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SUBALGEBRAS OF A q-ANALOG FOR THE VIRASORO ALGEBRA

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ABSTRACT. We define subalgebras $V_q^{mZ\times nZ}$ of V_q where V_q are in the paper [4]. We show that the Lie algebra $V_q^{mZ\times nZ}$ is simple and maximally abelian decomposing. We may define a Lie algebra is maximally abelian decomposing, if it has a maximally abelian decomposition of it. The F-algebra automorphism group of the Laurent extension of the quantum plane is found in the paper [4], so we find the Lie automorphism group of $V_q^{mZ\times nZ}$ in this paper.

1. Preliminaries

Let N be the set of all negative integers and Z be the set of all integers. Let F be a field of characteristic zero. Let $q \in F$ be a fixed non-root of unity ($q^n \neq 1$ for any $n \in N$.) Throughout the paper, let us denote that mN and nN are additive submonoids of Z, where m and n are fixed non-negative integers. Similarly, let mZ and nZ denote additive subgroups of Z, where m and n are fixed non-negative integers. The skew polynomial ring $F_q[x,y]$, where $yx = qxy$, has been called the quantum plane and it has the Laurent extension $F_q[x^{\pm 1}, y^{\pm 1}]$. $F_q[x^{\pm 1}, y^{\pm 1}]$ has a subring $F_q[x^{mN}, y^{nN}] = \{x^{ma}y^{nb} | a, b \in N\}$, where m and n are fixed non-negative integers.

$F_q[x^{mN}, y^{nN}]$ can be localized at the Ore set of powers of x and y to give a ring of Laurent polynomials $F_q[x^{mZ}, y^{nZ}]$. The center of $F_q[x^{mN}, y^{nN}]$ or $F_q[x^{mZ}, y^{nZ}]$ is F.

Let A be an associative F-algebra. We may define a Lie algebra $A_{[,]}$ on A using the commutator $[,]$ on A defined by $[a,b] = ab - ba$ for any

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$a, b \in A$. The F-algebra $F_q[x^{\pm 1}, y^{\pm 1}]$ gives the Lie algebra $F_{q,[,]}[x^{\pm 1}, y^{\pm 1}]$, and it has the Lie algebra decompositions as follows;

$$F_q[x^{\pm 1}, y^{\pm 1}] = F \bigoplus V_q^{mZ \times nZ}, \tag{1}$$

where the Lie subalgebra $V_q^{mZ \times nZ}$ has the standard basis $\{x^{mi}y^{nj} | i, j \in Z$, at least i or j is not zero $\}$. The Lie algebra V_q in the paper [4] is isomorphic to the Lie algebra $V_q^{Z^2}$. The Lie algebra $V_q^{mZ \times nZ}$ is generated by $x^{-m}y^{-n}, x^m$, and y^n [4].

2. Simplicity

The monomial of the form $x^{mi}y^{nj}$ form a vector space basis of $F_q[x^{mZ}, y^{nZ}]$.

THEOREM 1. *The F-algebra $F_q^{mZ \times nZ}$ is simple.*

Proof. It is straightforward. So let us omit the proof of the theorem. □

THEOREM 2. *The Lie algebra $V_q^{mZ \times nZ}$ is simple.*

Proof. The proof of this theorem is similar to the proof of the Theorem 1.3 in the paper [4] or it can be easily proved by induction on the number of terms of an element in a non-zero ideal of $V_q^{mZ \times nZ}$ [6]. Let us omit the proof of the theorem. □

COROLLARY 1. *The Lie algebra $V_q^{Z^2}$ is simple.*

Proof. This is the Theorem 1.3 in the paper [4]. □

Theorem 2 can be proved by a result of I. N. Herstein [2], [4], i.e. $F_q[x^{\pm 1}, y^{\pm 1}] = F \bigoplus V_q^{mZ \times nZ}$.

