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A LOCAL APPROXIMATION METHOD FOR THE SOLUTION OF K–POSITIVE DEFINITE OPERATOR EQUATIONS

C. E. Chidume and S. J. Aneke

Abstract. In this paper we extend the definition of K-positive definite operators from linear to Fréchet differentiable operators. Under this setting, we derive from the inverse function theorem a local existence and approximation results corresponding to those of Theorems 1 and 2 of the authors [8], in an arbitrary real Banach space. Furthermore, an asymptotically K-positive definite operator is introduced and a simplified iteration sequence which converges to the unique solution of an asymptotically K-positive definite operator equation is constructed.

1. Introduction

Let H_1 be a dense subspace of a Hilbert space, H. An operator T with domain $D(T) \supseteq H_1$ is called *continuously H_1 invertible* if the range of T, $R(T)$, with T considered an operator restricted to H_1 is dense in H and T has a bounded inverse on $R(T)$. Let H be a complex and separable Hilbert space and A be a linear unbounded operator defined on a dense domain $D(A)$ in H with the property that there exists a continuously $D(A)$–invertible closed linear operator K with $D(A) \subseteq D(K)$, and a constant $c > 0$ such that

$$\langle Au, Ku\rangle \geq c||Ku||^2, \qquad u \in D(A), \tag{1.1}$$

then A is called *K-positive definite* (*Kpd*) (see e.g. [13]). If $K = I$ (the identity operator) inequality (1.1) reduces to $\langle Au, u\rangle \geq c||u||^2$, and in this case, A is called *positive definite.* If in addition $c = 0$, A is called *positive* operator (or *accretive* operator). Positive definite operators have been studied by various authors (see, e.g. [1, 2, 3, 6, 7, 15]. It is clear that

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the class of Kpd operators contains among others, the class of positive definite operators, and also contains the class of invertible operators (when $K = A$ as its subclasses. Furthermore, Petryshyn [13] remarked that for a proper choice of K, the ordinary differential operators of odd order, the weakly elliptic partial differential operators of odd order, are members of the class of Kpd operators . Moreover, if the operators are bounded, the class of Kpd operators forms a subclass of *symmetrizable operators* studied by Reid [15].

In [13], Petryshyn proved the following theorem.

THEOREM P. *If A is a Kpd operator and $D(A) = D(K)$, then there exists a constant $\alpha > 0$ such that for all $u \in D(K)$,*

$$||Au|| \leq \alpha||Ku||.$$

Furthermore, the operator A is closed, $R(A) = H$ and the equation $Au = f$, $f \in H$, has a unique solution.

In the case that K is bounded and A is closed, F. E. Browder [3] obtained a result similar to the second part of Theorem P.

In [8], the authors extended the notion of a K-positive definite (Kpd) operator to *real separable Banach spaces*, X. In particular, if X is a real separable Banach space with a *strictly convex dual*, we proved that the equation $Au = f, f \in X$, where A is a Kpd operator with the same domain as K has a unique solution. Furthermore, if $X = Lp$ (*or* lp), $p \geq 2$, and is separable, we constructed an iteration process which converges strongly to this solution.

Precisely, the following theorems were proved in [8].

THEOREM CA1. *Let X be a real separable Banach space with a strictly convex dual and let A be a Kpd operator with $D(A) = D(K)$. Suppose that for all $x, y \in D(K)$,*

$$\langle Ax, j(Ky)\rangle = \langle Kx, j(j(Ay)\rangle,$$

then there exists a constant $\omega > 0$ such that for $x \in D(A)$,

$$||Ax|| \leq \omega||Kx||.$$

Furthermore, the operator A is closed, $R(A) = X$ and the equation $Ax = h$, for each $h \in X$, has a unique solution.

THEOREM CA2. *Suppose $X = Lp$ or $l_p, p \geq 2$, and is separable. Suppose $A : D(A) \subseteq X \to X$ is a Kpd operator with $D(A) = D(K) =$*

$R(K)$ and that for all $x, y \in D(A)$, $\langle Ax, j(Ky)\rangle = \langle Kx, j(Ay)\rangle$. Define the sequence $\{x_n\}$ iteratively by

$$x_0 \in D(K) \tag{1.2}$$

$$x_{n+1} = x_n + t_n K^{-1} r_n, \ n \geq 0, \tag{1.3}$$

$$t_n = \frac{\langle Br_n, j(Kr_n)\rangle}{(p-1)||Br_n||^2}, \text{ where } B = KAK^{-1} \tag{1.4}$$

and

$$r_n = f - Ax_n, \qquad f \in R(K). \tag{1.5}$$

If A and K commute, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to the unique solution of $Ax = f$ in X.

In [10], the authors extended the above result to a larger space, the *q-uniformly smooth Banach spaces.*

Let K be a subset of a real Banach space E. A map $T : K \to K$ is called a *strict contraction* if there exists $k \in [0, 1)$ such that $||Tx - Ty|| \leq k||x - y||$, and it is called *nonexpansive* if, for arbitrary $x, y \in K$, $||Tx - Ty|| \leq ||x - y||$. The map T is called *pseudocontractive* if, for each $x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq ||x - y||^2.$$

In 1972, Goebel and Kirk [11] introduced a class of mappings generalizing the class of nonexpansive operators.

Let K be a nonempty subset of a normed space E. A mapping $T : K \to K$ is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$, $k_n \geq 1$, such that $\lim_{n\to\infty} k_n = 1$, and $||T^n x - T^n y|| \leq k_n||x - y||$ for each x, y in K and for each integer $n \geq 1$.

