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ON INJECTIVITY AND P-INJECTIVITY, IV

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Abstract. This note contains the following results for a ring A : (1) A is simple Artinian if and only if A is a prime right YJ-injective, right and left V-ring with a maximal right annihilator ; (2) if A is a left quasi-duo ring with Jacobson radical J such that ${}_AA/J$ is p-injective, then the ring A/J is stongly regular ; (3) A is von Neumann regular with non-zero socle if and only if A is a left $p.p.$ring containing a finitely generated p-injective maximal left ideal satisfying the following condition : if e is an idempotent in A, then eA is a minimal right ideal if and only if Ae is a minimal left ideal ; (4) If A is left non-singular, left YJ-injective such that each maximal left ideal of A is either injective or a two-sided ideal of A, then A is either left self-injective regular or strongly regular ; (5) A is left continuous regular if and only if A is right p-injective such that for every cyclic left A-module M, ${}_AM/Z(M)$ is projective. ((5) remains valid if ≪continuous≫ is replaced by ≪self-injective≫ and ≪cyclic≫ is replaced by ≪finitely generated≫). Finally, we have the following two equivalent properties for A to be von Neumann regular : (a) A is left non-singular such that every finitely generated left ideal is the left annihilator of an element of A and every principal right ideal of A is the right annihilator of an element of A ; (b) Change ≪left non-singular≫ into ≪right non-singular≫ in (a).

Introduction

The concept of p-injective modules was introduced in 1974 to study von Neumann regular rings, V-rings, self-injective rings and their generalizations ([16], [17]). This was later generalized to YJ-injective modules [24]. Von Neumann regular rings are sometimes called absolutely flat rings in the sense that all left (right) modules are flat (M. Harada

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(1956) ; M. Auslander (1957)). Similarly, we may say that von Neumann regular rings are absolutely p-injective since all left (right) modules are p-injective (cf. [13], [14], [16], [30]). Many authors have studied injective modules over non-semi-simple Artinian rings and flat modules over non-von Neumann regular rings (cf. for example [1], [2], [3], [5], [6], [8], [11], [12]). We are thus motivated to study p-injective modules over rings which are not necessarily von Neumann regular. Throughout, A denotes an associative ring with identity and A-modules are unital. J will always denote the Jacobson radical of A. An ideal of A will always mean a two-sided ideal of A. A is called left quasi-duo (following S. H. Brown) if every maximal left ideal of A is an ideal of A. A is called reduced if it contains no non-zero nilpotent element. For a left A-module $M, Z(M) = \{z \in M \backslash l(z)$ is an essential left ideal of $A\}$ is called the left singular submodule of M. M is called left non-singular (resp. singular) if $Z(M) = 0$ (resp. $Z(M) = M$).

The left singular ideal of A is $Z({}_AA)$ which will be noted Z. A is left non-singular if and only if $Z = 0$. It is well-known that A is left non-singular if and only if every left annihilator is a complement left ideal of A. J and Z are fundamental concepts in ring theory ([3], [4], [5], [7], [9], [12]). Note that A is von Neumann regular if and only if for every left A-module M, $Z(M)$ is flat [19, Theorem 5]. But if $Z(M)$ is injective for every left A-module M, A needs not be semi-simple Artinian (cf. [3], [14] and the example below).

The study of non-singular rings has been motivated by the following well-known facts (among others) : (1) A is left non-singular if and only if A has a von Neumann regular maximal left quotient ring Q (R. E. Johnson). In that case, Q is a left self-injective ring and ${}_AQ$ is the injective hull of ${}_AA$. (2) The classes of non-singular rings include von Neumann regular rings, hereditary rings and prime rings with non-zero socle. In 1967, F. L. Sandomierski proved that if A is left non-singular and has left finite Goldie dimension, then the homomorphic image of every injective left A-module contains its singular submodule as a direct summand (cf. the bibliography of [3]). Answering in the negative a question raised by Sandomierski, we showed (1969) that the hypothesis on Goldie dimension is superfluous (cf. Abraham ZAKS' comment in Math. Reviews 40(1970)#5664 and [23]). A standard reference for non-singular rings and modules is K. R. Goodearl [4].

In [28], we prove the following results : (1) If A is commutative, then every factor ring of A is quasi-Frobeniusean if and only if every factor ring of A is p-injective with maximum condition on annihilators; (2)

Every factor ring of A is left self-injective regular with non-zero socle if and only if every factor ring of A is semi-prime with an injective maximal left ideal. Non-singular rings and p-injectivity are concerned in some way or other in the results of the present sequel to [28]. In particular, we characterize von Neumann regular rings with non-zero socle, continuous regular rings and self-injective regular ring in terms of p-injectivity.

