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GENERALIZATION ON PROPER G-SPACES FOR LOCALLY COMPACT LIE GROUP G

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ABSTRACT. In this paper we study the relation of Euler characteristic with respect to cohomology with compact support in $\mathbb{F}_p$-coefficient for the fibration of ENR's and generalize some properties on proper G-spaces for locally compact Lie group G.

0. Introduction

Let G be a Lie group. Then the transformation group theory on compact G has been developed with lots of properties. If G is allowed to be anything more general than a compact group, theorems about G-spaces become extremely scarce. To recover some theory on spaces with noncompact group action, there must be some restriction on G-action. The Cartan G-spaces are those G-spaces which make many statements valid which apply when G is compact ([8]). In this paper we generalize some properties which are also satisfied on compact case for Cartan G-spaces and more restrictive proper G-spaces where G is a locally compact Lie group. We also study the property regarding ENR's (Euclidean Neighborhood Retract) and then we show that if X is a proper G-ENR, then the orbit space X/G is an ENR. This paper is organized as followings. In section 1, we study the relation of Euler characteristic with respect to cohomology with compact support in $\mathbb{F}_p$-coefficient for the fibration of ENR's. In section 2, we generalize some properties on proper G-spaces for locally compact Lie group G and apply the proper G-action to ENR's.

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1. Property of Euler characteristic of ENR's

Let X be an ENR (Euclidean Neighborhood Retract) and X^+ denotes its one point compactification. Let $\chi_c(X)$ denote the Euler Characteristic of X with respect to the cohomology with compact support in coefficient ring R. In this section we give the relation of Euler characteristic with respect to cohomology with compact support in $\mathbb{F}_p$-coefficient for the fibration of ENR's.

LEMMA 1.1. *Let X and Y be ENR's. Then, for disjoint union of X and Y,*

$$\chi_c(X \sqcup Y) = \chi_c(X) + \chi_c(Y).$$

Proof. It is trivial. □

We set

$$\widetilde{\chi}(X^+) = \sum_{i=0}^{\infty}(-1)^i \text{ rank } \widetilde{H}^i(X^+; R).$$

LEMMA 1.2. *Let X and Y be ENR's. Then*

$$\chi_c(X \times Y) = \chi_c(X) \cdot \chi_c(Y).$$

Proof. This is obvious from the Kunneth formula for the Alexander Spainer cohomology with compact supports ;

$$H_c^n(X \times Y; \mathbb{F}_p) \approx \sum_{p+q=n} H_c^p(X; \mathbb{F}_p) \otimes H_c^p(Y; \mathbb{F}_p).$$

We also show the result by the following way.

For the Euler Characteristic of $X \times Y$ with compact support,

$$\chi_c(X \times Y) = \widetilde{\chi}(X^+ \wedge Y^+).$$

Now we have

$$\widetilde{H}^*(X^+ \times Y^+; R) \cong \widetilde{H}^*(X^+; R) \oplus \widetilde{H}^*(Y^+; R) \oplus \widetilde{H}^*(X^+ \wedge Y^+; R).$$

Hence

$$\begin{aligned}\widetilde{\chi}(X^+ \times Y^+) &= \widetilde{\chi}(X^+) + \widetilde{\chi}(Y^+) + \widetilde{\chi}(X^+ \wedge Y^+) \\ &= \chi_c(X) + \chi_c(Y) + \chi_c(X \times Y).\end{aligned}$$

However

$$\begin{aligned}\widetilde{\chi}(X^+ \times Y^+) &= \widetilde{\chi}(X^+) \cdot \widetilde{\chi}(Y^+) + \widetilde{\chi}(X^+) + \widetilde{\chi}(Y^+) \\ &= \chi_c(X) \cdot \chi_c(Y) + \chi_c(X) + \chi_c(Y).\end{aligned}$$

Therefore

$$\chi_c(X \times Y) = \chi_c(X) \cdot \chi_c(Y).$$

□

If X is an ENR, $\chi_c(X)$ may not be defined if $H_c^*(X; \mathbb{F}_p)$ is infinite dimesional.

