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UNITARY INTERPOLATION PROBLEMS IN CSL-ALGEBRA ALG$\mathcal{L}$

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ABSTRACT. Given vectors x and y in a Hilbert space, an interpolating operator is a bounded operator T such that $Tx = y$. An interpolating operator for n-vectors satisfies the equation $Ax_i = y_i$ for $i = 1, 2, \cdots, n$. In this article, we investigate unitary interpolation problems in CSL-Algebra Alg$\mathcal{L}$: Let $\mathcal{L}$ be a commutative subspace lattice on a Hilbert space $\mathcal{H}$. Let x and y be vectors in $\mathcal{H}$. When does there exist a unitary operator A in Alg$\mathcal{L}$ such that $Ax = y$?

1. Introduction

Let $\mathcal{C}$ be a collection of operators acting on a Hilbert space $\mathcal{H}$ and let x and y be vectors on $\mathcal{H}$. *An interpolation question* for $\mathcal{C}$ asks for which x and y is there a bounded operator $T \in \mathcal{C}$ such that $Tx = y$. A variation, the 'n-vector interpolation problem', asks for an operator T such that $Tx_i = y_i$ for fixed finite collections $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$. The n-vector interpolation problem was considered for a C^*-algebra $\mathcal{U}$ by Kadison [5]. In case $\mathcal{U}$ is a nest algebra, the (one-vector) interpolation problem was solved by Lance [7]: his result was extended by Hopenwasser [1] to the case that $\mathcal{U}$ is a CSL-algebra. Munch [8] obtained conditions for interpolation in case T is required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra. Hopenwasser [2] once again extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra.

We establish some notations and conventions. A subspace lattice $\mathcal{L}$ is a strongly closed lattice of projections acting on a Hilbert space $\mathcal{H}$. A subspace lattice $\mathcal{L}$ is a commutative subspace lattice, or CSL if all

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projections in $\mathcal{L}$ are commutative. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If $\mathcal{L}$ is CSL, then Alg$\mathcal{L}$ is the algebra of all bounded linear operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$ and called a CSL-algebra. Let x and y be vectors in a Hilbert space. Then $\langle x, y\rangle$ means the inner product of vectors x and y. In this paper, we use the convention $\frac{0}{0} = 0$, when necessary.

We investigate unitary interpolation problems in CSL-Algebra Alg$\mathcal{L}$: Given two vectors x and y in a Hilbert space $\mathcal{H}$, when does there exist a unitary operator in CSL-Algebra Alg$\mathcal{L}$ such that $Ax = y$?

2. Results

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a commutative subspace lattice of orthogonal projections acting on $\mathcal{H}$ containing 0 and I. Let M be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of M. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers. An operator U is *unitary* if $UU^* = U^*U = I$, where I is the identity operator acting on $\mathcal{H}$.

THEOREM 1. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$. Let x and y be vectors in $\mathcal{H}$. If there is an operator A in Alg$\mathcal{L}$ such that $Ax = y$, A is unitary and every E in $\mathcal{L}$ reduces A, then*

$$\sup\left\{\frac{\|\sum_{i=1}^{n}\alpha_i E_i y\|}{\|\sum_{i=1}^{n}\alpha_i E_i x\|} : n \in \mathbb{N}, \alpha_i \in \mathbb{C} \text{ and } E_i \in \mathcal{L}\right\} < \infty,$$

$$\sup\left\{\frac{\|\sum_{i=1}^{n}\alpha_i E_i x\|}{\|\sum_{i=1}^{n}\alpha_i E_i y\|} : n \in \mathbb{N}, \alpha_i \in \mathbb{C} \text{ and } E_i \in \mathcal{L}\right\} < \infty$$

and

$$\langle Ey, y\rangle = \langle Ex, x\rangle$$

for all E in $\mathcal{L}$.

