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THE EMPIRICAL LIL FOR THE KAPLAN-MEIER INTEGRAL PROCESS

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Abstract. We prove an empirical LIL for the Kaplan-Meier integral process constructed from the random censorship model under bracketing entropy and mild assumptions due to censoring effects. The main method in deriving the empirical LIL is to use a weak convergence result of the sequential Kaplan-Meier integral process whose proofs appear in Bae and Kim [2]. Using the result of weak convergence, we translate the problem of the Kaplan Meier integral process into that of a Gaussian process. Finally we derive the result using an empirical LIL for the Gaussian process of Pisier [6] via a method adapted from Ossiander [5]. The result of this paper extends the empirical LIL for IID random variables to that of a random censorship model.

1. Introduction

In the present paper we investigate an empirical law of the iterated logarithm(LIL) for the Kaplan-Meier integral process constructed from the incomplete data of the usual random censorship model under the integrability assumption of metric entropy with bracketing and the assumptions due to censoring effects.

We review a weak convergence result for the sequential Kaplan-Meier integral process where the process is regarded as random elements of a Banach space of bounded functions defined on a product space.

We translate the problem of the Kaplan-Meier integral process into that of a Gaussian process and derive the main result via a method adapted from Ossiander [5] where the empirical LIL for the IID random

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variables is dealt with. By specializing our result of the random censorship model into the case of no censoring at all, we will regain an integral version of the empirical LIL for the IID model. In Section 2, we review a weak convergence result for the sequential Kaplan-Meier integral process and translate the problem of the Kaplan-Meier integral process into that of a Gaussian process. In Section 3, we deal with the proofs of the empirical LIL for the Kaplan-Meier integral process using a result for a Gaussian process of Pisier [6].

Let X be a random variable defined on a probability space $(\Omega, \mathcal{T}, P)$ and let $\{X_i : i \geq 1\}$ be a sequence of independent copies of X. Consider a class $\mathcal{F} \subseteq \mathcal{L}^2(P)$ of real valued measurable functions defined on a measurable space $(\mathbb{R}, \mathcal{B})$. Consider an empirical process $\{S_n(f) : f \in \mathcal{F}\}$ defined by

$$S_n(f) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(X_i) \text{ for } f \in \mathcal{F}.$$

It is known that $\mathcal{F}$ satisfies an empirical LIL of Strassen type (see, for example, Kuelbs and Dudley [4]) under certain metric entropy integrability conditions. This problem consists of showing the relative compactness of

$$\left\{ \frac{\sum_{i=1}^{n} f(X_i)}{\sqrt{2n \log\log n}} : f \in \mathcal{F}, n \geq 3 \right\}$$

and specifying the set of its limit points. An approach to solve the problem is to use the weak convergence of the process of following types

$$S_n(t, f) := \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} f(X_i) \text{ for } (t, f) \in [0,1] \otimes \mathcal{F} \tag{1.1}$$

to a Gaussian process. The weak convergence of $\{S_n(t, f) : (t, f) \in [0,1] \otimes \mathcal{F}\}$ to a Gaussian process $\{\mathbf{W}(t, f) : (t, f) \in [0,1] \otimes \mathcal{F}\}$ essentially means that $\mathcal{L}(S_n(t, f) : (t, f) \in [0,1] \otimes \mathcal{F}) \to \mathcal{L}(\mathbf{W}(t, f) : (t, f) \in [0,1] \otimes \mathcal{F})$, where the processes are indexed by $[0,1] \otimes \mathcal{F}$ and are considered as random elements in $B([0,1] \otimes \mathcal{F})$, the space of the bounded real-valued functions on $[0,1] \otimes \mathcal{F}$. The process $(\mathbf{W}(t, f) : (t, f) \in [0,1] \otimes \mathcal{F})$, known as *Kiefer-Muller* process, will be mean zero Gaussian and covariance function

$$cov(\mathbf{W}(t, f), \mathbf{W}(s, g)) = (t \wedge s)(Pfg - PfPg). \tag{1.2}$$

See Ossiander [5] for the invariance principle approach to solve an empirical LIL problem. See also Bae [1] for the problem of stationary martingale differences.