PROPOSITION 1.3. *Let E and B be ENR's and $f : E \to B$ be a fibration with typical ENR fiber F. Assume $\chi_c(F)$ and $\chi_c(B)$ are defined. Then $\chi_c(E)$ is defined and*

$$\chi_c(E) = \chi_c(F) \cdot \chi_c(B),$$

where χ_c is the Euler Characteristic with respect to cohomology with compact support and coefficients are finite field $\mathbb{F}_p$.

Proof. Since ENR's are locally contractible, we use Alexander Spainer cohomology. Then the fibratin $f : E \to B$ gives a Leray spectral sequence with E_2-term

$$E_2^{p,q} = H_c^p(B; H_c^q(F)),$$

where the coefficient $H_c^q(F)$ are considered as a local coefficient system on B. If this local coefficient system is trivial, then

$$H_c^p(B; H_c^q(F)) \cong H_c^p(B) \otimes H_c^q(F)$$

and

$$\begin{aligned}\chi_c(E_2) &= \sum_{p,q} (-1)^{p+q} dim E_2^{p,q} \\ &= \sum_{p,q} (-1)^{p+q} dim(H_c^p(B) \otimes H_c^q(F)) \\ &= \sum_{p,q} (-1)^{p+q} dim(\widetilde{H}^p(B^+) \otimes \widetilde{H}^q(F^+)) \\ &= \widetilde{\chi}(B^+) \cdot \widetilde{\chi}(F^+) \\ &= \chi_c(B) \cdot \chi_c(F).\end{aligned}$$

Since $E_2^{p,q} = 0$ if p and q are large enough and the same is true of $E_r^{p,q}$ for any r. $E_\infty = E_r$ for large r and

$$\chi_c(E_\infty) = \chi_c(B) \cdot \chi_c(F).$$

Now we have

$$dim[H^n(E)] = \sum_{s+t=n} dim E_\infty^{s,t},$$

and hence

$$\chi_c(E) = \chi_c(E_\infty) = \chi_c(B) \cdot \chi_c(F).$$

If the local coefficient system is nontrivial, we reduce the system to be trivial by using finite covering over B since $H_c^q(F; \mathbb{F}_p)$ is finite. Let $\pi_1(B) = G$ and $\rho : G \to GL_n(\mathbb{F}_p)$. If $K = Ker\rho$, then G/K is a finite covering of $\widetilde{B} \to B$ where $\widetilde{B}$ is a universal covering space corresponding to K. Hence $\chi_c(\widetilde{B}) = N \cdot \chi_c(B)$ where $|G/K| = N$ ([2] 5.3.3). For the induced covering $\widetilde{E}$ over E, $\chi_c(\widetilde{E}) = N \cdot \chi_c(E)$. Now $\chi_c(\widetilde{E}) = \chi_c(F) \cdot \chi_c(\widetilde{B})$ by trivial local coefficient system since $\pi_1(\widetilde{B}, b_0)$ acts trivially on $H_c^q(F, \mathbb{F}_p)$. Hence $N \cdot \chi_c(E) = \chi_c(F) \cdot N \cdot \chi_c(B)$. Therefore

$$\chi_c(E) = \chi_c(F) \cdot \chi_c(B).$$

□

2. Proper action of locally compact Lie group

Now we study some properties on spaces with noncompact group action. We consider a complete regular space X with a fixed action on G. If G is a compact Lie group then a lot of general theory of G-spaces has been developed. For the noncompact case we need to give some condition on G-space for which theory can be applied reasonably. For our purpose, we study proper G-space for locally compact Lie group G. Then many of the statements which hold when G is compact are valid in this case ([8]).

Let G be a locally compact Lie group with identity e which acts on complete regular space X. We recall some definitions and facts from [8]. We define the subsets of G

$$((U, V)) = \{g \in G \mid gU \cap V \neq \emptyset\},$$

where U and V are the subsets of G-space X. If U and V are the subsets of a G-space X then we say that U is *thin* relative to V if $((U, V))$ has compact closure in G. If U is thin relative to itself then we say that U is thin.

DEFINITION 2.1. A G-space X is *Cartan G-space* if every point of X has a thin neighborhood.

DEFINITION 2.2. A subset S of a G-space X is a *small subset* of X if each point of X has a neighborhood which is thin relative to S. A G-space X is *proper* if each point of X has a small neighborhood.

We state some important relation between Cartan G-space and proper G-space.

