mathrm{G}(I))$ denotes the a-invariant of $\mathrm{G}(I)$ ([4], Definition (3.1.4)). We denote by $\mathcal{X}_A$ the set of good ideals I in A. The concept of good ideals was first introduced by S. Goto, S. Iai, and K. Watanabe and they intensively studied $\mathfrak{m}$-primary

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good ideals in a given Gorenstein local ring and gave many inspiring results ([3]).

Let $Q = (a_1, a_2, \cdots, a_d)$ be a fixed parameter ideal in A. Let $S = A[Q/a_1]$ and $P = \mathfrak{m}S$. We denote by $\mathcal{X}_A(Q)$ the set of good ideals I in $\mathcal{X}_A$ such that I contains Q as a reduction. With this notation the main result of this paper is stated as follows.

THEOREM 1.1. *Let I ($\neq A$) be an ideal in A. Suppose that I contains a parameter ideal $Q = (a_1, \cdots, a_d)$ as a reduction. Then the following conditions are equivalent.*

(1) $I^2 = QI$ *and* $I = Q : I$.
(2) $I^2S = a_1IS$ *and* $IS = a_1S :_S IS$.
(3) $I^2S_P = a_1IS_P$ *and* $IS_P = a_1S_P :_{S_P} IS_P$.

COROLLARY 1.2. *Let I ($\neq A$) be an ideal in A. Suppose that I contains a parameter ideal $Q = (a_1, \cdots, a_d)$ as a reduction. Then the following conditions are equivalent.*

(1) $I \in \mathcal{X}_A(Q)$.
(2) $IS_P \in \mathcal{X}_{S_P}(QS_P)$.

In what follows, let $(A, \mathfrak{m})$ be a Gorenstein local ring and $d = \dim A$. Let $K = Q(A)$ be the total quotient ring of A. We denote by $\mu_A(*)$ the number of generators and $\ell_A(*)$ the length.

Let $B = \oplus_{n \in \mathbb{Z}} B_n$ be a Noetherian graded ring and assume that B contains a unique graded maximal ideal $\mathfrak{M}$. We denote by $\mathrm{H}^i_{\mathfrak{M}}(*)$ ($i \in \mathbb{Z}$) the $i^{\underline{th}}$ local cohomology functor of B with respect to $\mathfrak{M}$. For each graded B-module E and $n \in \mathbb{Z}$, let $[\mathrm{H}^i_{\mathfrak{M}}(E)]_n$ denote the homogeneous component of the graded B-module $\mathrm{H}^i_{\mathfrak{M}}$ of degree n. Let E be a graded B-module. For each $n \in \mathbb{Z}$ let $E(n)$ stand for the graded B-module, whose underlying B-module coincides with that of E and whose graduation is given by $[E(n)]_i = E_{n+i}$ for all $i \in \mathbb{Z}$. We refer the reader to [5], [1], or [6] for any unexplained notation or terminology.

2. Preliminaries

Let $(A, \mathfrak{m})$ be a d-dimensional Gorenstein local ring with $d \geq 2$ and $K = Q(A)$ be the total quotient ring of A. Let $Q = (a_1, \cdots, a_d)$ be a fixed parameter ideal for A. Let $S = A[Q/a_1](= \cup_{n \geq 0} Q^n/a_1^n)$ and

$P = \mathfrak{m}S$. Then $A \subseteq S \subseteq K$ and we have the isomorphism

$$S \cong \frac{A[T_2, T_3, \cdots, T_d]}{(a_1T_2 - a_2, a_1T_3 - a_3, \cdots, a_1T_d - a_d)},$$

where $T_2, T_3, \cdots, T_d$ denote indeterminates over A. Hence S is a d-dimensional Gorenstein ring, since $a_1T_2 - a_2, a_1T_3 - a_3, \cdots, a_1T_d - a_d$ is a regular sequence ([2]). Moreover P is a height 1 prime ideal of S, because $S/P \cong (A/\mathfrak{m})[T_2, T_3, \cdots, T_d]$ is a $(d-1)$-dimensional regular domain, whence S_P is a 1-dimensional Gorenstein local ring. For the proof of our result we need the following lemmas.