In this paper we solve the problem of an empirical LIL for the Kaplan-Meier integral process by considering the weak convergence results of an integral process. We begin with considering an integral process, instead of (1.1), $\{S_n(t,f) : (t,f) \in \mathbb{R} \otimes \mathcal{F}\}$ defined by

$$S_n(t,f) := \sqrt{n} \int_{-\infty}^{t} f(x)(\mathbf{P}_n - P)(dx) \text{ for } (t,f) \in \mathbb{R} \otimes \mathcal{F}, \tag{1.3}$$

where $\mathbf{P}_n(\cdot) = n^{-1} \sum_{i=1}^{n} \delta_{X_i}(\cdot)$ denotes the empirical measure.

REMARK 1. The two processes of (1.1) and (1.3) are slightly different. For instance, the limiting Gaussian process of (1.3) is given by

$$\begin{aligned} &Cov(\mathbf{W}(t,f), \mathbf{W}(s,g)) \\ &= \int_{-\infty}^{t \wedge s} f(x)g(x)P(dx) - \int_{-\infty}^{t} f(x)P(dx) \int_{-\infty}^{s} g(x)P(dx), \end{aligned}$$

which is different from (1.2).

We define the metric entropy with bracketing. See, for example, van der Vaart and Wellner [9].

DEFINITION 1. For a metric space $(\mathcal{F}, d)$ and $\delta > 0$ we define the covering number with bracketing $N_{[\,]}(\delta, \mathcal{F}, d)$ as the smallest n for which there exists $\{f_{0,\delta}^l, f_{0,\delta}^u, \cdots, f_{n,\delta}^l, f_{n,\delta}^u\}$ so that for every $f \in \mathcal{F}$ there exist some $0 \le i \le n$ satisfying $f_{i,\delta}^l \le f \le f_{i,\delta}^u$ and $d(f_{i,\delta}^l, f_{i,\delta}^u) < \delta$. Define the metric entropy with bracketing to be $\log N_{[\,]}(\delta, \mathcal{F}, d)$. We also define the associated integral of the metric entropy with bracketing to be

$$J(\delta) := \int_0^{\delta} [\log N_{[\,]}(u, \mathcal{F}, d)]^{\frac{1}{2}} du \textit{ for } 0 < \delta \le 1.$$

For a function $\psi : \mathcal{F} \to \mathbb{R}$, we let $||\psi||_{\mathcal{F}} = \sup_{f \in \mathcal{F}} |\psi(f)|$ denote the sup of $|\psi|$ over $\mathcal{F}$. We write $||\cdot||$ in stead of $||\cdot||_{\mathcal{F}}$ when there is no risk of ambiguity. We define

$$\mathcal{M} = \left\{ f \in \mathcal{L}^2(P) : \int f dP = 0 \right\}.$$

It is easy to see that $\mathcal{M}$ is a closed subspace of the Hilbert space $\mathcal{L}^2(P)$, and hence $\mathcal{M}$ is also a Hilbert space. Let $\mathcal{U}$ be the unit ball of $\mathcal{M}$,

$$\mathcal{U} = \left\{ g \in \mathcal{M} : ||g||^2 = \int g^2 dP \le 1 \right\}.$$

Then $\mathcal{U}$ defines a set $\mathcal{U}(\mathcal{F})$ of functions on $\mathcal{F}$:

$$\mathcal{U}(\mathcal{F}) = \left\{ f \to \int f \cdot g dP : f \in \mathcal{F}, g \in \mathcal{U} \right\}.$$

The Ossiander's result states that if $J(1) < \infty$, then

$$\left\{ \frac{\sum_{i=1}^{n} f(X_i)}{\sqrt{2n \log\log n}} : f \in \mathcal{F}, n \geq 3 \right\}$$

is relatively compact with respect to $||\cdot||_{\mathcal{F}}$ with probability 1, and the set of its limit points is $\mathcal{U}(\mathcal{F})$.

