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ON THE GENERAL SOLUTION OF A QUARTIC FUNCTIONAL EQUATION

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ABSTRACT. In this paper, we determine the general solution of the quartic equation $f(x+2y)+f(x-2y)+6f(x) = 4[f(x+y)+f(x-y)+6f(y)]$ for all $x, y \in \mathbb{R}$ without assuming any regularity conditions on the unknown function f. The method used for solving this quartic functional equation is elementary but exploits an important result due to M. Hosszú [3]. The solution of this functional equation is also determined in certain commutative groups using two important results due to L. Székelyhidi [5].

1. Introduction

In this paper, we determine the general solution of the functional equation

$$f(x+2y) + f(x-2y) + 6f(x) = 4\left[f(x+y) + f(x-y) + 6f(y)\right] \tag{1.1}$$

for all $x, y \in \mathbb{R}$ (the set of reals). We will solve the above functional equation using an elementary technique but without using any regularity conditions. Recently, J. M. Rassias [6] investigated the Hyers-Ulam stability of the functional equation (1.1). It was mentioned in [6] that $f(x) = x^4$ is a solution of the above functional equation because of the identity

$$(x+2y)^4 + (x-2y)^4 + 6x^4 = 4\left[(x+y)^4 + (x-y)^4 + 6y^4\right].$$

For the obvious reason, he called the above functional equation a quartic functional equation and any solution of the above equation a quartic function. He proves the following result: Let X be a normed linear

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space and Y be a real complete normed linear space. If $f : X \to Y$ satisfies the inequality

$$||f(x+2y)+f(x-2y)+6f(x)-4\left[f(x+y)+f(x-y)+6f(y)\right]|| \leq \varepsilon$$

for all $x, y \in X$ with a constant $\varepsilon \geq 0$ (independent of x and y), then there exists a unique quartic function $F : X \to Y$ such that $||F(x) - f(x)|| \leq \frac{17}{180}\varepsilon$ for all $x \in X$.

A function $A : \mathbb{R} \to \mathbb{R}$ is said to be *additive* if $A(x+y) = A(x)+A(y)$ for all x, $y \in \mathbb{R}$ (see [2]). Let $n \in \mathbb{N}$ (the set of natural numbers). A function $A_n : \mathbb{R}^n \to \mathbb{R}$ is called *n-additive* if it is additive in each of its variable. A function A_n is called *symmetric* if $A_n(x_1, x_2, \ldots, x_n) = A_n(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ for every permutation $\{\pi(1), \pi(2), \ldots, \pi(n)\}$ of $\{1, 2, \ldots, n\}$. If $A_n(x_1, x_2, \ldots, x_n)$ is a n-additive symmetric map, then $A^n(x)$ will denote the diagonal $A_n(x, x, \ldots, x)$. Further the resulting function after substitution $x_1 = x_2 = \cdots = x_\ell = x$ and $x_{\ell+1} = x_{\ell+2} = \cdots = x_n = y$ in $A_n(x_1, x_2, \ldots, x_n)$ will be denoted by $A^{\ell,n-\ell}(x, y)$. A 2-additive map is said to be *biadditive* map. The *diagonal* of a biadditive map B is the map $x \mapsto B(x, x)$. A function $Q : \mathbb{R} \to \mathbb{R}$ is said to be *quadratic* if $Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$ for all x, $y \in \mathbb{R}$. It is well known (see [1]) that a quadratic function from $\mathbb{R}$ into $\mathbb{R}$ is the diagonal of a symmetric biadditive map.

For $f : \mathbb{R} \to \mathbb{R}$, let Δ_h be the difference operator defined as follows:

$$\Delta_h f(x) = f(x+h) - f(x) \qquad \text{for } h \in \mathbb{R}.$$

Further, let $\Delta_h^0 f(x) = f(x)$, $\Delta_h^1 f(x) = \Delta_h f(x)$ and $\Delta_h \circ \Delta_h^n f(x) = \Delta_h^{n+1} f(x)$ for all $n \in \mathbb{N}$ and all $h \in \mathbb{R}$. Here $\Delta_h \circ \Delta_h^n$ denotes the composition of the operators Δ_h and Δ_h^n. For any given $n \in \mathbb{N} \cup \{0\}$, the functional equation

