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TWIN POSITIVE SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS FOR THE ONE-DIMENSIONAL p-LAPLACIAN

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ABSTRACT. For the boundary value problem (BVP) of second order functional differential equations for the one-dimensional p-Laplacian:

$$(\Phi_p(y'))'(t) + m(t)f(t,y^t) = 0 \quad \text{for } t \in [0,1],$$
$$y(t) = \eta(t) \quad \text{for } t \in [-\sigma, 0],$$
$$y'(t) = \xi(t) \quad \text{for } t \in [1,d],$$

suitable conditions are imposed on $f(t,y^t)$ which yield the existence of at least two positive solutions. Our result generalizes the main result of Avery, Chyan and Henderson.

1. Introduction

In this paper, we are concerned with the existence of twin positive solutions for the boundary value problems (BVP) of a second order functional differential equation for the one-dimensional p-Laplacian

$$(\Phi_p(y'))'(t) + m(t)f(t,y^t) = 0 \qquad \text{for } t \in [0,1], \tag{1.1}$$

$$y(t) = \eta(t) \qquad \text{for } t \in [-\sigma, 0], \tag{1.2}$$

$$y'(t) = \xi(t) \qquad \text{for } t \in [1,d], \tag{1.3}$$

where $\Phi_p(s) = |s|^{p-2}s$, $p > 1$, σ and $\lambda = d-1$ are nonnegative constants, $m(t)$ is a nonnegative continuous function on $(0,1)$, $m(t)$ is allowed to have singularity at $t = 0$ or 1, $f \in C([0,1] \times D, [0,+\infty))$, $D = C([-\sigma,\lambda],\mathbf{R})$ for every $t \in [0,1]$, $y^t \in D$ is defined by $y^t(l) = y(t+l)$, $l \in [-\sigma,\lambda]$, $\eta \in C([-\sigma,0],\mathbf{R})$, $\xi \in C([1,d],\mathbf{R})$, and $\eta(0) = \xi(1) = 0$.

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There is much current attention focused on questions of positive solutions of boundary value problems for ordinary differential equations and functional differential equations, see [2], [3], [6]-[13], to name a few. Much of this interest is due to the applicability of certain Krasnosel'skii fixed-point theorems or the Leggett-Williams multiple fixed-point theorem, or a synthesis of both to obtain positive solutions or multiple positive solutions which lie in a cone. The recent book by Agarwal, Wong and O'Regan [1] gives a good overview for much of the work which has been done and the methods used.

If $\sigma = \lambda = 0$, $p = 2$, $m(t) \equiv 1$ on $[0,1]$ and $f(t, y^t) \equiv f(y)$ for $t \in [0,1]$, then BVP (1.1)-(1.3) be reduced to the following second-order boundary value problem :

$$y'' + f(y) = 0, \quad 0 \le t \le 1, \tag{1.4}$$

$$y(0) = 0 = y'(1), \tag{1.5}$$

where $f : \mathbf{R} \to [0, \infty)$ is continuous. In [5], Avery and Henderson used fixed-point index theory in establishing a new twin fixed-point theorem for a completely continuous operator on a cone in a Banach space. Very recently, Avery, Chyan and Henderson [4] imposed conditions on $f(y)$ to yield at least two positive solutions to (1.4), (1.5) by applying the twin fixed-point theorem in [5].

In this paper, we imposed growth conditions on $f(t, y^t)$ which allow us to apply the twin fixed-point theorem in obtaining at least two positive solutions for BVP (1.1)-(1.3). Here, we point out that the main difficulty that appears when passing from $p = 2$ to $p \neq 2$ is that, for the first case, there is a well-known Green's function for the right focal boundary value problem, which is used to prove the Theorem 3.1 in [4]. However, for $p \neq 2$, there cannot be a Green's function because the differential operator $(\Phi_p(y'))'$ is nonlinear.