2. The empirical LIL for the Kaplan-Meier integral process

Let X be a random variable defined on a probability space $(\Omega, \mathcal{T}, P)$ with distribution function F and let $\{X_i : i \geq 1\}$ be a sequence of independent copies of X. Consider the random censorship model where one observes the incomplete data $\{Z_i, \delta_i\}$. The $\{Z_i\}$ are independent copies of Z whose distribution is H. The $\{Z_i, \delta_i\}$ are obtained by the equations $Z_i = \min(X_i, Y_i)$ and $\delta_i = \{X_i \leq Y_i\}$ where the $\{Y_i\}$ are independent copies of the censoring random variable Y with distribution G which is also assumed to be independent of F, the distribution of IID random variables $\{X_i\}$ of original interest in a statistical inference. Let $F\{a\} = F(a) - F(a-)$ denote the jump size of F at a and let A be the set of all atoms of H which is an empty set when H is continuous. Let $\tau_H = \inf\{x : H(x) = 1\}$ denote the supremum of the support of H. In order to describe the minimal assumptions due to censoring effects which are originated by Stute [8], we need to consider the following sub distribution functions

$$\tilde{H}^0(z) = P(Z \leq z, \delta = 0) = \int_{-\infty}^{z} (1 - F(y))G(dy),$$

and

$$\tilde{H}^1(z) = P(Z \leq z, \delta = 1) = \int_{-\infty}^{z} (1 - G(y-))F(dy), \ z \in \mathbb{R}.$$

Define

$$\gamma(x) = \exp\left\{ \int_{-\infty}^{x-} \frac{\tilde{H}^0(dz)}{1 - H(z)} \right\},$$

and

$$C(x) = \int_{-\infty}^{x-} \frac{G(dy)}{[1 - H(y)][1 - G(y)]}.$$

Notice that $\gamma(x)$ does not depend on f.

Let $\mathcal{F} \subseteq \mathcal{L}^2(P)$ be a class of functions which are real-valued measurable defined on $\mathbb{R}$. The following two assumptions will be imposed on the main result of the paper.

$$\int f^2(x)\gamma^2(x)\tilde{H}^1(dx) = \int [f(Z)\gamma(Z)\delta]^2\, dP < \infty \text{ for each } f \in \mathcal{F} \tag{2.4}$$

and

$$\int |f(x)|\, C^{1/2}(x)\tilde{F}(dx) < \infty \text{ for each } f \in \mathcal{F}. \tag{2.5}$$

Recall that $\tilde{H}^0$ and $\tilde{H}^1$ are sub-distribution functions which represent censoring effects. Throughout the paper events are identified with their indicator functions when there is no risk of ambiguity. In order to consider the dependence of γ_1 and γ_2 on f we let

$$\gamma_1^f(x) = \frac{1}{1 - H(x)} \int \{x < w\} f(w)\gamma(w)\tilde{H}^1(dw),$$

and

$$\gamma_2^f(x) = \int\int \{v < x, v < w\} \frac{f(w)\gamma(w)}{[1 - H(v)]^2} \tilde{H}^0(dv)\tilde{H}^1(dw).$$

We consider the sequential Kaplan-Meier integral process $\{U_n(t, f) : (t, f) \in \mathbb{R} \otimes \mathcal{F}\}$ defined by

$$U_n(t, f) = n^{1/2} \int_{-\infty}^{t} f(x)(\hat{F}_n - \tilde{F})(dx) \text{ for } (t, f) \in \mathbb{R} \otimes \mathcal{F}\ , \tag{2.6}$$

where $\hat{F}_n$ is the usual Kaplan-Meier estimator constructed from the random censorship model and $\tilde{F}$ is a sub-distribution function defined by

$$\tilde{F}(x) = F(x)\{x < \tau_H\} + [F(\tau_H -) + \{\tau_H \in A\}F\{\tau_H\}]\{x \geq \tau_H\}.$$