$$\Delta_h^{n+1} f(x) = 0$$

for all $x, h \in \mathbb{R}$ is well studied. It is known (see Kuczma [4]) that in the case where one deals with functions defined in $\mathbb{R}$ the last functional equation is equivalent to the Fréchet functional equation

$$\Delta_{h_1,\ldots,h_{n+1}} f(x) = 0 \tag{1.2}$$

where $x, h_1, \ldots, h_{n+1} \in \mathbb{R}$ and $\Delta_{h_1,\ldots,h_k} = \Delta_{h_k} \circ \cdots \circ \Delta_{h_1}$ for $k = 2, 3, \ldots, n+1$.

2. Some preliminary results

In this section, we will prove a couple of lemmas that will be needed to solve the quartic functional equation.

LEMMA 2.1. *If any function f satisfies the quartic functional equation (1.1) for all $x, y \in \mathbb{R}$, then it also satisfies the functional equation*

$$f(x+2y)+f(x-2y)=f(2x+y)+f(2x-y)+30f(y)-30f(x) \tag{2.1}$$

for all $x, y \in \mathbb{R}$.

Proof. Letting $x=0=y$ in (1.1), we obtain $f(0)=0$. Next, letting $x=0$ and $y=-x$ in (1.1), we have

$$f(-2x)+f(2x)-4f(x)-28f(-x)=0 \tag{2.2}$$

for all $x \in \mathbb{R}$. Similarly, letting $x=0$ and $y=x$ in (1.1) and then using $f(0)=0$, we obtain

$$f(2x)+f(-2x)-4f(-x)-28f(x)=0. \tag{2.3}$$

Hence the equations (2.2) and (2.3) imply $f(x)=f(-x)$ for all $x \in \mathbb{R}$, that is f is an even function. Next, we interchange x with y in (1.1) and using the fact that f is even, we have

$$f(2x+y)+f(2x-y)+6f(y)=4\left[f(x+y)+f(x-y)+6f(x)\right]. \tag{2.4}$$

From (1.1) and (2.4), we have the asserted result and the proof of the lemma is now complete. □

In the following lemma, we reduce the functional equation (2.1) to the functional equation (1.2) when $n=4$.

LEMMA 2.2. *If the function $F:\mathbb{R}\to\mathbb{R}$ satisfies the functional equation*

$$F(x+2y)+F(x-2y)=F(2x+y)+F(2x-y)+30F(y)-30F(x) \tag{2.5}$$

for all $x, y \in \mathbb{R}$, then F also satisfies the Fréchet functional equation

$$\Delta_{x_1,\ldots,x_5}F(x_0)=0 \tag{2.6}$$

for all $x_0, x_1, \ldots, x_5 \in \mathbb{R}$.

Proof. Let us write (2.5) as

$$F(x+2y)+F(x-2y)=F(2x+y)+F(2x-y)+F_1(x)+F_2(y) \tag{2.7}$$

where $F_1(x) = -30F(x)$ and $F_2(y) = 30F(y)$. Now we substitute $x_0 = x + 2y$ and $y_1 = x - 2y$ that is $x = \frac{1}{2}(x_0 + y_1)$ and $y = \frac{1}{4}(x_0 - y_1)$ in (2.7) to get

$$\begin{aligned} F(x_0) + F(y_1) &= F\left(\frac{5}{4}x_0 + \frac{3}{4}y_1\right) + F\left(\frac{3}{4}x_0 + \frac{5}{4}y_1\right) \\ &\quad + F_1\left(\frac{1}{2}(x_0 + y_1)\right) + F_2\left(\frac{1}{4}(x_0 - y_1)\right). \end{aligned} \tag{2.8}$$

Replacing x_0 by $x_0 + x_1$ in (2.8), we obtain

$$\begin{aligned} &F(x_0 + x_1) + F(y_1) \\ &= F\left(\frac{5}{4}(x_0 + x_1) + \frac{3}{4}y_1\right) + F\left(\frac{3}{4}(x_0 + x_1) + \frac{5}{4}y_1\right) \\ &\quad + F_1\left(\frac{1}{2}(x_0 + x_1 + y_1)\right) + F_2\left(\frac{1}{4}(x_0 + x_1 - y_1)\right). \end{aligned} \tag{2.9}$$