LEMMA 2.1. *Let I $(\neq A)$ be an ideal in A. Suppose that I contains Q as a reduction. Then*

(1) *IS is a P-primary ideal in S.*
(2) $IS_P \cap A = I$.
(3) $IS \cap A = I$.
(4) $\ell_{S_P}(S_P/IS_P) = \ell_A(A/I)$ *and* $\ell_{S_P}(S_P/QS_P) = \ell_A(A/Q)$.

Proof. Notice that $QS = a_1S$ and $\sqrt{QS} = \sqrt{IS} = P$.
(1) $S/IS \cong (A/I)[T_2, T_3, \cdots, T_d]$, since $IA[T_2, T_3, \cdots, T_d] \supseteq (a_1T_2 - a_2, \cdots, a_1T_d - a_d)$. Hence $\mathrm{Ass}_S(S/IS) = \{\mathfrak{m}S\}$, because $\mathrm{Ass}(A[T_2, \cdots, T_d]/IA[T_2, \cdots, T_d]) = \{\mathfrak{m}A[T_2, \cdots, T_d]\}$. Thus IS is a P-primary ideal in S.
(2) $IS_P \cap S = I$ by (1). Hence we have $IS_P \cap A = (IS_P \cap S) \cap A = I \cap A = I$.
(3) Let $\alpha \in IS \cap A$ and write $\alpha = \beta \frac{g}{a_1^\ell}$ with $\beta \in I$ and $g \in Q^\ell$ for some $\ell \geq 0$. Since $\alpha \in A$, we get $\alpha a_1^\ell = \beta g \in IQ^\ell = I(a_1^\ell + (a_2, a_3, \cdots, a_d)Q^{\ell-1})$. Now we write $\alpha a_1^\ell = \omega(a_1^\ell + f\sum_{i=2}^d x_i a_i)$ with $\omega \in I$, $f \in Q^{\ell-1}$, and $x_i \in A$ for $i = 2, \cdots, d$. Then $a_1^\ell(\alpha - \omega) = \omega f \sum_{i=2}^d x_i a_i \in (a_2, a_3, \cdots, a_d)$ so that $\alpha - \omega \in (a_2, a_3, \cdots, a_d) : a_1^\ell = (a_2, a_3, \cdots, a_d)$, since $a_1, a_2, \cdots, a_d$ is a regular sequence. hence $\alpha \in \omega + (a_2, a_3, \cdots, a_d) \in I$. The other inclusion is obvious and hence $IS \cap A = I$.
(4) We have the following isomorphisms

$$\frac{S_P}{IS_P} \cong \left(\frac{A[T_2, T_3, \cdots, T_d]}{IA[T_2, T_3, \cdots, T_d]}\right)_{\mathfrak{m}A[T_2, T_3, \cdots, T_d]}$$
$$\cong \frac{A[T_2, T_3, \cdots, T_d]_{\mathfrak{m}A[T_2, T_3, \cdots, T_d]}}{IA[T_2, T_3, \cdots, T_d]_{\mathfrak{m}A[T_2, T_3, \cdots, T_d]}},$$

where $\overline{\mathfrak{m}A[T_2, T_3, \cdots, T_d]} = \frac{\mathfrak{m}A[T_2,T_3,\cdots,T_d]}{IA[T_2,T_3,\cdots,T_d]}$. Hence $\ell_{S_P}(S_P/IS_P) = \ell_A(A/I)$, because $A[T_2, T_3, \cdots, T_d]_{\mathfrak{m}A[T_2,T_3,\cdots,T_d]}$ is faithfully flat over A. Similarly, we have $\ell_{S_P}(S_P/QS_P) = \ell_A(A/Q)$. This completes the proof of Lemma (2.1). □

LEMMA 2.2. ([3], Proposition (2.2)) *Let I be an $\mathfrak{m}$-primary ideal in A and assume that I contains Q as a reduction. Then the following conditions are equivalent.*

(1) $I \in \mathcal{X}_A$.
(2) $I^2 = QI$, $I = Q : I$.
(3) $I^2 = QI$, $\ell_A(A/I) = \frac{1}{2}\ell_A(A/Q)$.
(4) $I^3 \subseteq Q^2$ *and* $I = Q : I$.
(5) *The algebra* $\mathrm{R}'(I) = \oplus_{n\geq 0} I^n t^n$ *is a Gorenstein ring and* $\mathrm{K}_{\mathrm{R}'(I)} \cong \mathrm{R}'(I)(2-d)$ *as graded* $\mathrm{R}'(I)$*-modules, where* $\mathrm{K}_{\mathrm{R}'(I)}$ *denotes the canonical module of* $\mathrm{R}'(I)$.