The index (t, f) ranges over $\mathbb{R} \otimes \mathcal{F}$. We rewrite $U_n(t, f)$ in (2.6) using the Kaplan-Meier empirical measure as follow. See Pollard [7] for a construction of the Kaplan-Meier empirical measures.

$$U_n(t, f) = n^{1/2} \int_{-\infty}^{t} f(x)(\mathbf{K}_n - \tilde{P})(dx) \text{ for } (t, f) \in \mathbb{R} \otimes \mathcal{F}\ ,$$

where $\mathbf{K}_n$ is the Kaplan-Meier empirical measure and $\tilde{P}$ is the measure induced by the sub-distinction $\tilde{F}$.

Write $\mathcal{S} := \mathbb{R} \otimes \mathcal{F}$. We use the following definition of weak convergence.

DEFINITION 2. A sequence of $B(\mathcal{S})$-valued random functions $\{Y_n\}$ converges in law to a $B(\mathcal{S})$-valued random function Y whose law concentrates on a separable subset of $B(\mathcal{S})$ if

$$Eg(Y) = \lim_{n\to\infty} E^* g(Y_n) \;\forall\; g \in C(B(\mathcal{S}), ||\cdot||_{\mathcal{F}}),$$

where $C(B(\mathcal{S}), ||\cdot||_{\mathcal{S}})$ is the set of all bounded, continuous functions from $(B(\mathcal{S}), ||\cdot||_{\mathcal{S}})$ into $\mathbb{R}$. Here E^* denotes the upper expectation with respect to the outer probability P^*. We denote this convergence by $Y_n \Rightarrow Y$.

Write for each $t \in \mathbb{R}$ and $f \in \mathcal{F}$

$$\xi(t,f) = f(Z)\{Z \le t\}\gamma(Z)\delta - \int f(x)\{x \le t\}\, d\tilde{F}(x)$$
$$+\gamma_1(t,f)(Z)(1-\delta) - \gamma_2(t,f)(Z),$$

where

$$\gamma_1(t,f)(x) = \frac{1}{1-H(x)} \int \{x < w \wedge t\} f(w)\gamma(w)\tilde{H}^1(dw),$$
$$\gamma_2(t,f)(x) = \int\int \{v < x, v < w \wedge t\} \frac{f(w)\gamma(w)}{[1-H(v)]^2} \tilde{H}^0(dv)\tilde{H}^1(dw).$$

Notice that $E\xi(t,f) = 0$ for each $t \in \mathbb{R}$ and $f \in \mathcal{F}$. With no censoring present, all δ's equal 1 so that each $\xi(t,f)$ collapses to

$$f(X)\{X \le t\} - \int f(x)\{x \le t\}\, dF(x).$$

We will use the following weak convergence result for the sequential Kaplan-Meier integral process whose proofs appear in Bae and Kim [2].

PROPOSITION 1. *Suppose that $J(1) < \infty$. Assume that (2.4) and (2.5). Then*

$$U_n \Rightarrow \mathbf{W}, \text{ as random elements of } B(\mathbb{R} \otimes \mathcal{F}),$$

where $\{\mathbf{W}(t,f) : (t,f) \in \mathbb{R} \otimes \mathcal{F}\}$ is a Gaussian process with the mean $E\mathbf{W}(t,f) = 0$ and the covariance function is given by

$$cov(\mathbf{W}(t,f), \mathbf{W}(s,g)) = cov(\xi(t,f), \xi(s,g)).$$

The Gaussian process $\mathbf{W}$ has uniformly continuous sample path with respect to the metric ρ defined by

$$\rho((t,f),(s,g)) = \max\{|t-s|, d(f,g)\},$$

where $\mathcal{L}^2$ metric d defined by $d(f,g) = \left[\int (f-g)^2 dP\right]^{1/2}$.