Subtracting (2.8) from (2.9), we have

$$\begin{aligned} &F(x_0 + x_1) - F(x_0) \\ &= F\left(\frac{5}{4}(x_0 + x_1) + \frac{3}{4}y_1\right) - F\left(\frac{5}{4}x_0 + \frac{3}{4}y_1\right) \\ &\quad + F\left(\frac{3}{4}(x_0 + x_1) + \frac{5}{4}y_1\right) - F\left(\frac{3}{4}x_0 + \frac{5}{4}y_1\right) \\ &\quad + F_1\left(\frac{1}{2}(x_0 + x_1 + y_1)\right) - F_1\left(\frac{1}{2}(x_0 + y_1)\right) \\ &\quad + F_2\left(\frac{1}{4}(x_0 + x_1 - y_1)\right) - F_2\left(\frac{1}{4}(x_0 - y_1)\right). \end{aligned} \tag{2.10}$$

Letting $y_2 = \frac{5}{4}x_0 + \frac{3}{4}y_1$ (that is, $y_1 = \frac{4}{3}y_2 - \frac{5}{3}x_0$) in (2.10), we see that

$$\begin{aligned} &F(x_0 + x_1) - F(x_0) \\ &= F\left(y_2 + \frac{5}{4}x_1\right) - F(y_2) \\ &\quad + F\left(\frac{5}{3}y_2 - \frac{4}{3}x_0 + \frac{3}{4}x_1\right) - F\left(\frac{5}{3}y_2 - \frac{4}{3}x_0\right) \\ &\quad + F_1\left(\frac{2}{3}y_2 - \frac{1}{3}x_0 + \frac{1}{2}x_1\right) - F_1\left(\frac{2}{3}y_2 - \frac{1}{3}x_0\right) \\ &\quad + F_2\left(-\frac{1}{3}y_2 + \frac{2}{3}x_0 + \frac{1}{4}x_1\right) - F_2\left(-\frac{1}{3}y_2 + \frac{2}{3}x_0\right). \end{aligned} \tag{2.11}$$

Now replacing x_0 by $x_0 + x_2$ in (2.11) and subtracting (2.11) from the resulting expression, we obtain

$$\begin{aligned}
& F(x_0 + x_1 + x_2) - F(x_0 + x_1) - F(x_0 + x_2) + F(x_0) \\
= \ & F\left(\frac{5}{3}y_2 - \frac{4}{3}(x_0 + x_2) + \frac{3}{4}x_1\right) - F\left(\frac{5}{3}y_2 - \frac{4}{3}x_0 + \frac{3}{4}x_1\right) \\
& - F\left(\frac{5}{3}y_2 - \frac{4}{3}(x_0 + x_2)\right) + F\left(\frac{5}{3}y_2 - \frac{4}{3}x_0\right) \\
& + F_1\left(\frac{2}{3}y_2 - \frac{1}{3}(x_0 + x_2) + \frac{1}{2}x_1\right) - F_1\left(\frac{2}{3}y_2 - \frac{1}{3}x_0 + \frac{1}{2}x_1\right) \\
& - F_1\left(\frac{2}{3}y_2 - \frac{1}{3}(x_0 + x_2)\right) + F_1\left(\frac{2}{3}y_2 - \frac{1}{3}x_0\right) \\
& + F_2\left(-\frac{1}{3}y_2 + \frac{2}{3}(x_0 + x_2) + \frac{1}{4}x_1\right) - F_2\left(-\frac{1}{3}y_2 + \frac{2}{3}x_0 + \frac{1}{4}x_1\right) \\
& - F_2\left(-\frac{1}{3}y_2 + \frac{2}{3}(x_0 + x_2)\right) + F_2\left(-\frac{1}{3}y_2 + \frac{2}{3}x_0\right).
\end{aligned} \tag{2.12}$$