Recall that a left A-module M is p-injective if for any principal left ideal P of A, every left A-homomorphism of P into M extends to one of A into M. ([3, p.122], [12, p.340]], [13], [14], [16]). ${}_AM$ is YJ-injective if, for any $0 \neq a \in A$, there exists a positive integer n such that $a^n \neq 0$ and every left A-homomorphism of Aa^n into M extends to one of A into M ([13], [24], [26], [30]). P-injectivity and YJ-injectivity are similarly definited on the right side. A is called left p-injective (resp. YJ-injective) if and only if ${}_AA$ is p-injective (resp. YJ-injective). P-injectivity is also noted principal injectivity in the literature ([8], [10], [30]). (But the term $\ll$ p-injective module $\gg$ is used in R. Wisbauer [12] and C. Faith [3]).

We know that A is a right YJ-injective ring if and only if for every $0 \neq b \in A$, there exists a positive integer n such that Ab^n is a non-zero left annihilator [24, Lemma 3]. Also if A is right YJ-injective, then Y, the right singular ideal of A, coincides with J [22, Proposition 1(1)]. Further, if A is reduced right YJ-injective, then A is strongly regular [22, Proposition 1(2)]. We first characterize simple Artinian rings in terms of YJ-injectivity and a maximal annihilator. Y will stand for the right singular ideal of A.

THEOREM 1. *The following conditions are equivalent :* (1) *A is simple Artinian;* (2) *A is a prime right YJ-injective, right and left V-ring with a maximal right annihilator.*

Proof. It is evident that (1) implies (2).
Assume (2). Let $R = r(c)$ be a maximal right annihilator, where $0 \neq c \in A$. Suppose that $Soc(A)$, the socle of A, is zero. Given any maximal left ideal E of A, E must be an essential left ideal of A. If $Ec = 0$, then $E = l(c)$ and Ac is a minimal left ideal of A, contradicting $Soc(A) = 0$. Therefore $Ec \neq 0$. Let $0 \neq b \in E \cap Ec$. Since A is right YJ-injective, there exists a positive integer n such that Ab^n is a non-zero left annihilator [24, Lemma 3]. Then $b^n = dc$ for some $d \in E$. Now $r(c) = r(dc)$ (in as much as $R = r(c)$ is a maximal right annihilator), which yields $r(b^n) = r(c)$. We have $c \in l(r(c)) = l(r(b^n)) = l(r(Ab^n)) = Ab^n$ (because Ab^n is a left annihilator). But $Ab^n \subseteq E$ which proves that c belongs to every maximal left ideal of A. Therefore $c \in J = Y$ by [22,

Proposition 1(1)]. Since A is prime, $ckc \neq 0$ for some $k \in A$ and since $c \in Y$, $r(c)$ is an essential right ideal of A. Then $r(c) \cup kcA \neq 0$ and there exists $t \in A$ such that $0 \neq kct \in r(c)$. Therefore $t \in r(ckc) = r(c)$ (again because $r(c)$ is a maximal right annihilator), yielding $ct = 0$, which contradicts $kct \neq 0$. This proves that $Soc(A) \neq 0$. Since A is a prime left and right V-ring, A contains injective, projective, simple, faithful left and right modules, then A is simple Artinian by a result of J.P. Jans [Pac. J. Math. 9(1959), 1103-1108 (Corollary 2.2)]. Thus (2) implies (1). □

The next lemma is due to HuaPing Yu [15].

LEMMA 2. *If A is left quasi-duo, then A/J is a reduced ring.*

PROPOSITION 3. *Let A be a left quasi-duo ring such that ${}_AA/J$ is p-injective. Then A/J is a strongly regular ring.*

Proof. Let $B = A/J$, $b \in B$, $b = a + J$, $a \in A$, $f : Bb \to B$ a left B-homomorphism. Then $f : (Aa + J)/J \to A/J$ and $f(a + J) = d + J$ for some $d \in A$. Define a left A-homomorphism $g : Aa \to A/J$ by $g(ca) = cd + J$ for all $c \in A$. It is easily seen that g is a well-defined left A-homomorphism. Since ${}_AA/J$ is p-injective, there exists $u \in A$ such that $g(ca) = cau + J$ for all $c \in A$. Therefore $f(ca+J) = (c+J)f(a+J) = (c + J)(d + j) = cd + J = g(ca) = cau + J = (ca + J)(u + J)$ for all $c \in A$. This proves that $B = A/J$ is a left p-injective ring. By Lemma 2, B is a reduced ring. Therefore B is a strongly regular ring by [22, Proposition 1(2)]. □