LEMMA 1. *The map $D_{\alpha,\beta}(x^{mi}y^{nj}) = (\alpha mi + \beta nj)x^{mi}y^{nj}$ induces a derivation of $F_q[x^{mN}, y^{nN}]$ for any $\alpha, \beta \in F$. The derivation $D_{\alpha,\beta}$ is not an inner derivation of $F_q[x^{mN}, y^{nN}]$.*

Proof. It is standard (please refer the proof of Lemma 1.1 in the paper [4].) □

THEOREM 3. *Every derivation of the F-algebra $F_q[x^{mN}, y^{nN}]$ is the sum of inner and $D_{\alpha,\beta}$ where $D_{\alpha,\beta}$ is the derivation in the Lemma 1.*

Proof. It is standard by Lemma 1. □

COROLLARY 2. *The Lie algebra of derivations of F-algebra $F_q[x^{mN}, y^{nN}]$ is generated by the inner and the derivations $D_{\alpha,\beta}$ in Lemma 1.*

Proof. It is standard (please refer the proof of Theorem 1.2 in the paper [4].) □

3. Abelian decomposition of a Lie algebra

A Lie algebra L has a decomposition of abelian subalgebras of it, if $L = \bigoplus_{i\in I} A_i$ such that A_i, $i \in I$, is an abelian subalgebra of L and I is an index set.

A Lie algebra L is maximally abelian decomposing, if it holds the following two conditions:
(i) $L = \oplus_{i\in I} A_i$ such that A_i, $i \in I$, is an abelian subalgebra of L and I is an index set.
(ii) If any element $l \in L$ such that $l = \sum_{l_i \in A_i, i\in J\subset I} l_i$ and $|J| \geq 2$, then the centralizer of l is one dimensional vector space.

The Lie algebra $V_q^{mZ\times nZ}$ is maximally abelian decomposing, since

$$V_q^{mZ\times nZ} = \oplus_{(i,j)\in Z\times N} A_{(i,j)} \tag{2}$$

is a required maximally abelian decomposition where $A_{(i,j)}$ is the subalgebra of it spanned by $x^{mik}y^{njk}$, $k \in Z$, where we may choose the minimal integers i and j for $A_{(i,j)}$ using the absolute values of them.

LEMMA 2. *The units of $F_q^{mZ\times nZ}$ are monomials of the form cx^iy^j for $0 \neq c \in F$, $i \in mZ$, and $j \in mZ$.*

Proof. It is standard (please refer the proof of Lemma 1.4 in the paper [4].) □

Each element $(\alpha, \beta) \in F^\bullet \times F^\bullet$ induces an automorphism of $F_q^{mZ\times nZ}$, namely $\sigma_{(\alpha,\beta)}(x^{mi}y^{nj}) = (\alpha x)^{mi}(\beta y)^{nj}$. Each element of $Sl(2, Z)$ induces an automorphism of $F_q^{mZ\times nZ}$: $\begin{pmatrix} h & k \\ i & j \end{pmatrix}$ corresponds to the automorphism which maps to x^m to $x^{mh}y^{nk}$ and y^n maps to $x^{mi}y^{nj}$ (note that $\sigma(x^my^n) = q^{njmh}x^{mi+mh}y^{nk+nk}$ and $\sigma(y^nx^m) = q^{mn+njmh}x^{mi+mh}y^{nk+nk}$, we have that $jh - ik = 1$.) We show that these automorphisms generate the automorphism group.

THEOREM 4. *The automorphism group $Aut_F(F_q^{mZ\times nZ})$ of $F_q^{mZ\times nZ}$ is isomorphic to the semidirect of $Sl(2, Z)$ and $F^\bullet \times F^\bullet$.*

Proof. If σ is an automorphism of $F_q^{mZ\times nZ}$ it take x^m and y to units. So by Lemma 2, $\sigma(x^m) = \lambda x^{mh}y^{nk}$ and $\sigma(y^n) = \mu x^{mi}y^{nj}$ for $\lambda, \mu \in F^\bullet \times F^\bullet$ and $h, k, i, j \in Z$. As the above note, we have that $det \begin{pmatrix} h & k \\ i & j \end{pmatrix} = 1$. Since

$$\sigma_{(\alpha,\beta)} \begin{pmatrix} h & k \\ i & j \end{pmatrix} = \begin{pmatrix} h & k \\ i & j \end{pmatrix} \sigma_{(\alpha^h\beta^k, \alpha^i\beta^j)}, \tag{3}$$

it follows that the product is a semidirect. □

The following lemma for the Lie automorphism group $V_q^{mZ\times nZ}$ of corresponds to the Lemma 2 for the automorphism group of $F_q^{mZ\times nZ}$.