We need the following

DEFINITION 3. A sequence of $B(\mathcal{S})$-valued random functions $\{Y_n : n \geq 1\}$ converges in probability to 0, denoted $Y_n \to^P 0$, if

$$\lim_{n\to\infty} P^*\{|Y_n| > \epsilon\} = 0 \text{ for every } \epsilon > 0.$$

Let $(\mathbf{W}(f) : f \in \mathcal{F})$ be a Gaussian process which has mean zero and the covariance function

$$cov(\mathbf{W}(f), \mathbf{W}(g)) = cov(\xi(f), \xi(g)),$$

where $\xi(f) := \xi(\infty, f)$ for each $f \in \mathcal{F}$.

The following Theorem 1 is a restatement of Proposition 1. See Theorem 1.3 of Dudley and Philipp [3]. See also Theorem 4.1 of Ossiander [5]. We will use this restated result in the proof of Theorem 2.

THEOREM 1. *Under the assumptions of Proposition 1, there exists a sequence* $\mathbf{W}_1, \mathbf{W}_2, \ldots$ *IID copies of* $\{\mathbf{W}(f) : f \in \mathcal{F}\}$ *with*

$$\bar{\mathbf{W}}_n = \frac{\mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_n}{n}$$

such that

$$n^{\frac{1}{2}} \sup_{t\in\mathbb{R}} \sup_{f\in\mathcal{F}} \left| \int_{-\infty}^{t} f(x)(\mathbf{K}_n - \tilde{P} - \bar{\mathbf{W}}_n)(dx) \right| \to^P 0, \text{ as } n \to \infty.$$

The $\mathbf{W}_i$*'s can also be chosen such that, with probability* 1 *for some measurable* Y_n

$$n^{\frac{1}{2}} \sup_{f\in\mathcal{F}} \left| \int f(x)(\mathbf{K}_n - \tilde{P} - \bar{\mathbf{W}}_n)(dx) \right| \leq Y_n = o(\sqrt{\log\log n}).$$

Consider the process $\{\tilde{\mathbf{W}}_n(f) : f \in \mathcal{F}\}$ defined by

$$\tilde{\mathbf{W}}_n(f) = n^{1/2} \int f(x)\bar{\mathbf{W}}_n(dx) \text{ for } f \in \mathcal{F}\,. \tag{2.7}$$

Then $\{\tilde{\mathbf{W}}_n(f) : f \in \mathcal{F}\}$ is a Gaussian process with mean zero and the covariance function is given by

$$cov(\tilde{\mathbf{W}}_n(f), \tilde{\mathbf{W}}_n(g)) = cov(\xi(f), \xi(g)).$$

The following corollary easily follows from the above proposition.

COROLLARY 1. *Consider the Kaplan-Meier integral process* $\{U_n(f) : f \in \mathcal{F}\}$ *given by*

$$U_n(f) = n^{1/2} \int f(x)(\mathbf{K}_n - \tilde{P})(dx) \text{ for } f \in \mathcal{F}\,.$$

Then, under the assumptions of Theorem 1, there exists a sequence $\{\tilde{\mathbf{W}}_n : n \geq 1\}$*, with bounded and continuous sample paths, of copies of a Gaussian process* $\{\mathbf{W}(f) : f \in \mathcal{F}\}$ *defined on* $(\Omega, \mathcal{T}, P)$ *such that* $||U_n - \tilde{\mathbf{W}}_n|| \to^P 0$ *as* $n \to \infty$*. The* $\mathbf{W}_i'$*s can also be chosen such that, with probability* 1 *for some measurable* Y_n, $||U_n - \tilde{\mathbf{W}}_n|| \leq Y_n = o(\sqrt{\log\log n})$.