We say a function $y(t)$ is a positive solution of BVP (1.1)-(1.3), if $y(t) > 0$ on $(0,1)$, $y(t) \ge 0$ on $[-\sigma, 0] \cup [1, d]$, $y(t)$ is continuous on $[-\sigma, \lambda]$, $y(t) = \eta(t)$ for $t \in [-\sigma, 0]$, $y(t) = y(1) + \int_1^t \xi(s)ds$ for $t \in [1, d]$, $\Phi_p(y')(t)$ is locally absolutely continuous in $(0,1)$, and the equality $(\Phi_p(y'))'(t) = -m(t) f(t, y^t)$ holds almost everywhere in $(0,1)$.

2. Twin positive solutions

Let E be a real Banach space, $P \subset E$ is a cone.

DEFINITION 2.1. A functional $\varphi : P \to \mathbf{R}$ is said to be increasing on P, provided

$$\varphi(x) \leq \varphi(y) \quad \text{for all } x, y \in P \text{ with } x \leq y.$$

DEFINITION 2.2. Given a *nonnegative continuous functional* γ *on a* cone P, we define for each $d > 0$, the set

$$P(\gamma, d) = \{x \in P : \gamma(x) < d\}.$$

In obtaining Twin positive solutions of BVP (1.1)-(1.3), the following fixed point theorem of Avery and Henderson will be fundamental.

THEOREM 2.1 [4, 5]. *Let P be a cone in a Banach space E. Let α and γ be increasing, nonnegative, continuous functionals on P, and let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that for some $c > 0$ and $M > 0$,*

$$\gamma(x) \leq \theta(x) \leq \alpha(x) \quad \text{and} \quad \|x\| \leq M\gamma(x)$$

for all $x \in \overline{P(\gamma, c)}$. Suppose there exist a completely continuous operator $A : \overline{P(\gamma, c)} \to P$ and $0 < a < b < c$ such that

$$\theta(\lambda x) \leq \lambda\theta(x) \quad \text{for } 0 \leq \lambda \leq 1 \quad \text{and} \quad x \in \partial P(\theta, b),$$

and

(i) *$\gamma(Ax) > c$ for all $x \in \partial P(\gamma, c)$;*

(ii) *$\theta(Ax) < b$ for all $x \in \partial P(\theta, b)$;*

(iii) *$P(\alpha, a) \neq \emptyset$, and $\alpha(Ax) > a$ for all $x \in \partial P(\alpha, a)$.*

Then A has at least two fixed points x_1 and x_2 belonging to $\overline{P(\gamma, c)}$ such that

$$a < \alpha(x_1), \quad \text{with } \theta(x_1) < b,$$

and

$$b < \theta(x_2), \quad \text{with } \gamma(x_2) < c.$$

Let $D = C([-\sigma, \lambda], \mathbf{R})$ $(\lambda = d - 1)$ be a space with a norm $\|\psi\|_{[-\sigma,\lambda]} = \max_{-\sigma \leq x \leq \lambda} |\psi(x)|$ for $\psi \in D$. Let

$$D^+ = \{\psi \in D : \psi(x) \geq 0,\ x \in [-\sigma, \lambda]\}.$$

Next, let $E = C([-\sigma, d], \mathbf{R})$ be endowed with the maximum norm, $\|y\|_{[-\sigma,d]} = \max_{-\sigma\le t\le d} |y(t)|$ for $y \in E$. Let $P \subset E$ be a cone defined by

$$P = \left\{ \begin{array}{l} y \in E : y \text{ is nonnegative valued on } [-\tau, d],\ y \text{ is nondecreasing} \\ \text{on } [0,1], \text{ and } y(t) \ge t\|y\|_{[-\sigma,d]} \text{ for } t \in [0,1] \end{array} \right\}.$$

For the remainder of this section, fix $0 < h \le k < r < 1$, such that

$$\int_0^r \Phi_q\left(\int_s^r m(\tau)d\tau\right) ds \le \int_0^k \Phi_q\left(\int_s^1 m(\tau)d\tau\right) ds, \tag{2.1}$$

where Φ_q is the inverse function to Φ_p $(p > 1)$, that is $\Phi_q(s) = |s|^{q-2}s$ $(q = p/(p-1) > 1)$. We now define the nonnegative, increasing, continuous functionals, γ, θ, and α, by