LEMMA 3. *For any Lie automorphism θ of $V_q^{mZ\times nZ}$, $\theta(x^m) = x^{mh}y^{nk}$ and $\theta(y^n) = x^{mi}y^{nj}$ hold for some $h, k, i,$ and $j \in Z$.*

Proof. Let θ be an Lie automorphism of $V_q^{mZ\times nZ}$. It is enough to show that $\theta(x^m) = x^{mh}y^{nk}$ for some $h, k \in Z$. Since $V_q^{mZ\times nZ}$ is maximally abelian decomposing, $\theta(x^m)$ should be in $A_{(i,j)}$ by 4 for some $(i,j) \in Z \times N$. Similarly, for any element $x^uy^v \in V_q^{mZ\times nZ}$, $\theta(x^uy^v) \in A_{(i,j)}$ for some $(i,j) \in Z \times N$. Assume that there is $x^uy^v \in V_q^{mZ\times nZ}$, such that $\theta(x^uy^v)$ has p non-zero terms in $A_{(i,j)}$ such that p is maximal, i.e. for any element $x^by^s \in V_q^{mZ\times nZ}$, the number of terms of $\theta(x^by^s)$ is less than or equal to p, $p > 1$. There is $t \in mZ$ such that $\theta([x^t, x^uy^v]) = (1 - q^{vt}\theta(x^{t+u}y^v)$ has p terms, since $V_q^{mZ\times nZ}$ is maximally abelian decomposing. $\theta([x^uy^v, x^{t+u}y^v])$ has p^2 terms. Since $p^2 > p$, this contradicts the fact that $\theta(x^uy^v)$ has the maximal number of non-zero terms. This shows that $p = 1$. Therefore we have proven the lemma. □

Let $Aut_{Lie}(V_q^{mZ\times nZ})$ be the group of all Lie automorphisms of $V_q^{mZ\times nZ}$.

Each element $(\alpha, \beta) \in F^\bullet \times F^\bullet$ induces an automorphism of $V_q^{mZ\times nZ}$, namely $\sigma_{(\alpha,\beta)}(x^{mi}y^{nj}) = (\alpha x)^{mi}(\beta y)^{nj}$. Each element of $Sl(2, Z)$ induces an automorphism of $V_q^{mZ\times nZ}$: $\begin{pmatrix} h & k \\ i & j \end{pmatrix}$ corresponds to the automorphism which maps to x^m to $x^{mh}y^{nk}$ and y maps to $x^{mi}y^{nj}$. From $\theta([x^m, y^n]) = \theta(x^my^n - y^nx^m) = (1 - q^{mn})\theta(x^my^n)$, we have that $[\theta(x^m), \theta(y^n)] = (1 - q^{mn})\theta(x^my^n)$. This implies that $[x^{mh}y^{nk}, x^{mi}y^{nj}] = (1 - q^{mn})\ x^{mh}y^{nk}x^{mi}y^{nj}$. This implies that $q^{njmh} = q^{nkmi+nm}$. We have that $nm(det \begin{pmatrix} h & k \\ i & j \end{pmatrix}) = nm$. This implies that $det \begin{pmatrix} h & k \\ i & j \end{pmatrix} = 1$. We show that these Lie automorphisms generate the automorphism group of $V_q^{mZ\times nZ}$.

THEOREM 5. *$Aut_{Lie}(V_q^{mZ\times nZ})$ is isomorphic to the semidirect of $Sl(2,Z)$ and $F^\bullet \times F^\bullet$.*

Proof. The proof of this theorem is similar to the proof of Theorem 4 by Lemma 3 and the above note. Thus let us omit the proof of the theorem. □

LEMMA 4. *For F-algebra isomorphism θ from $F_{q_1}[x^{\pm 1}, y^{\pm 1}]$ to $F_{q_2}[u^{\pm 1}, v^{\pm 1}]$, $\theta(x) = c_1 u^h v^k$ and $\theta(y) = c_2 u^i v^j$ hold for some h, k, i, and $j \in Z$ where c_1 and c_2 are non-zero scalars.*

Proof. It is straightforward from the that the unit of $F_q[u^{\pm 1}, v^{\pm 1}]$ has the form $u^s v^t$ for $s, t \in Z$. Therefore we have proven the lemma. □