Now we substitute $y_3 = \frac{5}{3}y_2 - \frac{4}{3}x_0$ (that is $y_2 = \frac{3}{5}y_3 + \frac{4}{5}x_0$) in (2.12) to get

$$\begin{aligned}
& F(x_0 + x_1 + x_2) - F(x_0 + x_1) - F(x_0 + x_2) + F(x_0) \\
= \ & F\left(y_3 - \frac{4}{3}x_2 + \frac{3}{4}x_1\right) - F\left(y_3 + \frac{3}{4}x_1\right) - F\left(y_3 - \frac{4}{3}x_2\right) \\
& + F(y_3) + F_1\left(\frac{2}{5}y_3 + \frac{1}{5}x_0 + \frac{1}{2}x_1 - \frac{1}{3}x_2\right) \\
& - F_1\left(\frac{2}{5}y_3 + \frac{1}{5}x_0 + \frac{1}{2}x_1\right) \\
& - F_1\left(\frac{2}{5}y_3 + \frac{1}{5}x_0 - \frac{1}{3}x_2\right) + F_1\left(\frac{2}{5}y_3 + \frac{1}{5}x_0\right) \\
& + F_2\left(-\frac{1}{5}y_3 + \frac{2}{5}x_0 + \frac{1}{4}x_1 + \frac{2}{3}x_2\right) \\
& - F_2\left(-\frac{1}{5}y_3 + \frac{2}{5}x_0 + \frac{1}{4}x_1\right) \\
& - F_2\left(-\frac{1}{5}y_3 + \frac{2}{5}x_0 + \frac{2}{3}x_2\right) \\
& + F_2\left(-\frac{1}{5}y_3 + \frac{2}{5}x_0\right).
\end{aligned} \tag{2.13}$$

Again we replace x_0 by $x_0 + x_3$ in (2.13) and then subtracting (2.13) from the resulting expression, we have

$$\begin{aligned}
& F(x_0+x_1+x_2+x_3) \\
& - F(x_0+x_1+x_2) - F(x_0+x_1+x_3) - F(x_0+x_2+x_3) \\
& + F(x_0+x_1) + F(x_0+x_2) + F(x_0+x_3) - F(x_0) \\
= \; & F_1\left(\frac{2}{5}y_3 + \frac{1}{5}(x_0+x_3) + \frac{1}{2}x_1 - \frac{1}{3}x_2\right) \\
& - F_1\left(\frac{2}{5}y_3 + \frac{1}{5}x_0 + \frac{1}{2}x_1 - \frac{1}{3}x_2\right) \\
& - F_1\left(\frac{2}{5}y_3 + \frac{1}{5}(x_0+x_3) + \frac{1}{2}x_1\right) + F_1\left(\frac{2}{5}y_3 + \frac{1}{5}x_0 + \frac{1}{2}x_1\right) \\
& - F_1\left(\frac{2}{5}y_3 + \frac{1}{5}(x_0+x_3) - \frac{1}{3}x_2\right) \\
& + F_1\left(\frac{2}{5}y_3 + \frac{1}{5}x_0 - \frac{1}{3}x_2\right) \\
& + F_1\left(\frac{2}{5}y_3 + \frac{1}{5}(x_0+x_3)\right) \\
& - F_1\left(\frac{2}{5}y_3 + \frac{1}{5}x_0\right) \\
& + F_2\left(-\frac{1}{5}y_3 + \frac{2}{5}(x_0+x_3) + \frac{1}{4}x_1 + \frac{2}{3}x_2\right) \\
& - F_2\left(-\frac{1}{5}y_3 + \frac{2}{5}x_0 + \frac{1}{4}x_1 + \frac{2}{3}x_2\right) \\
& - F_2\left(-\frac{1}{5}y_3 + \frac{2}{5}(x_0+x_3) + \frac{1}{4}x_1\right) \\
& + F_2\left(-\frac{1}{5}y_3 + \frac{2}{5}x_0 + \frac{1}{4}x_1\right) \\
& - F_2\left(-\frac{1}{5}y_3 + \frac{2}{5}(x_0+x_3) + \frac{2}{3}x_2\right) + F_2\left(-\frac{1}{5}y_3 + \frac{2}{5}x_0 + \frac{2}{3}x_2\right) \\
& + F_2\left(-\frac{1}{5}y_3 + \frac{2}{5}(x_0+x_3)\right) - F_2\left(-\frac{1}{5}y_3 + \frac{2}{5}x_0\right).
\end{aligned}$$