PROPOSITION 2.3. ([8]) *A G-space X is proper if and only if X is a Cartan G-space and X/G is regular.*

Now we prove the following properties on Cartan G-space for locally compact Lie group G.

PROPOSITION 2.4. *If X is a locally compact space then X/G is locally compact.*

Proof. The orbit map $\pi : X \rightsquigarrow X/G$ is open. For $x \in U \subset X$, let $\overline{U}$ be a compact closure of U. Then $\pi(x) \in \pi(U) \subset \pi(\overline{U})$, where $\pi(\overline{U})$ is a compact closure containing $\pi(x)$. □

PROPOSITION 2.5. ([8]) *If X is a proper G-space and N is a closed normal subgroup of G, then X/N is a proper G/N-space.*

PROPOSITION 2.6. *If X is a proper G-space and N is a closed normal subgroup of G, then X^N is a proper G/N-space.*

Proof. Since X is a proper G-space, X/G is regular by Proposition 2.3. For every $x \in X$, x has a thin neighborhood U such that $((U, U))$ is relatively compact in G. Recall G/N acts on X^N by $(gN)(x) = gNx = gx$. Then G/N action on X^N is equivalent to G-action on X^N and every subspace of a regular space is regular, and hence $X^N/(G/N)$ is regular. To show for every $x \in X^N$, x has a thin neighborhood U^* such that $((U^*, U^*))$ is relatively compact in G/N, we take $U^* = \{x \in U \mid nx = x$ for every $n \in N\} = U \cap X^N$ which is open in X^N. Moreover if p is the canonical map of G onto G/N, it can be easily checked $p((U, U)) = ((U^*, U^*))$ since

$$\begin{aligned}((U^*, U^*)) &= \{gN \mid gNU^* \cap U^* \neq \emptyset\} \\ &= \{gN \mid gN(U \cap X^N) \cap (U \cap X^N) \neq \emptyset\}.\end{aligned}$$

□

PROPOSITION 2.7. ([8]) *Let X be a proper G-space. If X is separable metric then X/G is also separable metric.*

Now we apply some property to G-ENR for locally compact Lie group G. We define a G-ENR(Euclidean Neighborhood Retract) to be a G-space X which is (G-homeomorphic to) a G-retract of some open G-subset in a G-module V. If we have no group G acting we simply talk about ENR's.

PROPOSITION 2.8. ([3, 4]) *If $X \subset R^n$ is locally $(n-1)$ connected and locally compact then X is an ENR.*

A separable metric space of dimension $\leq n$ can be embedded in R^{2n+1} [5]. Hence a space is an ENR if and only if it is locally compact, separable metric, finite dimensional and locally contractible.

Now we give the example of the proper action on a space.

EXAMPLE. We give the action of Z/p^∞ on Z/p^∞ by left translation. Then this is the proper action ([3], p.31). To show this is Z/p^∞-ENR, we need to check it is locally compact, separable metric, finite dimensional and locally contractible. Generally the following fact is known in [2], [3], [5] and point set topology. Every discrete space is locally compact and locally contractible. Every contractible discrete space is separable. Finite or countable space is 0-dimensional space hence finite dimensional space. Therefore Z/p^∞ is a proper Z/p^∞-ENR.

The idea of the proof of the following result is coming from [3, p.159].

PROPOSITION 2.9. *Let X be a proper G-ENR. Then the orbit space X/G is an ENR.*

Proof. Since X is G-ENR, X is a retract of some open G subset U in a G-module, i.e. $X \xrightarrow{i} U \xrightarrow{r} X$ and $r \circ i = id_X$. A retract of an ENR is an ENR. Hence we prove the proposition for X a differential G-manifold and then apply it to the manifold U. Let $\pi : X \to X/G$ be the quotient map. Then X/G is locally compact by Proposition 2.4 and separable metric by Proposition 2.7. By dimension theory [5], $dim X/G \leq dim X$. Hence X/G is finite dimensional. To show X/G is locally contractible, given $\bar{x} \in V \subset X/G$, V open, $\pi^{-1}(V)$ is open. Since X is locally contractible, $\pi^{-1}(V)$ contains G-invariant neighborhood W of the orbit $\pi^{-1}(\bar{x}) = xG$ which is null homotopic. Hence πW is also contractible in X/G. Therefore X/G is locally contractible. □