If $d \geq 1$, we may add the following.

(6) $I^n = Q^n : I$ *for all* $n \in \mathbb{Z}$.

When this is the case, we have $\mathrm{r}(A/I) = \mu_A(I/Q) = \mu_A(I) - d \geq 1$ *and* $\mathrm{e}_I(A) = 2\ell_A(A/I)$, *where* $\mathrm{r}(A/I)$ *denotes the Cohen-Macaulay type of* A/I *and* $\mathrm{e}_I(A)$ *denotes the multiplicity of A with respect to I.*

3. Proof of Theorem 1.1

Proof of Theorem 1.1. (1)⇒(2) Since $QS = a_1S$, we have $I^2S = QIS = a_1IS$. Let $f \in a_1S :_S IS$ with $f \in S$. Then $fx \in a_1S$ with $x \in I$ and write $fx \in a_1(Q^\ell/a_1^\ell)$ for some $\ell \geq 0$, since $S = A[Q/a_1] = \cup_{n\geq 0} Q^n/a_1^n$. Since $f \in S$, we have $x\frac{h}{a_1^u} = a_1\frac{g}{a_1^\ell}$ with $h \in Q^u$ and $g \in Q^\ell$ for some $u \geq 0$. We may assume that $\ell = u$. Hence $xh = a_1g \in Q^{\ell+1}$. Since $x \in I$, we have $h \in Q^{\ell+1} : I = I^{\ell+1} = Q^\ell I$ by Lemma 2.2.(6), whence $f = \frac{h}{a_1^\ell} \in I\frac{Q^\ell}{a_1^\ell} \subseteq IS$. Thus $IS = a_1S :_S IS$.

(2)⇒(3) This is clear.

(3)⇒(2) Suppose that $I^2S \nsubseteq a_1IS$. Then there exists a prime ideal $\mathfrak{p} \in \mathrm{Ass}_S(S/a_1IS)$ such that $I^2S_\mathfrak{p} \nsubseteq a_1IS_\mathfrak{p}$. If $\mathfrak{p} = P$, then $I^2S_P = a_1IS_P$, which is impossible. Hence $\mathfrak{p} \supsetneq P$, whence $\mathrm{ht}_S\mathfrak{p} \geq 2$. We look at the exact sequences

$$(*) \qquad 0 \to (IS)_\mathfrak{p} \xrightarrow{a_1} S_\mathfrak{p} \to (S/a_1IS)_\mathfrak{p} \to 0,$$

$$(**) \qquad 0 \to (IS)_{\mathfrak{p}} \to S_{\mathfrak{p}} \to (S/IS)_{\mathfrak{p}} \to 0$$

of $S_{\mathfrak{p}}$-modules. Apply functors $\mathrm{H}^i_{\mathfrak{m}}(-)$ to (**) and we have $\mathrm{depth}(IS)_{\mathfrak{p}} \geq 2$, because $S_{\mathfrak{p}}$ is a Gorenstein local ring of $\dim S_{\mathfrak{p}} \geq 2$ and $\mathrm{depth}(S/IS)_{\mathfrak{p}} \geq 1$, since $\mathfrak{p} \supsetneq P$ and IS is a P-primary ideal. Now apply functors $\mathrm{H}^I_{\mathfrak{m}}(-)$ to (*) and we have $\mathrm{depth}(S/a_1IS)_{\mathfrak{p}} \geq 1$, when $\mathfrak{p} \notin \mathrm{Ass}_S(S/a_1IS)$. This is impossible, because $\mathfrak{p} \in \mathrm{Ass}_S(S/a_1IS)$ by our assumption. Thus $I^2S = a_1IS$. Suppose that $IS \subsetneq a_1S :_S IS$. Then there exists a prime ideal $\mathfrak{q} \in \mathrm{Ass}_S(S/IS)$ such that $IS_{\mathfrak{q}} \subsetneq a_1S_{\mathfrak{q}} :_{S_{\mathfrak{q}}} IS_{\mathfrak{q}}$. Since $\mathrm{Ass}_S(S/IS) = \{\mathfrak{p}\}$, we have $\mathfrak{q} = \mathfrak{p}$. This is a contradiction to our assumption. Hence $IS = a_1S :_S IS$.