Letting $y_4 = \frac{2}{5}y_3 + \frac{1}{5}x_0$ (so that $y_3 = \frac{5}{2}y_4 - \frac{1}{2}x_0$) in the last equation, we obtain

$$
\begin{aligned}
& F(x_0+x_1+x_2+x_3) \\
& - F(x_0+x_1+x_2) - F(x_0+x_1+x_3) - F(x_0+x_2+x_3) \\
& + F(x_0+x_1) + F(x_0+x_2) + F(x_0+x_3) - F(x_0) \\
= \; & F_1\left(y_4+\frac{1}{2}x_1-\frac{1}{3}x_2+\frac{1}{5}x_3\right) - F_1\left(y_4+\frac{1}{2}x_1-\frac{1}{3}x_2\right) \\
& - F_1\left(y_4+\frac{1}{2}x_1+\frac{1}{5}x_3\right) + F_1\left(y_4+\frac{1}{2}x_1\right) \\
& - F_1\left(y_4-\frac{1}{3}x_2+\frac{1}{5}x_3\right) + F_1\left(y_4-\frac{1}{3}x_2\right) \\
& + F_1\left(y_4+\frac{1}{5}x_3\right) - F_1\left(y_4\right) \\
& + F_2\left(-\frac{1}{2}y_4+\frac{1}{2}x_0+\frac{1}{4}x_1+\frac{2}{3}x_2+\frac{2}{5}x_3\right) \\
& - F_2\left(-\frac{1}{2}y_4+\frac{1}{2}x_0+\frac{1}{4}x_1+\frac{2}{3}x_2\right) \\
& - F_2\left(-\frac{1}{2}y_4+\frac{1}{2}x_0+\frac{1}{4}x_1+\frac{2}{5}x_3\right) + F_2\left(-\frac{1}{2}y_4+\frac{1}{2}x_0+\frac{1}{4}x_1\right) \\
& - F_2\left(-\frac{1}{2}y_4+\frac{1}{2}x_0+\frac{2}{3}x_2+\frac{2}{5}x_3\right) + F_2\left(-\frac{1}{2}y_4+\frac{1}{2}x_0+\frac{2}{3}x_2\right) \\
& + F_2\left(-\frac{1}{2}y_4+\frac{1}{2}x_0+\frac{2}{5}x_3\right) - F_2\left(-\frac{1}{2}y_4+\frac{1}{2}x_0\right).
\end{aligned}
$$

Now we replace x_0 by x_0+x_4 in last equation and then subtracting the last equation from the resulting expression to get

$$
\begin{aligned}
& F(x_0+x_1+x_2+x_3+x_4) - F(x_0+x_1+x_2+x_3) \\
& - F(x_0+x_1+x_2+x_4) - F(x_0+x_1+x_3+x_4) \\
& - F(x_0+x_2+x_3+x_4) + F(x_0+x_1+x_2) \\
& + F(x_0+x_1+x_3) + F(x_0+x_1+x_4) + F(x_0+x_2+x_3) \\
& + F(x_0+x_2+x_4) + F(x_0+x_3+x_4) \\
& - F(x_0+x_1) - F(x_0+x_2) - F(x_0+x_3) - F(x_0+x_4) + F(x_0) \\
= \; & F_2\left(-\frac{1}{2}y_4+\frac{1}{2}(x_0+x_4)+\frac{1}{4}x_1+\frac{2}{3}x_2+\frac{2}{5}x_3\right) \\
& - F_2\left(-\frac{1}{2}y_4+\frac{1}{2}x_0+\frac{1}{4}x_1+\frac{2}{3}x_2+\frac{2}{5}x_3\right)
\end{aligned}
$$