3. Structure of the Burnsid module of the locally compact Lie group

Minami [7] generalized Tom Dieck's Burnside ring of compact Lie groups [2] to the relative case. In this section, we extend the structure of the relative Burnside module for locally compact Lie groups. Let G be a locally compact Lie group of the type $G = F \times L$ where L is a normal subgroup of G. We assume all $F \times L$-action to be proper. We define $A(F, L)$ to be the set of equivalence classes of proper $F \times L$-ENR with a free L-action under the equivalence relation

$$X \sim Y \quad \text{if and only if} \quad \chi_c(X^S/N_L S) = \chi_c(Y^S/N_L S)$$

for any $S \subset F \times L$, where $N_L S = N_{F\times L} S \cap L$ acts free on X^S and Y^S so that the following diagrams commute.

$$\begin{array}{ccc} N_L S \times X^S & \longrightarrow & X^S \\ \downarrow & & \downarrow \\ (F \times L) \times X & \longrightarrow & X \end{array} \qquad \begin{array}{ccc} N_L S \times Y^S & \longrightarrow & Y^S \\ \downarrow & & \downarrow \\ (F \times L) \times Y & \longrightarrow & Y \end{array}$$

Here χ_c is the Euler characteristic with respect to the cohomology with compact support in $\mathbb{F}_p$-coefficient. Given $H < F$, a closed subgroup H and $\phi : H \to L$ a homomorphism, $(H, \phi) = \{(h, \phi(h)) | h \in H\}$ is a closed subgroup of $F \times L$. Let $C(F, L)$ be the set of the conjugacy classes of closed subgroups (H, ϕ).

LEMMA 3.1. (i) *Let $F \to E \to B$ be a proper G-fiber bundle such that a closed normal subgroup $N \triangleleft G$ acts trivially on B, then $F^N \to E^N \to B$ is a proper G/N-fiber bundle.*

(ii) *Let $F \to E \to B$ be a proper G-fiber bundle such that G acts trivially on B, then $F/G \to E/G \to B$ is a proper fiber bundle.*

Proof. This can be easily checked by using the local triviality and Proposition 2.5 and 2.6. □

THEOREM 3.2. *$A(F, L)$ is a free abelian group with basis $[(F \times L)/(H, \phi)]$ for each $(H, \phi) \in C(F, L)$. For any proper $F \times L$-ENR X with free L-action,*

$$[X] = \sum_{(H,\phi)} \chi_c(X_{(H,\phi)}/(F \times L))[(F \times L)/(H, \phi)] \in A(F, L)$$

where (H, ϕ) runs over $C(F, L)$.

Proof. The addition is given by disjoint union and the inverse form of $[X]$ is

$$-[X] = [X \times K],$$

where K is a $F \times L$-ENR with trivial action such that $\chi_c(K) = -1$.

Now we want to express any $[X] \in A(F, L)$ as a linear combination of $[(F \times L)/(H, \phi)]$ where $(H, \phi) \in C(F, L)$. Since X is the disjoint union of its orbit bundles [3], $X = \coprod_{(H,\phi)\in C(F,L)} X_{(H,\phi)}$, by additivity of the Euler Characteristic

$$\chi_c(X^S/N_L S) = \sum_{(H,\phi)\in C(F,L)} \chi_c(X^S_{(H,\phi)}/N_L S).$$

We consider the fiber bundle

$$F \times L/(H, \phi) \to X_{(H,\phi)} \to X_{(H,\phi)}/F \times L.$$

By applying Lemma 3.1, (i) for $S \lhd N_{F\times L}S \subset F \times L$ and (ii) for $N_L S \subset N_{F\times L}S$, we obtain the following fiber bundle

$$(F \times L/(H, \phi))^S/N_L S \to X^S_{(H,\phi)}/N_L S \to X_{(H,\phi)}/F \times L.$$

If we apply Proposition 1.3 to this bundle, we have

$$\chi_c(X^S_{(H,\phi)}/N_L S) = \chi_c((F \times L/(H, \phi))^S/N_L S) \cdot \chi_c(X_{(H,\phi)}/F \times L).$$