$$\gamma(y) = \min_{h\le t\le r} y(t),$$

$$\theta(y) = \max_{0\le t\le k} y(t), \quad \text{and}$$

$$\alpha(y) = \max_{0\le t\le r} y(t).$$

We observe that, for each $y \in P$,

$$\gamma(y) \le \theta(y) \le \alpha(y).$$

In addition, for each $y \in P$, $\gamma(y) = y(h) \ge h\|y\|_{[-\sigma,d]}$. Thus,

$$\|y\|_{[-\sigma,d]} \le \frac{1}{h}\gamma(y) \qquad \text{for all } y \in P.$$

Finally, we also note that

$$\theta(\lambda y) = \lambda\theta(y), \cdot\ 0 \le \lambda \le 1, \quad \text{and} \quad y \in \partial P(\theta, b).$$

Suppose that $y(t)$ is a solution of BVP(1.1)-(1.3), then it can be written as

$$y(t) = \begin{cases} \eta(t), & -\sigma \le t \le 0; \\ \int_0^t \Phi_q\left(\int_s^1 m(\tau)f(\tau, y^\tau)d\tau\right) ds, & 0 \le t \le 1; \\ y(1) + \int_1^t \xi(s)ds, & 1 \le t \le d. \end{cases}$$

Throughout this paper we assume that $x_0(t)$ is the solution of (1.1)-(1.3) with $f \equiv 0$. Clearly, $x_0(t)$ can be expressed as follows:

$$x_0(t) = \begin{cases} \eta(t), & -\sigma \le t \le 1; \\ 0, & 0 \le t \le 1; \\ \int_1^t \xi(s)ds, & 1 \le t \le d. \end{cases}$$

Let $y(t)$ be a solution of BVP(1.1)-(1.3) and $x(t) = y(t) - x_0(t)$. Noting that $x(t) \equiv y(t)$ for $0 \le t \le 1$, we have

$$x(t) = \begin{cases} 0, & -\sigma \le t \le 0; \\ \int_0^t \Phi_q\left(\int_s^1 m(\tau) f(\tau,\ x^\tau + x_0^\tau)d\tau\right) ds, & 0 \le t \le 1; \\ \int_0^1 \Phi_q\left(\int_s^1 m(\tau) f(\tau,\ x^\tau + x_0^\tau)d\tau\right) ds, & 1 \le t \le d. \end{cases}$$

For convenience, setting

$$N_1 = \Phi_p\left(\frac{1}{h\Phi_q(\int_h^1 m(\tau)d\tau)}\right), \qquad N_2 = \Phi_p\left(\frac{1}{\int_0^k \Phi_q(\int_s^1 m(\tau)d\tau)ds}\right),$$

and

$$N_3 = \Phi_p\left(\frac{1}{\int_0^r \Phi_q(\int_s^r m(\tau)d\tau)ds}\right).$$

We now present our main result of this paper.

THEOREM 2.2. *Suppose that there exist positive numbers a, b, and c such that*

$$(2.2) \qquad 0 < a < \frac{\int_0^r \Phi_q(\int_s^r m(\tau)d\tau)ds}{\int_0^k \Phi_q(\int_s^1 m(\tau)d\tau)ds} b < \frac{h\int_0^r \Phi_q(\int_s^r m(\tau)d\tau)ds}{\int_0^k \Phi_q(\int_s^1 m(\tau)d\tau)ds} c.$$

Let the following conditions are satisfied:

(H_1) $f \in C([0,1] \times D^+, [0,\infty))$, $\quad \eta \in C([-\sigma,0],[0,\infty))$, $\xi \in C([1,d],[0,\infty))$,

(H_2) $m \in C((0,1),[0,\infty))$, $\quad 0 < \int_0^1 m(t)dt < \infty$, $0 < \int_0^1 \Phi_q(\int_s^1 m(\tau)d\tau)ds < \infty$,

(H_3) $f(t,\psi) > N_1\Phi_p(c)$, $\quad$ *if* $(t,\psi) \in [h,1] \times D^+_{[c,\ c/h+W_0]}$,

(H_4) $f(t,\psi) < N_2\Phi_p(b)$, $\quad$ *if* $(t,\psi) \in [0,1] \times D^+_{[0,\ b/k+W_0]}$,