REMARK 1. In Proposition 3, we do not have a von Neumann regular ring A/J if ≪left quasi-duo≫ is withdrawn. Indeed, Beidar-Wisbauer [2, Example 4.8] showed that if A is a semi-prime left (and right) p-injective, *P.I.* ring, then A is not necessarily von Neumann regular. (Note that $J = Z = 0$ here). They answered in the negative a question raised in 1981.

QUESTION 1. Is it true that $Z \cap J = 0$ if every simple singular left A-module is YJ-injective? (The answer is ≪yes≫ if ≪ YJ-injective ring,≫ is replaced by ≪ p-injective≫ (cf. [18, Proposition 3]).

The next remark improves [26, Proposition 6].

REMARK 2. If A is a semi-prime right YJ-injective ring, then the centre of A is von Neumann regular. (But A itself is not necessarily regular as confirmed by [2, Example 4.8]).

A well-known theorem of I. Kaplansky asserts that a commutative ring A is von Neumann regular if and only if every simple A-module is injective. (This remains true if ≪injective≫ is weakened to ≪ YJ-injective≫).

As usual, A is called a right (resp. left) SF-ring if every simple right (resp. left) A-module is flat.

NOTATION. Write ≪ A satisfies (*)≫ if A has a finite number of maximal right ideals whose product is contained in J.

PROPOSITION 4. *If A is a right SF-ring, then*

(1) Any left regular element is right invertible in A ; Consequently, A coincides with its classical right (and left) quotient ring.

(2) $Z \subseteq J$.

Proof. (1) Let $c \in A$ such that $l(c) = 0$. If we suppose that $cA \neq A$, let M be a maximal right ideal containing cA. Then A/M_A is flat which implies $c = dc$ for some $d \in M$. Therefore $1 - d \in l(c) = 0$ which yields $1 = d \in M$, contradicting $M \neq A$. This proves that $cu = 1$ for some $u \in A$. For any non-zero-divisor $c, c = cuc, u \in A$, and $1 - uc \in r(c) = 0$, whence $uc = cu = 1$, proving that every non-zero-divisor is invertible in A. It follows that A coincides with its classical right (and left) quotient ring.

(2) was proved by YuFei Xiao [One sided SF-rings with certain chain conditions, Canad. Math. Bull. 37 (1994), 272–277]. □

COROLLARY 5. *If A is a right SF-ring whose simple left modules are either p-injective or projective, then $Z = 0$.*

Proof. By [18, Proposition 3], $Z \cap J = 0$. Now apply Proposition 4 (2). □

PROPOSITION 6. *Let A be a right SF-ring satisfying (*). Then $Z = J = 0$.*

Proof. Let $M_1, \cdots, M_n$ be maximal right ideals of A such that $M_1 M_2 \cdots M_{n-1} M_n \subseteq J$. If $u \in J$, since $u \in M_n$ and A/M_{nA} is flat, then $u = u_n u$ for some $u_n \in M_n$. Since $u_n u \in J \subseteq M_{n-1}$ and A/M_{n-1A} is flat, $u = u_n u = u_{n-1} u_n u$ for some $u_{n-1} \in M_{n-1}$ and so on. Finally, we have $u_i \in M_i$ $1 \leq i \leq n$, such that $u_1 u_2 \cdots u_{n-1} u_n \in M_1 M_2 \cdots M_{n-1} M_n \subseteq J$ and $u = u_1 u_2 \cdots u_{n-1} u_n u$. Now $v(1 - u_1 u_2 \cdots u_n) = 1$ for some $v \in A$ which yields $u = 1u = v(1 - u_1 u_2 \cdots u_n)u = 0$. Thus $J = 0$. $Z = 0$ by Proposition 4(2). □

COROLLARY 7. *If A is a left or right self-injective, right SF-ring satisfying (*), then A is von Neumann regular.*

Now an example of a ring satisfying (*).

EXAMPLE. Set $A = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$ where K is a field. Then A has only two maximal right 0 K ideals : $M = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}$.