Therefore

$$[X] = \sum_{(H,\phi)\in C(F,L)} \chi_c(X_{(H,\phi)}/F \times L)[(F \times L)/(H, \phi)].$$

Next we show $\{[(F\times L)/(H, \phi)]\}$ are linearly independent. Let us assume $\sum_{(H,\phi)\in C(F,L)} a_{(H,\phi)}[(F \times L)/(H, \phi)] = 0$ where $a_{(H,\phi)} = \chi_c(X_{(H,\phi)}/F \times L)$. We suppose $\{[(F \times L)/(H, \phi)]\}$ are linearly dependent. We take (H', ϕ') to be maximal among those such that $a_{(H,\phi)} \neq 0$. Then we get

$$\begin{aligned} 0 &= \chi_c\Big(\sum_{(H,\phi)\in C(F,L)} a_{(H,\phi)}[(F \times L)/(H, \phi)]^{(H',\phi')}/N_L(H', \phi')\Big) \\ &= a_{(H',\phi')} \cdot \chi_c([(F \times L)/(H', \phi')]^{(H',\phi')}/N_L(H', \phi')). \end{aligned}$$

But $a_{(H',\phi')} \neq 0$ and $\chi_c([(F \times L)/(H', \phi')]^{(H',\phi')}/N_L(H', \phi')) \neq 0$ since $[(F \times L)/(H', \phi')]^{(H',\phi')}/N_L(H', \phi')$ is finite. This is a contradiction. Therefore $\{[(F \times L)/(H, \phi)]\}$ are linearly independent. Now we claim $[(F\times L)/(H', \phi')]^{(H',\phi')}/N_L(H', \phi')$ is finite. We consider the fiber bundle

$$\begin{aligned} &N_{F\times L}(H, \phi)/(H, L) \cap N_{F\times L}(H, \phi) \\ &\to N_{F\times L}(H, L)/(H, L) \to N_{F\times L}(H, L)/N_{F\times L}(H, \phi) \cdot (H, L). \end{aligned}$$

Then the base space $N_{F\times L}(H, L)/N_{F\times L}(H, \phi) \cdot (H, L)$ is finite since $N_{F\times L}(H, \phi) \supset C_{F\times L}(H, L)$ and $N_{F\times L}(H, L)/N_{F\times L}(H, L) \cdot (H, L)$ is finite [4]. Now the total space $N_{F\times L}(H, L)/(H, L) \approx N_F H/H$ is finite.

Therefore the fiber $N_{F\times L}(H,\phi)/(H,L)\cap N_{F\times L}(H,\phi)$ is finite and we have the following relation

$$\begin{aligned}
&[(F\times L)/(H,\phi)]^{(H,\phi)}/N_L(H,\phi)\\
&= N_L(H,\phi)\setminus[(F\times L)/(H,\phi)]^{(H,\phi)}\\
&= N_L(H,\phi)\setminus N_{F\times L}(H,\phi)/(H,\phi)\\
&= N_{F\times L}(H,\phi)/(H,\phi)\cdot N_L(H,\phi)\\
&= N_{F\times L}(H,\phi)/(H,L)\cap N_{F\times L}(H,\phi).
\end{aligned}$$

This completes our claim. □

References

[1] G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972.

[2] T. tom Dieck, *Transformation groups and representation theory*, Lecture Notes in Math. **766**, Springer-Verlag, Berlin, 1979.

[3] ———, *Transformation groups*, Walter de Gruyter, Berlin, 1987.

[4] A. Dold, *Lectures on algebraic topology*, Springer, Heidelberg-New York, 1972.

[5] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Mathematical Series, Vol. 4, Princeton Univ. Press, 1941.

[6] H. -S. Lee, *Induced maps in homology for p-toral compact Lie groups*, Topology Appl. **87** (1998), 189–197.

[7] N. Minami, *The relative Burnside ring and the stable maps between classifying spaces of compact Lie groups*, Trans. Amer. Math. Soc. **347** (1995), no. 2, 461–498.

[8] R. S. Palais, *On the existence of slices for actions of noncompact Lie groups*, Ann. of Math. **73** (1961), no. 2, 295–323.

[9] ———, *The classification of G-spaces*, Mem. Amer. math. Soc., No. 36, 1960.

[10] E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

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