(H_5) $f(t,\psi) > N_3\Phi_p(a)$, $\quad$ *if* $(t,\psi) \in [0,r] \times D^+_{[0,\ a/r+W_0]}$,

where

$$D^+_{[u,\ v]} = \{\psi \in D^+ : u \le \|\psi\|_{[-\sigma,\lambda]} \le v\},$$

$W_0 = \|x_0\|_{[-\sigma,\ d]}$. *Then, the boundary value problem* (1.1)-(1.3) *has at least two positive solutions* y_1 *and* y_2, *such that*

$$a < \max_{0<t<r} y_1(t), \qquad \text{with} \quad \max_{0<t<k} y_1(t) < b,$$

and

$$b < \max_{0\le t\le k} y_2(t), \qquad \textit{with} \quad \min_{h\le t\le r} y_2(t) < c.$$

Proof. Define a operator $A : P \to E$ as follows:

$$Ax(t) := \begin{cases} 0, & -\sigma \le t \le 0; \\ \displaystyle\int_0^t \Phi_q\left(\int_s^1 m(\tau)f(\tau,\ x^\tau + x_0^\tau)d\tau\right) ds, & 0 \le t \le 1; \\ \displaystyle\int_0^1 \Phi_q\left(\int_s^1 m(\tau)f(\tau,\ x^\tau + x_0^\tau)d\tau\right) ds, & 1 \le t \le d. \end{cases}$$

For $x \in \overline{P(\gamma,c)}$, we have from ($H_1$), ($H_2$) that $Ax(t) \ge 0$ on $[-\sigma, d]$, and $(Ax)'(t) = \Phi_q(\int_t^1 m(\tau)f(\tau, x^\tau + x_0^\tau)d\tau) \ge 0$ for $t \in [0,1]$, that is, Ax is nonnegative and nondecreasing, for $t \in [-\sigma, d]$ and $t \in [0,1]$, respectively. Let $B(t) = (Ax)(t) - t\|Ax\|_{[-\sigma,d]}$, then $B(0) = (Ax)(0) = 0$, $B(1) = (Ax)(1) - \|Ax\|_{[-\sigma,d]} = 0$, and

$$\begin{aligned} B'(t) = \Phi_q&\left(\int_t^1 m(\tau)f(\tau,\ x^\tau + x_0^\tau)d\tau\right) \\ &- \int_0^1 \Phi_q\left(\int_s^1 m(\tau)f(\tau,\ x^\tau + x_0^\tau)d\tau\right) ds. \end{aligned}$$

By the first mean value theorem, there exists at least $\mu \in [0,1]$ such that

$$\int_0^1 \Phi_q\left(\int_s^1 m(\tau)f(\tau,\ x^\tau + x_0^\tau)u\right) ds = \Phi_q\left(\int_\mu^1 m(\tau)f(\tau,\ x^\tau + x_0^\tau)d\tau\right).$$

Thus, $B'(\mu) = 0$, and $B'(t) \ge 0$ for $0 \le t \le \mu$, $B'(t) \le 0$ for $\mu < t \le 1$. Hence $B(t) \ge 0$ for $t \in [0,1]$, that is, $(Ax)(t) \ge t\|Ax\|$. Consequently, $A : \overline{P(\gamma,c)} \to P$. From [9], we easy to check that A is completely continuous. It is well known that if $x + x_0$ is a solution of (1.1)-(1.3), then x is a fixed point of A and conversely.

By (2.1) and (2.2), we know that $0 < a < b < c$. We now show that the conditions of Theorem 2.1 are satisfied. First, for $u \in E$, $\forall s \in [0,1]$, we have $u^s \in D$, and

$$\|u^s\|_{[-\sigma,\lambda]} = \max_{l\in[-\sigma,\lambda]} |u(s+l)| \le \max_{t\in[-\sigma,d]} |u(t)| = \|u\|_{[-\sigma,d]}. \tag{2.3}$$