Proof. Define $\tilde{\mathbf{W}}_n$ as in (2.7) whose existence is guaranteed by Theorem 1. Observe that $U_n \Rightarrow \mathbf{W}$ as a random elements of $B(\mathcal{F})$. Observe also that

$$||U_n - \tilde{\mathbf{W}}_n|| = n^{\frac{1}{2}} \sup_{f \in \mathcal{F}} \left| \int f d(\mathbf{K}_n - \tilde{P} - \bar{\mathbf{W}}_n) \right|.$$

Since the Gaussian processes $\mathbf{W}$ and $\tilde{\mathbf{W}}$ have the same mean and covariance structure, they have the same distribution. Proposition 1 implies the results. □

We define

$$\mathcal{M} = \left\{ f \in \mathcal{L}^2(P) : \int \xi(f) dP = 0 \right\}.$$

Notice that $\mathcal{M}$ is a Hilbert space. Let $\mathcal{G}$ be the unit ball of $\mathcal{M}$,

$$\mathcal{G} = \left\{ g \in \mathcal{M} : ||\xi(g)||^2 = \int \xi(g)^2 dP \leq 1 \right\}.$$

Then $\mathcal{G}$ defines a set $\mathcal{G}(\mathcal{F})$ of functions on $\mathcal{F}$:

$$\mathcal{G}(\mathcal{F}) = \left\{ f \to \int \xi(f) \cdot \xi(g) dP : f \in \mathcal{F}, g \in \mathcal{G} \right\}.$$

We are ready to state an empirical LIL for the Kaplan-Meier integral process.

THEOREM 2. (*An Empirical LIL for the Kaplan-Meier integral process*) *Suppose that* $J(1) < \infty$*. Assume that* (2.4) *and* (2.5)*. Then*

$$\left\{ \frac{n^{\frac{1}{2}} \int f(x)(\mathbf{K}_n - \tilde{P})(dx)}{\sqrt{2\log\log n}} : f \in \mathcal{F}, n \geq 3 \right\}$$

is relatively compact with respect to $|| \cdot ||_{\mathcal{F}}$ *with probability* 1*, and the set of its limit points is* $\mathcal{G}(\mathcal{F})$.

The following restatement of Theorem 4.2 of Ossiander [5] will be a special case of Theorem 1.

COROLLARY 2. *(An Empirical LIL for IID Random Variables) Suppose that* $J(1) < \infty$. *Then*

$$\left\{ \frac{n^{\frac{1}{2}} \int f(x)(\mathbf{P}_n - P)(dx)}{\sqrt{2 \log \log n}} : f \in \mathcal{F}, n \geq 3 \right\}$$

is relatively compact with respect to $|| \cdot ||_{\mathcal{F}}$ *with probability* 1, *and the set of its limit points is* $\mathcal{U}(\mathcal{F})$, *where*

$$\mathcal{U}(\mathcal{F}) = \left\{ f \to \int f(x) g(x) P(dx) : f \in \mathcal{F}, g \in \mathcal{U} \right\}$$

with

$$\mathcal{U} = \left\{ g \in \mathcal{L}^2(P) : \int g^2(x) P(dx) \leq 1 \right\}.$$

Proof. When there is no censoring at all, we may formally set $G = \delta_\infty$ = Dirac at infinity. In this case, all $\delta's$ equal to 1. Furthermore,

$$\tilde{H}^0 = 0, \ \tilde{H}^1 = H = F, \ \gamma = 1 \text{ and } \gamma_2^f = 0.$$

Consequently, the Kaplan-Meier integral process $\int f(x)(\mathbf{K}_n - \tilde{P})(dx)$ collapses to the empirical process $\int f(x)(\mathbf{P}_n - P)(dx)$ and the limit set $\mathcal{G}(\mathcal{F})$ boils down to $\mathcal{U}(\mathcal{F})$. Conditions (2.4) and (2.5) automatically satisfied because $\mathcal{F} \subseteq \mathcal{L}^2(P)$. Apply Theorem 2 to complete the proof of Corollary 2. □

The following corollary can be considered as a log log law for the Kaplan-Meier integral for the random censorship model.