Proof. By Theorem 1 [4],

$$\sup\left\{\frac{\|\sum_{i=1}^{n}\alpha_i E_i y\|}{\|\sum_{i=1}^{n}\alpha_i E_i x\|} : n \in \mathbb{N}, \alpha_i \in \mathbb{C} \text{ and } E_i \in \mathcal{L}\right\} < \infty$$

and

$$\sup\left\{\frac{\|\sum_{i=1}^{n}\alpha_i E_i x\|}{\|\sum_{i=1}^{n}\alpha_i E_i y\|} : n\in\mathbb{N}, \alpha_i\in\mathbb{C} \text{ and } E_i\in\mathcal{L}\right\}<\infty.$$

Since $Ax = y$ and A is unitary, $x = A^*y$. So $\langle Ey, y\rangle = \langle EAx, y\rangle = \langle Ex, A^*y\rangle = \langle Ex, x\rangle$ for all E in $\mathcal{L}$. □

THEOREM 2. *Let $\mathcal{L}$ be a commutative subspace lattice on a Hilbert space $\mathcal{H}$. Let x and y be vectors in $\mathcal{H}$. Assume that*

$$\mathcal{M}_0 = \left\{\sum_{i=1}^{n}\alpha_i E_i x : n\in\mathbb{N}, \alpha_i\in\mathbb{C} \text{ and } E_i\in\mathcal{L}\right\}$$

and

$$\mathcal{M}_1 = \left\{\sum_{i=1}^{n}\alpha_i E_i y : n\in\mathbb{N}, \alpha_i\in\mathbb{C} \text{ and } E_i\in\mathcal{L}\right\}$$

are dense in $\mathcal{H}$. If

$$\sup\left\{\frac{\|\sum_{i=1}^{n}\alpha_i E_i y\|}{\|\sum_{i=1}^{n}\alpha_i E_i x\|} : n\in\mathbb{N}, \alpha_i\in\mathbb{C} \text{ and } E_i\in\mathcal{L}\right\}<\infty,$$

$$\sup\left\{\frac{\|\sum_{i=1}^{n}\alpha_i E_i x\|}{\|\sum_{i=1}^{n}\alpha_i E_i y\|} : n\in\mathbb{N}, \alpha_i\in\mathbb{C} \text{ and } E_i\in\mathcal{L}\right\}<\infty$$

and

$$\langle Ey, y\rangle = \langle Ex, x\rangle$$

for all E in $\mathcal{L}$, then there is an operator A in Alg$\mathcal{L}$ such that $Ax = y$, A is unitary and every E in $\mathcal{L}$ reduces A.

Proof. By Theorem 2 [4], there are operators A and B in Alg$\mathcal{L}$ such that $Ax = y$, $x = By$, A and B are invertible and every E in $\mathcal{L}$ reduces A and B. We want to show that $A^* = B$. Since $\langle Ey, y\rangle = \langle Ex, x\rangle$ for all E in $\mathcal{L}$,

$$\begin{aligned}\left\langle A(\sum_{i=1}^{n}\alpha_i E_i x), y\right\rangle &= \left\langle \sum_{i=1}^{n}\alpha_i E_i y, y\right\rangle \\ &= \left\langle \sum_{i=1}^{n}\alpha_i E_i x, x\right\rangle,\end{aligned}$$

$n \in \mathbb{N}, \alpha_i \in \mathbb{C}$ and $E_i \in \mathcal{L}$. So $\langle Af, y\rangle = \langle f, x\rangle$ for all f in $\mathcal{M}_0$. Since $\mathcal{M}_0$ is dense in $\mathcal{H}$, $A^*y = x$. So

$$\begin{aligned} A^*(\sum_{i=1}^{n} \alpha_i E_i y) &= \sum_{i=1}^{n} \alpha_i E_i A^* y \\ &= \sum_{i=1}^{n} \alpha_i E_i x, \end{aligned}$$

$n \in \mathbb{N}, \alpha_i \in \mathbb{C}$ and $E_i \in \mathcal{L}$. Since $\mathcal{M}_1$ is dense in $\mathcal{H}$, $A^* = B$. Hence A is unitary. □

THEOREM 3. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$. and let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ be vectors in $\mathcal{H}$. If there is an operator A in Alg$\mathcal{L}$ such that $y_j = Ax_j (j = 1, 2, \cdots, n)$, every E in $\mathcal{L}$ reduces A and A is unitary, then*

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|} : m_i \in \mathbb{N}, l \leq n, \alpha_{k,i} \in \mathbb{C} \text{ and } E_{k,i} \in \mathcal{L}\right\} < \infty,$$

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|} : m_i \in \mathbb{N}, l \leq n, \alpha_{k,i} \in \mathbb{C} \text{ and } E_{k,i} \in \mathcal{L}\right\} < \infty$$

and $\langle Ey_p, y_q\rangle = \langle Ex_p, x_q\rangle$ for all E in $\mathcal{L}$ and $p, q = 1, 2, \cdots, n$.