By (2.3), $\forall x \in \partial P(\gamma, c)$, we have

$$\begin{aligned}
\|x^s + x_0^s\|_{[-\sigma,\lambda]} &= \|(x+x_0)^s\|_{[-\sigma,\lambda]} \\
&\leq \|x + x_0\|_{[-\sigma,d]} \leq \|x\|_{[-\sigma,d]} + \|x_0\|_{[-\sigma,d]} \\
&\leq \frac{1}{h}\gamma(x) + W_0 \\
&= c/h + W_0, \qquad \forall s \in [0,1],
\end{aligned}$$

and for each $s \in [h, r]$,

$$\begin{aligned}
\|x^s + x_0^s\|_{[-\sigma,\lambda]} &= \max_{l\in[-\sigma,\lambda]} (x(s+l) + x_0(s+l)) \\
&\geq \max_{l\in[-\sigma,\lambda]} x(s+l) \quad (\text{Since } x_0(t) \geq 0 \text{ for } t \in [-\sigma, d]) \\
&\geq x(s) \geq \min_{t\in[h,r]} x(t) \\
&= \gamma(x) = c.
\end{aligned}$$

By assumption (H_3), we obtain that

$$\begin{aligned}
\gamma(Ax) &= Ax(h) \\
&= \int_0^h \Phi_q \left(\int_s^1 m(\tau) f(\tau, x^\tau + x_0^\tau) d\tau \right) ds \\
&\geq \int_0^h \Phi_q \left(\int_h^1 m(\tau) f(\tau, x^\tau + x_0^\tau) d\tau \right) ds \\
&> h\Phi_q \left(N_1 \Phi_p(c) \int_h^1 m(\tau) d\tau \right) \\
&= hc\Phi_q(N_1)\Phi_q \left(\int_h^1 m(\tau) d\tau \right) \\
&= c.
\end{aligned}$$

We conclude that (i) of Theorem 2.1 is satisfied.

We next address (ii) of Theorem 2.1. Let us choose $x \in \partial P(\theta, b)$, that is $b = \theta(x) = x(k) \geq k\|x\|_{[-\sigma,d]}$. So, we get

$$\begin{aligned}
\|x^s + x_0^s\|_{[-\sigma,\lambda]} &\leq \|x + x_0\|_{[-\sigma,d]} \leq \|x\|_{[-\sigma,d]} + \|x_0\|_{[-\sigma,d]} \\
&\leq b/k + W_0, \qquad \forall s \in [0,1].
\end{aligned}$$

Owing to (H_4), we have

$$\begin{aligned}\theta(Ax) &= (Ax)(k)\\ &= \int_0^k \Phi_q\left(\int_s^1 m(\tau)f(\tau,x^\tau+x_0^\tau)d\tau\right)ds\\ &< \int_0^k \Phi_q\left(N_2\Phi_p(b)\int_s^1 m(\tau)d\tau\right)ds\\ &= b\Phi_q(N_2)\int_0^k \Phi_q\left(\int_s^1 m(\tau)d\tau\right)ds\\ &= b.\end{aligned}$$

Finally, we turn to (iii) of Theorem 2.1. For this part, if we first define $y(t)=a/2$ for $-\sigma\le t\le d$, then $\alpha(y)=a/2$, and $P(\alpha,a)\neq\emptyset$.

Now, let us choose $x\in\partial P(\alpha,a)$. Similarly, by $a=\alpha(x)=x(r)\ge r\|x\|_{[-\sigma,d]}$, we have

$$\|x^s+x_0^s\|_{[-\sigma,\lambda]}\le a/r+W_0 \quad \text{for } s\in[0,1].$$

From assumption (H_5), we get

$$\begin{aligned}\alpha(Ax) &= (Ax)(r)\\ &= \int_0^r \Phi_q\left(\int_s^1 m(\tau)f(\tau,\ x^\tau+x_0^\tau)d\tau\right)ds\\ &\ge \int_0^r \Phi_q\left(\int_s^r m(\tau)f(\tau,\ x^\tau+x_0^\tau)d\tau\right)ds\\ &> \int_0^r \Phi_q\left(N_3\Phi_p(a)\int_s^r m(\tau)d\tau\right)ds\\ &= a\Phi_q(N_3)\int_0^r \Phi_q\left(\int_s^r m(\tau)d\tau\right)ds\\ &= a.\end{aligned}$$