PROPOSITION 1. *Let $F_{q_1}[x^{\pm 1}, y^{\pm 1}]$ and $F_{q_2}[u^{\pm 1}, v^{\pm 1}]$ be simple F-algebras. If $q_1 \neq q_2$, then $F_{q_1}[x^{\pm 1}, y^{\pm 1}]$ is not isomorphic to $F_{q_2}[u^{\pm 1}, v^{\pm 1}]$ as F-algebras.*

Proof. Let us assume that there is an isomorphism θ from $F_{q_1}[x^{\pm 1}, y^{\pm 1}]$ to $F_{q_2}[u^{\pm 1}, v^{\pm 1}]$. Then $\theta(x) = c_1 u^a v^i$ and $\theta(y) = c_2 u^b v^j$ hold by Lemma 4. By $\theta(xy) = q_1 \theta(yx)$, we have that $c_1 c_2 q_2^{bi} u^{a+b} v^{i+j} = c_1 c_2 q_1 q_2^{aj} u^{a+b} v^{i+j}$. This implies that

$$q_1 q_2^{aj} = q_2^{bi}. \tag{4}$$

By $\theta(xyx^2) = q_1^2 \theta(x^3 y)$, we have that

$$q_1^2 q_2^{2bi} = q_2^{2aj}. \tag{5}$$

By (4) and (5), we have that $q_1^4 = 1$. This contradicts the fact that q_1 is not a root of unity. Therefore, there does not exist such an isomorphism. Therefore, we have proven the proposition. □

COROLLARY 3. *Let $V_{q_1}^{Z\times Z}$ and $V_{q_2}^{Z\times Z}$ be given simple Lie algebras. If $q_1 \neq q_2$, then $V_{q_1}^{Z\times Z}$ is not isomorphic to $V_{q_2}^{Z\times Z}$.*

Proof. It is straightforward by Proposition 1 and Theorem 1. □

4. Subalgebras of skew polynomials

Let $\lambda_{ij} \in F$; the skew polynomial ring $R(\lambda) = F[x_1, \cdots, x_n]$ with relations $x_i x_j = \lambda_{ij} x_j x_i$ has been called a quasipolynomial ring [1]. The corresponding Laurent polynomial ring $P(\lambda) = F[x_1^{\pm 1}, \cdots, x_n^{\pm 1}] = S$

obtained by inverting the x_i was studied by the paper [5]. S is simple, if and only if the center is F, if and only if there does not exist $\mathbf{u} = (u_1, \cdots, u_n) \in Z^n$ with $\mathbf{u}$ non-zero such that for all j, $1 \leq j \leq n$, $(\lambda_{1j})^{u_1} \cdots (\lambda_{nj})^{u_n} = 1$. The Laurent polynomial ring $P(\lambda) = F[x_1^{\pm 1}, \cdots, x_n^{\pm 1}] = S$ has a subring $F[x_1^{m_1 Z}, \cdots, x_n^{m_n Z}]$ where $m_1, \cdots, m_n$ are fixed non-negative integers. Similarly, $F[x_1^{m_1 Z}, \cdots, x_n^{m_n Z}]$ is simple, if and only if the center is F, if and only if there does not exist $\mathbf{u} = (u_1, \cdots, u_n) \in Z^n$ with $\mathbf{u}$ non-zero such that for all j, $1 \leq j \leq n$, $(\lambda_{1j})^{u_1} \cdots (\lambda_{nj})^{u_n} = 1$. Using the commutator of $F[x_1^{m_1 Z}, \cdots, x_n^{m_n Z}]$, we define the Lie algebra $V_\lambda^{m_1 Z \times \cdots \times m_n Z}$ as $V_q^{mZ \times nZ}$ in Section 1.

THEOREM 6. *If F-algebra $F[x_1^{m_1 Z}, \cdots, x_n^{m_n Z}]$ is simple, then the Lie algebra $V_\lambda^{m_1 Z \times \cdots \times m_n Z}$ is simple.*

Proof. The proof of this theorem is similar to the proof of Theorem 2, so let us omit the proof of the theorem. □

Some infinite dimensional Lie algebra L has a proper subalgebra S of L such that there is a Lie isomorphism θ from S to L. So the following is an interesting problem.

QUESTION. Does there exist an isomorphism from $V_\lambda^{m_1 Z \times \cdots \times m_n Z}$ to $V_\lambda^{Z \times \cdots \times Z}$?

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