M_A is injective while L is the unique proper essential right ideal of A. Every simple right A-module is either injective or projective and the right (or left) singular ideal of A is zero. But $J = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ and $J^2 = 0$. A is neither semi-prime nor right p-injective. M, L are ideals of A and $LM \subseteq J$ but A is not right SF. Also, A is a ring whose singular right modules are injective but A is not von Neumann regular.

The above example shows that if A is a ring such that each maximal right ideal is either injective or an ideal of A, A needs neither be semi-prime nor right self-injective. It is well-known that if A is semi-prime, for any idempotent e, eA is a minimal right ideal if and only if Ae is a minimal left ideal of A.

THEOREM 8. *Let A be a left p.p. ring containing a finitely generated p-injective maximal left ideal such that for any idempotent e of A, eA is a minimal right ideal if and only if Ae is a minimal left ideal of A. Then A is von Neumann regular with non-zero socle.*

Proof. Let M be a finitely generated p-injective maximal left ideal of A. Since ${}_AM$ is p-injective, then ${}_AA/M$ is flat. Since M is finitely generated, then A/M is a finitely related flat left A-module which is therefore projective (S. U. Chase). Consequently, ${}_AM$ is a direct summand of ${}_AA$. Then $A = M \oplus U$, where U is a minimal left ideal of A, showing that A has non-zero socle. If $U = Au$, $u = u^2 \in A$, by hypothesis, uA is a minimal right ideal of A. First suppose that M is an ideal of A. Then $M = Ae$, where $e = 1 - u$, M is a maximal right ideal of A and $M \cap uA = 0$ (in as much as uA is a minimal right ideal of A). Therefore ${}_AA = M_A \oplus uA_A$ and A/M_A is simple projective. By [21, Lemma 1], ${}_AA/M$ is p-injective and ${}_AA = {}_AM \oplus {}_AU$ is p-injective. Now suppose that M is not an ideal of A. Then $A = M \oplus U$, where $U = Au$, $u = u^2 \in A$, $M = Ae$, $e = 1 - u$. If $MU = 0$, then $A = MA$ implies that $Au = MU = 0$, contradicting $u \neq 0$. Therefore $MU \neq 0$ and if $v \in U$ such that $Mv \neq 0$, we have $Mv = U$ and the map $g : M \to U$ defined by $g(m) = mv$ for all $m \in M$ yields ${}_AM/ker g \approx {}_AU$. Since ${}_AU$

is projective, $kerg$ is a direct summand of $_AM$ which yields $_AM/kerg$ p-injective (because $_AM$ is p-injective). Thus $_AU$ is p-injective which implies that $A = M \oplus U$ is left p-injective. In any case, A is left p-injective. Since A is a left $p.p.$ ring, then every quotient of a p-injective left A-module is p-injective [17, p.176] (cf. also [12, p.340]). Since $_AA$ is p-injective, every cyclic left A-module is p-injective which proves that A is von Neumann regular [16, Lemma 2].

Since a semi-prime principal left ideal ring is left hereditary, we get

COROLLARY 9. *A is semi-simple Artinian if and only if A is a semi-prime principal left ideal ring containing a p-injective maximal left ideal.*

In connection with Theorem 1, we have

REMARK 3. A is simple Artinian if and only if A is a prime right and left V-ring with a finitely generated p-injective maximal right ideal.

As before, A is called a left MI-ring if A contains an injective maximal left ideal. The proof of Theorem 8 yields the next lemma.

LEMMA 10. *Let A be a left MI-ring such that for any idempotent e, eA is a minimal right ideal of A if and only if Ae is a minimal left ideal of A. Then A is left self-injective.*

PROPOSITION 11. *Let A be a left non-singular, left YJ-injective ring such that each maximal left ideal is either injective or an ideal of A. Then A is either left self-injective regular or strongly regular.*

Proof. Since A is left YJ-injective, then by [22, Proposition 1(1)], $Z = J$. Therefore $J = 0$. First suppose that every maximal left ideal of A is an ideal of A. Since $J = 0$, by Proposition 2, A is reduced. Since A is left YJ-injective, then A is strongly regular by [22, Proposition 1(2)]. Now suppose there exists a maximal left ideal M of A which is not an ideal of A. By hypothesis, $_AM$ is injective and A is left MI. Since A is semi-prime (because $J = 0$), then by Lemma 11, A is left self-injective. Since $J = 0$, A is von Neumann regular.