Thus, (iii) of Theorem 2.1 is satisfied. Hence, A has at least two fixed points x_1 and x_2, belonging to $\overline{P(\gamma,c)}$. Thus, there exist at least two positive solutions $y_1=x_1+x_0$ and $y_2=x_2+x_0$ for the boundary value problem (1.1)-(1.3) such that

$$a<\alpha(x_1)=\max_{t\in[0,r]}x_1(t)=\max_{t\in[0,r]}y_1(t),$$

with

$$\max_{t\in[0,k]}y_1(t)=\max_{t\in[0,k]}x_1(t)=\theta(x_1)<b,$$

and

$$b < \theta(x_2) = \max_{t\in[0,k]} x_2(t) = \max_{t\in[0,k]} y_2(t),$$

with

$$\min_{t\in[h,r]} y_2(t) = \min_{t\in[h,r]} x_2(t) = \gamma(x_2) < c.$$

The proof is complete. □

REMARK 2.3. Let $G(t,s)$ be the Green's function of the differential equation

$$-u''(t) = 0, \qquad t \in (0,1)$$

subject to the following boundary condition:

$$\alpha u(t) - \beta u'(t) = 0, \qquad \gamma u(t) + \delta u'(t) = 0.$$

The following assumption (C_3) is necessary in the proof of main result (Theorem 2.1) of [9].

(C_3) $0 < \int_{E_M} G(1/2,s)ds < \infty$, where $E_M = \{s \in E : M \le s + \theta \le 1 - M\}$ for some small enough constant $M \in (0,1/2)$ and $\theta \in [-\tau, a]$.

From the proof process of Theorem 2.2, it is easy to know that the above assumption (C_3) in [9] can be dropped.

Noting that for $x \in \partial P(\alpha, a)$ and $s \in [0,r]$, $\|x^s + x_0^s\|_{[-\sigma,\lambda]} = \max_{l\in[-\sigma,\lambda]} (x(s+l) + x_0(s+l)) = x(s) \le x(r) = a$ if $\sigma = \lambda = 0$. Hence, setting $\sigma = \lambda = 0$, then theorem 2.2 reduces to the following:

COROLLARY 2.4. *Suppose that there exist positive numbers a, b, and c such that (2.2) holds. Let (H_2) holds. Suppose that f satisfies the following conditions:*

(H_1') $f \in C([0,1] \times [0,\infty), [0,\infty))$,

(H_3') $f(t,w) > N_1\Phi_p(c)$, *if* $(t,w) \in [h,1] \times [c, c/h]$,

(H_4') $f(t,w) < N_2\Phi_p(b)$, *if* $(t,w) \in [0,1] \times [0, b/k]$;

(H_5') $f(t,w) > N_3\Phi_p(a)$, *if* $(t,w) \in [0,r] \times [0,a]$.

Then, the right focal boundary value problem for the one-dimensional p-Laplacian

$$((\Phi_p(y'))' + m(t)f(t,y) = 0, \qquad t \in [0,1],$$

$$y(0) = y'(1) = 0.$$

has at least two positive solutions y_1 and y_2, such that

$$a < \max_{0\le t\le r} y_1(t), \qquad \textit{with} \quad \max_{0\le t\le k} y_1(t) < b,$$

and

$$b < \max_{0\le t\le k} y_2(t), \qquad \textit{with} \quad \min_{h\le t\le r} y_2(t) < c.$$

REMARK 2.5. Setting $p = 2$, $h = k = 1/2$, and $m(t) \equiv 1$ for $t \in [0,1]$, then

$$N_1 = \Phi_p\left(\frac{1}{h\Phi_q(\int_h^1 m(\tau)d\tau)}\right) = 4,$$

$$N_2 = \Phi_p\left(\frac{1}{\int_0^k \Phi_q(\int_s^1 m(\tau)d\tau)ds}\right) = \frac{1}{\int_0^k (1-s)ds} = \frac{8}{3},$$

and

$$N_3 = \Phi_p\left(\frac{1}{\int_0^r \Phi_q(\int_s^r m(\tau)d\tau)ds}\right) = \frac{1}{\int_0^r (r-s)ds} = \frac{2}{r^2}.$$

Thus, Theorem 3.1 of [4] is a special case of Corollary 2.4 here.

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