(2)⇒(1) $I^2 \subseteq I^2S \cap A = a_1IS \cap A \subseteq a_1S \cap A = QS \cap A = Q$, by the similar reason of Lemma 2.1.(3). Hence $I \subseteq Q : I$. By Lemma 2.1 (3), we have

$$\begin{aligned} I = IS \cap A &= (a_1S :_S IS) \cap A \\ &= (QS :_S IS) \cap A \\ &\supseteq (Q :_A I)^{ec} \\ &\supseteq Q :_A I. \end{aligned}$$

Hence $I = Q :_A I$. Finally, we want to show that $I^2 = QI$. Let $x \in I^2$ and write $x = \sum_{i=1}^d c_i a_i$ with $c_i \in A$, since $I^2 \subseteq Q$. Since $x \in I^2 \subseteq a_1IS$ and $S = A[Q/a_1] = \cup_{n \geq 0} Q^n/a_1^n$, we have $x \in a_1I(Q^\ell/a_1^\ell)$ for some $\ell \geq 0$, whence we write $x = a_1(y/a_1^\ell)$ where $y \in IQ^\ell$. Then $y/a_1^{\ell-1} = \sum_{i=1}^d c_i a_i$, whence $y = a_1^\ell c_1 + a_1^{\ell-1}a_2c_2 + \cdots + a_1^{\ell-1}a_dc_d$. Let t be an indeterminate over A. Then

$$yt^\ell = c_1(a_1t)^\ell + c_2(a_2t)(a_1t)^{\ell-1} + \cdots + c_d(a_dt)(a_1t)^{\ell-1} \in A[Qt].$$

Since $\mathrm{G}(Q) = A[Qt]/QA[Qt]$ and $\mathrm{G}(Q) \cong (A/Q)[T_1, T_2, \cdots T_d]$, where $\overline{a_it} \longmapsto T_i$ for $i = 1, 2, \cdots, d$, we have

$$\begin{aligned} &\overline{c_1(a_1t)^l + c_2(a_2t)(a_1t)^{l-1} + \cdots + c_d(a_dt)(a_1t)^{l-1}} \\ &= \overline{c_1}T_1^l + \overline{c_2}T_2T_1^{l-1} + \cdots + \overline{c_d}T_dT_1^{l-1}. \end{aligned}$$

Since $y \in IQ^l$, we write

$$y = \sum c_\alpha a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_d^{\alpha_d},$$

where $\{\alpha = (\alpha_1, \cdots, \alpha_d) | \alpha_1 + \cdots + \alpha_d = l \quad \text{and} \quad 0 \leq \alpha_i \in \mathbb{Z}\}$ and $c_\alpha \in I$. Then $yt^l = \sum c_\alpha (a_1 t)^{\alpha_1} (a_2 t)^{\alpha_2} \cdots (a_d t)^{\alpha_d}$, whence $\overline{yt^l} = \sum \overline{c_\alpha} T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_d^{\alpha_d}$ and hence

$$\overline{c_1} T_1^l + \overline{c_2} T_2 T_1^{l-1} + \cdots + \overline{c_d} T_d T_1^{l-1} = \sum \overline{c_\alpha} T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_d^{\alpha_d}.$$

Thus we have $\overline{c_i} = \overline{c_\alpha}$ for some $\alpha = (\alpha_1, \cdots, \alpha_d)$. Since $\overline{c_i} \in A/Q$ and $\overline{c_\alpha} \in I/Q$, we have $c_i - c_\alpha \in Q$, whence $c_i \in c_\alpha + Q \subseteq I$ and hence $x = \sum_{i=1}^d c_i a_i \in QI$. Therefore $I^2 = QI$. This completes the proof of Theorem 1.1. □

Proof of Corollary 1.2. Let I contain Q as a reduction. Hence I contains Q as a reduction if and only if $IS_\mathfrak{p}$ contains $QS_\mathfrak{p}$ as a reduction. Thus

$$I \in \mathcal{X}_A(Q) \Longleftrightarrow IS_\mathfrak{p} \in \mathcal{X}_{S_\mathfrak{p}}(QS_\mathfrak{p})$$

by Theorem 1.1. □

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