The proof of Proposition 11 together with [17, Lemma 1] and [1, Theorem 3.1] yields

PROPOSITION 12. *Let A be a ring whose simple right modules are p-injective and such that each maximal right ideal is either injective or an ideal of A. Then A is either right self-injective regular or strongly regular.*

In connection with Sandomierski's problem, we showed that if A is left non-singular, then (1) $Z(M)$ is injective for every injective left A-module M and (2) for any complement left ideal C of A, $Z(A/C) = 0$. Recall that A is left continuous (in the sense of Y. Utumi) if (a) every left ideal of A isomorphic to a direct summand of ${}_AA$ is a direct summand of ${}_AA$ and (b) every complement left ideal of A is a direct summand of ${}_AA$. In that case, $J = Z$ and A/Z is von Neumann regular by a result of Y. Utumi (1965).

THEOREM 13. *The following conditions are equivalent : (1) A is left continuous regular ; (2) A is a right p-injective ring such that for every cyclic left A-module M, ${}_AM/Z(M)$ is projective.*

Proof. Assume (1). Then $Z = 0$. In that case, for every cyclic left A-module, $Z(M/Z(M)) = 0$. Write $C = M/Z(M)$. Then $C = Ac$ (being cyclic). For every essential extension E of $l(c)$ in ${}_AA$, any $y \in E$, there exists an essential left ideal L of A such that $Ly \subseteq l(c)$, which implies that $L \subseteq l(yc)$, whence $yc \in Z(C) = 0$. Therefore $y \in l(c)$ which yields $E = l(c)$ proving that $l(c)$ is a complement left ideal of A. Since A is left continuous, $l(c)$ is a direct summand of ${}_AA$. Then ${}_AAc(\approx A/l(c))$ is projective which means that ${}_AM/Z(M)$ is projective. Thus (1) implies (2). Conversely, assume (2). Then ${}_AA/Z(A)$ is projective which implies that $Z = Z(A)$ is a direct summand of ${}_AA$, whence $Z = 0$(Z cannot contain a non-zero idempotent). Then for every complement left ideal of K of A, $Z(A/K) = 0$. By hypothesis, ${}_AA/K$ is projective which implies that ${}_AK$ is a direct summand of ${}_AA$. Since A is right p-injective, by Ikeda-Nakayama's theorem, every principal left ideal P of A is a left annihilator. Since $Z = 0$, P is a complement left ideal of A. In that case, ${}_AP$ is a direct summand of ${}_AA$. This proves that A is von Neumann regular. A is therefore left continuous and (2) implies (1).

We may now have a nice characterization of left self-injective regular rings.

THEOREM 14. *The following conditions are equivalent :*
(1) A is a left self-injective regular ring ;
(2) A is a right p-injective ring such that for every finitely generated left A-module M, ${}_AM/Z(M)$ is projective.

Proof. Assume (1). Since $Z = 0$, for any finitely generated left A-module M, $M/Z(M)$, is a non-singular left A-module. A finitely generated non-singular left A-module is projective by [29, Corollary 6]. Therefore ${}_AM/Z(M)$ is projective. Thus (1) implies (2). Assume (2).

Then A is left continuous regular by Theorem 14. Let ${}_AE$ denote the injective hull of ${}_AA$. For any $u \in E$, $B = A + Au$ is a finitely generated non-singular left A-module. By hypothesis, ${}_AB$ is projective. Since the left annihilator of any proper finitely generated right ideal of A is non-zero, by a well-known theorem of H. BASS, ${}_AA$ is a direct summand of ${}_AB$. But ${}_AA$ is essential in ${}_AB$ which yields $A = B$. This proves that $u \in A$ and hence $E = A$ is a left self-injective regular ring. Therefore (2) implies (1).

We know that if A is left p-injective, then any left ideal isomorphic to a direct summand of ${}_AA$ is itself a direct summand of ${}_AA$. The proof of Theorems 13 and 14 then yield. □

THEOREM 15. (1) *A is left continuous regular if and only if A is a left p-injective ring such that for every cyclic left A-module M, ${}_AM/Z(M)$ is projective ;* (2) *A is left self-injective regular if and only if A is a left p-injective ring such that for every finitely generated left A-module M, ${}_AM/Z(M)$ is projective.*

We now turn to new characteristic properties of von Neumann regular rings.