COROLLARY 3. *(A log log law for the Kaplan-Meier integral) Assume*

$$\int [\varphi(Z)\gamma(Z)\delta]^2 \, dP < \infty$$

and

$$\int |\varphi(x)| \, C^{1/2}(x) \tilde{F}(dx) < \infty.$$

Then the set of limit points of

$$\left\{ \frac{n^{\frac{1}{2}} \int \varphi(x)(\mathbf{K}_n - \tilde{P})(dx)}{\sqrt{2 \log \log n}} : n \geq 3 \right\}$$

is, with probability 1, *the closed interval* $\left[- \left(E\xi^2(\varphi)\right)^{1/2}, \left(E\xi^2(\varphi)\right)^{1/2} \right]$.

Proof. Apply Theorem 2 to the class $\mathcal{F} = \{\varphi\}$, a singleton class to get the result. □

3. Proof of Theorem 2

In the proof of Theorem 2 we will use the following Proposition.

PROPOSITION 2. (*Theorem 4.3 of Pisier* [6]) *Suppose $J(1) < \infty$. Let $\{\mathbf{W}_i : i \geq 1\}$ be a sequence of IID copies of a Gaussian process $\{\mathbf{W}(f) : f \in \mathcal{F}\}$ defined on $(\Omega, \mathcal{T}, P)$. Suppose $\{\mathbf{W}(f) : f \in \mathcal{F}\}$ has bounded and continuous sample paths with $E\mathbf{W}(f) = 0$ and $E||\mathbf{W}||^2 < \infty$. Then $\mathbf{W}$ satisfies the empirical LIL. That is,*

$$\left\{ \frac{\sum_{i=1}^n \mathbf{W}_i(f)}{\sqrt{2n \log\log n}} : f \in \mathcal{F}, n \geq 3 \right\}$$

is relatively compact with respect to $||\cdot||$ with probability 1, and the set of its limit points is

$$\mathcal{G}(\mathcal{F}) = \{f \to E\mathbf{W}(f)\mathbf{W}(g) : f \in \mathcal{F}, g \in \mathcal{G}\},$$

where $\mathcal{G} = \{g \in \mathcal{L}^2(P) : E\mathbf{W}^2(g) \leq 1\}$.

REMARK 2. $\mathbf{W}$ takes values in $C(\mathcal{F})$, the bounded and continuous functions from $\mathcal{F}$ to $\mathbb{R}$, forms a separable Banach space with the sup-norm $||\cdot||$.

Proof of Theorem 2. Let $\{\mathbf{W}(f) : f \in \mathcal{F}\}$ be a Gaussian process with bounded and continuous sample paths whose mean is zero and covariance function is

$$E\mathbf{W}(f)\mathbf{W}(g) = E(\xi(f) \cdot \xi(g)). \tag{3.8}$$

Apply Corollary 1 to choose a sequence $\{\mathbf{W}_i : i \geq 1\}$ of IID copies of $\{\mathbf{W}(f) : f \in \mathcal{F}\}$ and a measurable sequence $Y_n' s$ such that, with probability 1,

$$||\frac{U_n - \tilde{\mathbf{W}}_n}{\sqrt{2 \log\log n}}|| \leq Y_n = o(1). \tag{3.9}$$

Notice that

$$\tilde{\mathbf{W}}_n(f) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbf{W}_i.$$

By Proposition 2, $\mathbf{W}$ satisfies the empirical LIL. That is,

$$\left\{ \frac{\sum_{i=1}^{n} \mathbf{W}_i(f)}{\sqrt{2n \log\log n}} : f \in \mathcal{F}, n \geq 3 \right\}$$

is relatively compact with respect to $||\cdot||$ with probability 1, and the set of its limit points is

$$\mathcal{G}(\mathcal{F}) = \{f \to E\mathbf{W}(f)\mathbf{W}(g) : f \in \mathcal{F}, g \in \mathcal{G}\},$$

where $\mathcal{G} = \{g \in \mathcal{L}^2(P) : E\mathbf{W}^2(g) \leq 1\}$. This, together with (3.8) and (3.9), completes the proof of Theorem 2. □

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