$$
\begin{aligned}
&- F_2\left(-\frac{1}{2}y_4 + \frac{1}{2}(x_0 + x_4) + \frac{1}{4}x_1 + \frac{2}{3}x_2\right) \\
&+ F_2\left(-\frac{1}{2}y_4 + \frac{1}{2}x_0 + \frac{1}{4}x_1 + \frac{2}{3}x_2\right) \\
&- F_2\left(-\frac{1}{2}y_4 + \frac{1}{2}(x_0 + x_4) + \frac{1}{4}x_1 + \frac{2}{5}x_3\right) \\
&+ F_2\left(-\frac{1}{2}y_4 + \frac{1}{2}x_0 + \frac{1}{4}x_1 + \frac{2}{5}x_3\right) \\
&+ F_2\left(-\frac{1}{2}y_4 + \frac{1}{2}(x_0 + x_4) + \frac{1}{4}x_1\right) - F_2\left(-\frac{1}{2}y_4 + \frac{1}{2}x_0 + \frac{1}{4}x_1\right) \\
&- F_2\left(-\frac{1}{2}y_4 + \frac{1}{2}(x_0 + x_4) + \frac{2}{3}x_2 + \frac{2}{5}x_3\right) \\
&+ F_2\left(-\frac{1}{2}y_4 + \frac{1}{2}x_0 + \frac{2}{3}x_2 + \frac{2}{5}x_3\right) \\
&+ F_2\left(-\frac{1}{2}y_4 + \frac{1}{2}(x_0 + x_4) + \frac{2}{3}x_2\right) - F_2\left(-\frac{1}{2}y_4 + \frac{1}{2}x_0 + \frac{2}{3}x_2\right) \\
&+ F_2\left(-\frac{1}{2}y_4 + \frac{1}{2}(x_0 + x_4) + \frac{2}{5}x_3\right) - F_2\left(-\frac{1}{2}y_4 + \frac{1}{2}x_0 + \frac{2}{5}x_3\right) \\
&- F_2\left(-\frac{1}{2}y_4 + \frac{1}{2}(x_0 + x_4)\right) + F_2\left(-\frac{1}{2}y_4 + \frac{1}{2}x_0\right).
\end{aligned}
$$

First we let $y_5 = -\frac{1}{2}y_4 + \frac{1}{2}x_0$ and then replace x_0 by $x_0 + x_5$ in the last equation. Further subtracting the last equation from the resulting equation, we obtain

$$
\begin{aligned}
&F(x_0 + x_1 + x_2 + x_3 + x_4 + x_5) \\
&- F(x_0 + x_1 + x_2 + x_3 + x_4) - F(x_0 + x_1 + x_2 + x_3 + x_5) \\
&- F(x_0 + x_1 + x_2 + x_4 + x_5) - F(x_0 + x_1 + x_3 + x_4 + x_5) \\
&- F(x_0 + x_2 + x_3 + x_4 + x_5) \\
&+ F(x_0 + x_1 + x_2 + x_3) + F(x_0 + x_1 + x_2 + x_4) \\
&+ F(x_0 + x_1 + x_2 + x_5) + F(x_0 + x_1 + x_3 + x_4) \\
&+ F(x_0 + x_1 + x_3 + x_5) + F(x_0 + x_1 + x_4 + x_5) \\
&+ F(x_0 + x_2 + x_3 + x_4) + F(x_0 + x_2 + x_3 + x_5) \\
&+ F(x_0 + x_2 + x_4 + x_5) + F(x_0 + x_3 + x_4 + x_5) \\
&- F(x_0 + x_1 + x_2) - F(x_0 + x_1 + x_3) - F(x_0 + x_1 + x_4) \\
&- F(x_0 + x_1 + x_5) - F(x_0 + x_2 + x_3) - F(x_0 + x_2 + x_4)
\end{aligned}
$$

$$
\begin{aligned}
&- F(x_0 + x_2 + x_5) - F(x_0 + x_3 + x_4) - F(x_0 + x_3 + x_5) \\
&- F(x_0 + x_4 + x_5) + F(x_0 + x_1) + F(x_0 + x_2) \\
&+ F(x_0 + x_3) + F(x_0 + x_4) + F(x_0 + x_5) - F(x_0) \\
= {} & 0
\end{aligned}
$$

which is (2.6). The proof of the lemma is now complete. □

The following lemma is a special case of a more general result due to Hosszu [3], and will be instrumental in determining the general solution of (1.1).