Proof. By Theorem 3 [4],

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|} : m_i \in \mathbb{N}, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C}\right\} < \infty$$

and

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|} : m_i \in \mathbb{N}, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C}\right\} < \infty.$$

Since $Ax_p = y_p$, A is unitary and every E in $\mathcal{L}$ reduces A,

$$\begin{aligned} \langle Ey_p, y_q\rangle &= \langle EAx_p, y_q\rangle \\ &= \langle AEx_p, y_q\rangle \\ &= \langle Ex_p, A^*y_q\rangle \\ &= \langle Ex_p, x_q\rangle \end{aligned}$$

for all E in $\mathcal{L}$ and all $p, q = 1, 2, \cdots, n$. □

THEOREM 4. *Let $\mathcal{L}$ be a commutative subspace lattice on a Hilbert space $\mathcal{H}$. and let $x_1, \cdots, x_n, y_1, \cdots, y_n$ be vectors in $\mathcal{H}$. Assume that*

$$\mathcal{U}_0 = \left\{ \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i : m_i \in \mathbb{N}, l \leq n, \alpha_{k,i} \in \mathbb{C} \text{ and } E_{k,i} \in \mathcal{L} \right\}$$

and

$$\mathcal{U}_1 = \left\{ \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} y_i : m_i \in \mathbb{N}, l \leq n, \alpha_{k,i} \in \mathbb{C} \text{ and } E_{k,i} \in \mathcal{L} \right\}$$

are dense in $\mathcal{H}$. If

$$\sup \left\{ \frac{\| \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} y_i \|}{\| \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i \|} : m_i \in \mathbb{N}, l \leq n, \alpha_{k,i} \in \mathbb{C} \text{ and } E_{k,i} \in \mathcal{L} \right\} < \infty,$$

$$\sup \left\{ \frac{\| \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i \|}{\| \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} y_i \|} : m_i \in \mathbb{N}, l \leq n, \alpha_{k,i} \in \mathbb{C} \text{ and } E_{k,i} \in \mathcal{L} \right\} < \infty$$

and $\langle Ey_p, y_q \rangle = \langle Ex_p, x_q \rangle$ for all E in $\mathcal{L}$ and all $p, q = 1, 2, \cdots, n$, then there is an operator A in Alg$\mathcal{L}$ such that $Ax_j = y_j$ for all $j = 1, \cdots, n$, A is unitary and every E in $\mathcal{L}$ reduces A.

Proof. By Theorem 4 [4], there are operators A, B in Alg$\mathcal{L}$ such that $Ax_q = y_q$, $y_q = Bx_q$ $(q = 1, 2, \cdots, n)$, A and B are invertible and every E in $\mathcal{L}$ reduces A and B. We know that $A^{-1} = B$. We want to prove that $A^* = B$. Since $\langle Ey_p, y_j \rangle = \langle Ex_p, x_j \rangle$ for E in $\mathcal{L}$ and $p, j = 1, 2, \cdots, n$,

$$\begin{aligned} \left\langle A(\sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i), y_j \right\rangle &= \left\langle \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} A x_i, y_j \right\rangle \\ &= \left\langle \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} y_i, y_j \right\rangle \\ &= \left\langle \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i, x_j \right\rangle . \end{aligned}$$

Since $\mathcal{U}_0$ is dense in $\mathcal{H}$, $\langle Af, y_j \rangle = \langle f, x_j \rangle$ for all f in $\mathcal{H}$ and all $j = 1, 2, \cdots, n$. So $A^* y_j = x_j$ for all $j = 1, 2, \cdots, n$. Since

$$A^*(\sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} y_i) = \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i$$

and $\mathcal{U}_1$ is dense in $\mathcal{H}$, $A^* = B$. Hence A is unitary. □

If we modify the proofs of Theorems 3 and 4 a little bit, then we can get the following theorems. So we omit their proofs.