THEOREM 16. *The following conditions are equivalent :*
(1) *A is von Neumann regular ;*
(2) *A is a left non-singular ring such that every finitely generated left ideal is the left annihilator of an element of A and every principal right ideal of A is the right annihilator of an element of A;*
(3) *A is a right non-singular ring such that every finitely generated left ideal is the left annihilator of an element of A and every principal right ideal is the right annihilator of an element of A;*
(4) *A is a right SF-ring whose divisible torsionfree right modules are p-injective ;*
(5) *A is a right SF-ring whose divisible torsionfree left modules are p-injective.*

Proof. (1) implies (2) through (5) evidently.

Assume (2). Let F be a finitely generated left ideal of A. Then $F = l(u)$, $u \in A$, and $uA = r(w)$, $w \in A$. Therefore $F = l(uA) = l(r(w)) = l(r(Aw))$. Since Aw is a left annihilator, then $Aw = l(r(Aw))$ which implies that $F = Aw$ is a principal left ideal. Since every principal right ideal of A is a right annihilator, by Ikeda-Nakayama's theorem, A is left p-injective. In that case, the left singular ideal Z of A coincides with the Jacobson radical J of A (cf. [22, Proposition 1(1)]). Therefore

$J = 0$ which implies A semi-prime. Since every finitely generated left ideal of A is principal, by [3, Theorem 7.5B], A is left semi-hereditary. We have seen, in the proof of Theorem 9, that a left p-injective left $p.p.$ ring is von Neumann regular. Thus (2) implies (1). Similarly, (3) implies (1) (in as much as (3) implies that A is left and right p-injective). Either (4) or (5) implies (1) by [26. Theorem 3] and Proposition 4(1).

QUESTION 2. Is A von Neumann regular if A is left non-singular such that every principal one-sided ideal is the annihilator of an element of A?

A theorem of L. Levy (1963) and Proposition 4(1) yield an analogical result for semi-simple Artinian rings in terms of certain injective modules.

THEOREM 17. *The following conditions are equivalent :*
(1) *A is semi-simple, Artinian ;*
(2) *A is a right SF-ring whose divisible torsionfree right modules are injective ;*
(3) *A is a right SF-ring whose divisible torsionfree left modules are injective.*

We now give a characteristic property of semi-prime rings.

PROPOSITION 18. *The following conditions are equivalent :*
(1) *A is a semi-prime ring ;*
(2) *Every essential left ideal of A is a faithful left A-module.*

Proof. Assume (1). Let L be an essential left ideal of A. Since A is semi-prime, $L \cap l(L) = 0$. Now L essential implies that $l(L) = 0$. Thus (1) implies (2).
Assume (2). If T is a non-zero ideal of A such that $T^2 = 0$, then T is not an essential left ideal by hypothesis. Let K be a non-zero complement left ideal of A such that $L = T \oplus K$ is an essential left ideal of A (K exists by Zorn's Lemma). Then $TK \subseteq T \cap K = 0$ which implies that $TL = T(T \oplus K) = 0$, whence $l(L) \neq 0$, contradicting ${}_AL$ faithful. This proves that A must be semi-prime and (2) implies (1).

COROLLARY 19. *A is a left non-singular semi-prime ring if and only if for every essential left ideal L of A,* $l(L) = r(L) = 0$.

REMARK 4. If A is a right non-singular ring such that every essential left ideal of A is an essential right ideal, then A is semi-prime left non-singular.

REMARK 5. Let A be a reduced ring having a classical left quotient ring Q. If Q is a left MI-ring, then Q is left and right self-injective strongly regular and Q is also the classical right quotient ring of A.

Finally, we note that, answering positively two questions of the author, Zhang-Wu show that (1) Von Neumann regular rings are absolutely YJ-injective [30, Theorem 9] and (2) A is a $\prod$-regular ring if and only if every left A-module M has the following property : for any $a \in A$, there exists a positive integer n (depending on a) such that every left A-homomorphism of Aa^n into M extends to one of A into M [30, Theorem 3]. (Indeed, (2) confirms that $\prod$-regular rings are absolutely GP-injective (cf. [30] for the definition of GP-injectivity)).

For other results on various generalizations of injectivity, consult, for example,
(1) D. G. Wang, Rings characterized by injectivity classes, Comm. Alg. **24** (1996), 717–726.
(2) J. Y. Kim, H. S. Yang, N. K. Kim and S. B. Nam, Some comments on simple singular GP-injective modules, Kyungpook Math. J. **41** (2001), 23–27. (The definition of GP-injectivity here coincides with our definition of YJ-injectivity (cf. [24], [30])).

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