LEMMA 2.3. *The map F from $\mathbb{R}$ into $\mathbb{R}$ satisfies the functional equation (2.6) for all $x_0, x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}$ if and only if F is given by*

$$F(x) = A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x), \tag{2.14}$$

where $A^0(x) = A^0$ is an arbitrary constant and $A^n(x)$ is the diagonal of a n-additive symmetric function $A_n : \mathbb{R}^n \to \mathbb{R}$ for $n = 1, 2, 3, 4$.

3. Solution of equation (1.1) on reals

Now we are ready to prove our main theorem.

THEOREM 3.1. *The function $f : \mathbb{R} \to \mathbb{R}$ satisfies the quartic functional equation (1.1) for all $x, y \in \mathbb{R}$, if and only if f is of the form*

$$f(x) = A^4(x),$$

where $A^4(x)$ is the diagonal of a 4-additive symmetric function $A_4 : \mathbb{R}^4 \to \mathbb{R}$.

Proof. From Lemma 2.1 we see that the functional equation (1.1) implies equation

$$f(x + 2y) + f(x - 2y) = f(2x + y) + f(2x - y) + 30f(y) - 30f(x).$$

From Lemma 2.2 we see that f satisfies the Fréchet functional equation

$$\Delta_{x_1,\ldots,x_5} f(x_0) = 0 \tag{3.1}$$

for all $x_0, x_1, \ldots, x_5 \in \mathbb{R}$. The general solution of the equation (3.1) can be obtained from Lemma 2.3 as

$$f(x) = A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x), \tag{3.2}$$

where $A^0(x) = A^0$ is an arbitrary constant and $A^n(x)$ is the diagonal of the n-additive symmetric map $A_n : \mathbb{R}^n \to \mathbb{R}$ for $n = 1, 2, 3, 4$.

Since f is an even function (see Lemma 2.1), we have $A^3(x) \equiv 0$ and $A^1(x) \equiv 0$. Hence from (3.2), we have

$$f(x) = A^4(x) + A^2(x) + A^0. \tag{3.3}$$

Letting (3.3) into (1.1) and noting that

$$A^4(2y) = 16\,A^4(y),\;\; A^2(2y) = 4\,A^2(y),\;\; A^{2,2}(x, 2y) = 4\,A^{2,2}(x, y),$$

$$A^4(x + y) + A^4(x - y) = 2A^4(x) + 2A^4(y) + 12A^{2,2}(x, y),$$

and

$$A^2(x + y) + A^2(x - y) = 2A^2(x) + 2A^2(y)$$

for all $x, y \in \mathbb{R}$, we get

$$24A^2(y) + 24A^0 = 0 \tag{3.4}$$

for all $y \in \mathbb{R}$. Hence $A^2(y) \equiv 0$ and $A^0 = 0$. Thus from (3.3) we have $f(x) = A^4(x)$ and the proof of the theorem is now complete. □

4. Solution of equation (1.1) on commutative groups

In this section, we solve the functional equation (1.1) on commutative groups with some additional requirements.

A group $\mathbb{G}$ is said to be *divisible* if for every element $b \in \mathbb{G}$ and every $n \in \mathbb{N}$, there exists an element $a \in \mathbb{G}$ such that $na = b$. If this element a is unique, then $\mathbb{G}$ is said to be *uniquely divisible*. In a uniquely divisible group, this unique element a is denoted by $\frac{b}{n}$. The equation $na = b$ has a solution is equivalent to say that the multiplication by n is surjective. Similarly, the equation $na = b$ has a unique solution is equivalent to say that the multiplication by n is bijective. Thus the notions of n-divisibility and n-unique divisibility refer, respectively, to surjectivity and bijectivity of the multiplication by n.

The proof of Theorem 3.1 can be generalized to abstract structures by using the more general result of Hosszu [3] instead of Lemma 2.3. Since the proof of the following theorem is identical to the proof of Theorem 3.1, we omit its proof.

THEOREM 4.1. *Let $\mathbb{G}$ and $\mathbb{S}$ be uniquely divisible abelian groups. The function $f : \mathbb{G} \to \mathbb{S}$ satisfies the quartic functional equation* (1.1) *for all $x, y \in \mathbb{G}$, if and only if f is of the form*

$$f(x) = A^4(x),$$

where $A^4(x)$ is the diagonal of a 4-additive symmetric function $A_4 : \mathbb{G}^4 \to \mathbb{S}$.