THEOREM 5. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$. and let $\{x_n\}$ and $\{y_n\}$ be two infinite sequences of vectors in $\mathcal{H}$. If there is an operator A in Alg$\mathcal{L}$ such that $Ax_n = y_n$ $(n = 1, 2, \cdots)$, A is unitary and every E in $\mathcal{L}$ reduces A, then*

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|} : m_i, l \in \mathbb{N}, \alpha_{k,i} \in \mathbb{C} \text{ and } E_{k,i} \in \mathcal{L}\right\} < \infty,$$

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|} : m_i, l \in \mathbb{N}, \alpha_{k,i} \in \mathbb{C} \text{ and } E_{k,i} \in \mathcal{L}\right\} < \infty$$

and $\langle Ey_p, y_j\rangle = \langle Ex_p, x_j\rangle$ for all E in $\mathcal{L}$ and all $p, j = 1, 2, \cdots$.

THEOREM 6. *Let $\mathcal{L}$ be a commutative subspace lattice on a Hilbert space $\mathcal{H}$. Assume that*

$$\mathcal{K}_0 = \left\{\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i : m_i, l \in \mathbb{N}, \alpha_{k,i} \in \mathbb{C} \text{ and } E_{k,i} \in \mathcal{L}\right\}$$

and

$$\mathcal{K}_1 = \left\{\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i : m_i, l \in \mathbb{N}, \alpha_{k,i} \in \mathbb{C} \text{ and } E_{k,i} \in \mathcal{L}\right\}$$

are dense in $\mathcal{H}$. If

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|} : m_i, l \in \mathbb{N}, \alpha_{k,i} \in \mathbb{C} \text{ and } E_{k,i} \in \mathcal{L}\right\} < \infty,$$

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|} : m_i, l \in \mathbb{N}, \alpha_{k,i} \in \mathbb{C} \text{ and } E_{k,i} \in \mathcal{L}\right\} < \infty$$

and $\langle Ey_p, y_j\rangle = \langle Ex_p, x_j\rangle$ for all E in $\mathcal{L}$ and all $p, j = 1, 2, \cdots$, then there is an operator A in Alg$\mathcal{L}$ such that $Ax_n = y_n$ $(n = 1, 2, \cdots)$, A is unitary and every E in $\mathcal{L}$ reduces A.

In the following theorem, we consider a "corona-type" version of unitary interpolation problem in CSL-Algebra Alg$\mathcal{L}$.

THEOREM 7. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $x_1, \cdots, x_n$ and y be vectors in $\mathcal{H}$. Assume that*

$$\left\{\sum_{i=1}^{l}\alpha_iE_iy : l \in \mathbb{N}, \alpha_i \in \mathbb{C} \text{ and } E_i \in \mathcal{L}\right\}$$

and

$$\left\{\sum_{i=1}^{l} \alpha_i E_i x_k : l \in \mathbb{N}, \alpha_i \in \mathbb{C} \text{ and } E_i \in \mathcal{L}\right\}$$

are dense in $\mathcal{H}$ *for all* $k = 1, 2, \cdots, n$. *If*

$$\sup\left\{\frac{\|\sum_{i=1}^{l} \alpha_i E_i y\|}{\|\sum_{i=1}^{l} \alpha_i E_i x_k\|} : l \in \mathbb{N}, \alpha_i \in \mathbb{C} \text{ and } E_i \in \mathcal{L}\right\} < \infty,$$

$$\sup\left\{\frac{\|\sum_{i=1}^{l} \alpha_i E_i x_k\|}{\|\sum_{i=1}^{l} \alpha_i E_i y\|} : l \in \mathbb{N}, \alpha_i \in \mathbb{C} \text{ and } E_i \in \mathcal{L}\right\} < \infty$$

and $\frac{1}{n^2}\langle Ey, y\rangle = \langle Ex_k, x_k\rangle$ *for all* E *in* $\mathcal{L}$ *and all* $k = 1, 2, \cdots, n$, *then there are operators* $A_1, \cdots, A_n$ *in Alg*$\mathcal{L}$ *such that* $A_1x_1 + A_2x_2 + \cdots + A_nx_n = y$, A_k *is unitary and every* E *in* $\mathcal{L}$ *reduces* A_k *for each* $k = 1, 2, \cdots, n$.

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