Theorem 4.1 can be further strengthened using two important results due to Székelyhidi [5]. The results needed for this improvements are the followings (see [5], pp. 70–72):

THEOREM 4.2. *Let $\mathbb{G}$ be a commutative semigroup with identity, $\mathbb{S}$ a commutative group and n a nonnegative integer. Let the multiplication by $n!$ be bijective in $\mathbb{S}$. The function $f : \mathbb{G} \to \mathbb{S}$ is a solution of Fréchet functional equation*

$$\Delta_{x_1,\ldots,x_{n+1}} f(x_0) = 0 \qquad \forall\, x_0, x_1, \ldots, x_{n+1} \in \mathbb{G} \tag{4.1}$$

if and only if f is a polynomial of degree at most n.

THEOREM 4.3. *Let $\mathbb{G}$ and $\mathbb{S}$ be commutative groups, n a nonnegative integer, ϕ_i, ψ_i additive functions from $\mathbb{G}$ into $\mathbb{G}$ and $\phi_i(\mathbb{G}) \subseteq \psi_i(\mathbb{G})$ $(i = 1, 2, \ldots, n+1)$. If the functions $f, f_i : \mathbb{G} \to \mathbb{S}$ $(i = 1, 2, \ldots, n+1)$ satisfy*

$$f(x) + \sum_{i=1}^{n+1} f_i\left(\phi_i(x) + \psi_i(y)\right) = 0 \tag{4.2}$$

then f satisfies Fréchet functional equation (4.1).

Using these two theorems, Theorem 4.1 can be further improved.

THEOREM 4.4. *Let $\mathbb{G}$ and $\mathbb{S}$ be commutative groups. Let the multiplication by 2 be surjective in $\mathbb{G}$ and let the multiplication by 24 be bijective in $\mathbb{S}$. The function $f : \mathbb{G} \to \mathbb{S}$ satisfies the quartic functional equation* (1.1) *for all $x, y \in \mathbb{G}$, if and only if f is of the form*

$$f(x) = A^4(x), \tag{4.3}$$

where $A^4(x)$ is the diagonal of a 4-additive symmetric function $A_4 : \mathbb{G}^4 \to \mathbb{S}$.

Proof. Using the unique divisibility of $\mathbb{S}$ by 24 we can rewrite the functional equation (1.1) in the form

$$f(x) + \sum_{i=1}^{5} f_i\left(\phi_i(x) + \psi_i(y)\right) = 0 \tag{4.4}$$

where $f_1 = f_2 = \frac{1}{6}f$, $f_3 = f_4 = -\frac{2}{3}f$, $f_5 = -4f$, $\phi_1(x) = \phi_2(x) = \phi_3(x) = \phi_4(x) = x$, $\phi_5(x) = 0$, and $\psi_1(y) = 2y$, $\psi_2(y) = -2y$, $\psi_3(y) = \psi_5(y) = y$, $\psi_4(y) = -y$. From these ϕ_i and ψ_i we see that $\phi_i(\mathbb{G}) \subseteq \psi_i(\mathbb{G})$ for $i = 1, 2, 3, 4, 5$. Hence by Theorem 4.3, f satisfies the Fréchet functional equation (4.1). By Theorem 4.2, f is a polynomial of degree at most 4, that is f is of the form

$$f(x) = A^0(x) + A^1(x) + A^2(x) + A^3(x) + A^4(x),$$

where $A^0(x) = A^0$ is an arbitrary constant, $A_1 \in Hom(\mathbb{G}, \mathbb{S})$, and $A^n(x)$ is the diagonal of a n-additive symmetric function $A_n : \mathbb{G}^n \longrightarrow \mathbb{S}$, $n \in \{2, 3, 4\}$. Interchanging y with $-y$ in (1.1), one obtains $24f(y) = 24f(-y)$ and hence f is an even function. The same argument as used in the last ten lines of the proof of Theorem 3.1 shows that any function of the form (4.3) actually satisfies (1.1). □

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