Later in 1993, Bruck *et. al.* introduced and studied another class of asymptotic nonexpansive maps. A mapping $T : K \to K$ is called *asymptotically nonexpansive in the intermediate sense* (see e.g., Bruck *et. al.* [5]) provided T is uniformly continuous and

$$\limsup_{n\to\infty} \left\{ \sup_{x,y\in K} (||T^n x - T^n y|| - ||x - y||) \right\} \leq 0.$$

Asymptotic *pseudocontractive operators* have also been introduced and studied, first by Schu (see e.g., [16]) and then by a host of other authors, as a generalization of asymptotic nonexpansive maps. $T : K \to K$ is called *asymptotically pseudocontractive* if there exists a sequence $\{k_n\}$, $k_n \geq 1$, $\lim k_n = 1$ such that

$$\langle T^n x - T^n y, j(x - y)\rangle \leq k_n||x - y||^2$$

for each $x,\ y \in K$.

It is easy to see that asymptotically pseudocontractive maps include the asymptotic nonexpansive ones. These classes of maps have been studied by various authors.

Motivated by Goebel and Kirk [11], Bruck *et. al.* [5] and Schu [16], we now introduce the class of *asymptotically* K-positive definite operators.

DEFINITION 1.1. Let X be a Banach space and let A be a linear unbounded operator defined on a dense domain, $D(A)$, in X. The operator A will be called *asymptotically K-positive definite Kpd* if there exist a continuously $D(A)-$invertible closed linear operator K with $D(K) \supseteq D(A) \supseteq R(A)$, and a constant $c > 0$ such that for $j(Ku) \in J(Ku)$,

$$(1.6) \qquad \langle K^{n-1}Au, j(K^n u)\rangle \geq ck_n||K^n u||^2, \qquad u \in D(A),$$

where $\{k_n\}$ is a real sequence such that $k_n \geq 1, \lim_{n\to\infty} k_n = 1$.

It is our purpose in this paper to extend the notion of a *kpd* operator to Fréchet differentiable operators. Under this setting, a local existence theorem and an iterative scheme which converges to the unique solution of the *Kpd* operator equation in an *arbitrary Banach spaces,* are derived from the inverse function theorem. Moreover, we introduce and study a new notion-*asymptotically K-positive definite operators.*

2. Preliminaries

Let E be a real normed linear space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = ||x||^2 = ||f||^2\},$$

where $\langle .,.\rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single−valued and if E is uniformly smooth (equivalently if E^* is uniformly convex) then J is uniformly continuous on bounded subsets of E . We shall denote the single-valued duality mapping by j. The *modulus of smoothness* of E is the function $\rho_E :\ [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{||x+y|| + ||x-y||}{2} - 1 : ||x|| = 1, ||y|| = \tau\right\}.$$

E is said to be uniformly smooth if $\lim_{\tau\to 0^+} \frac{\rho_E(\tau)}{\tau} = 0$.

LEMMA 2.1. (see, e.g., [14]) *Let E be a real uniformly smooth Banach space and let J be the normalized duality map on E. Then for any given*

$x, y \in E$, the following inequality holds:

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x)\rangle + \max\{||x||, 1\}||y||b(||y||), \forall j(x) \in J(x),$$

where b is a continuous nondecreasing function satisfying the conditions: $b(0) = 0$, $b(ct) \leq cb(t), \forall c \geq 1$, where b is a continuous nondecreasing function satisfying the conditions: $b(0) = 0$, $b(ct) \leq cb(t), \forall c \geq 1$.

3. Main results

Now, we state the inverse function theorem and sketch its proof. We derive from the proof of the theorem that the iteration scheme in Theorem 2 of [8] converges to the unique solution of $Ax = f$ in an arbitrary real Banach space, provided $||f - Ax_0||$ is sufficiently small.

THEOREM 3.1. (The inverse function Theorem) *Suppose X, Y are Banach spaces and $A : X \to Y$ is such that A has uniformly continuous Fréchet derivatives in a neighborhood of some point x_0 of X. Then if $A'(x_0)$ is a linear homeomorphism of X onto Y, then A is a local homeomorphism of a neighborhood $U(x_0)$ of x_0 to a neighborhood of $A(x_0)$.*

Proof. Let $A(x_0) = y_0$. We first determine ρ so that $A(x_0 + \rho) = y$ provided $||y - y_0||$ is sufficiently small, or equivalently

$$A(x_0 + \rho) - A(x_0) = y - y_0. \tag{3.1}$$

Since A is C^1 at x_0 and $A'(x_0)$ is invertible, then (3.1) and Taylor's Theorem imply that $A'(x_0)\rho + R(x_0, \rho) = y - y_0$, i.e.,

$$\rho = [A'(x_0)]^{-1}[(y - y_0) - R(x_0, \rho)],$$

where the remainder

$$R(x_0, \rho) = A(x_0 + \rho) - A(x_0) - A'(x_0)\rho = o(||\rho||).$$

We show that (3.1) has one and only one solution for $||\rho||$ sufficiently small, by proving that the operator

$$T\rho = [A'(x_0)]^{-1}\{y - y_0 - R(x_0, \rho)\}$$

is a contraction mapping of a sphere $S(0,\epsilon)$ in X into itself, for some ϵ sufficiently small. For any $\rho_1, \rho_2 \in S(0,\epsilon)$,

$$\begin{aligned} & A'(x_0)(T\rho_2 - T\rho_1) \\ = \; & R(x_0,\rho_1) - R(x_0,\rho_2) \\ = \; & A(x_0+\rho_1) - A(x_0+\rho_2) - A'(x_0)(\rho_1-\rho_2) \\ = \; & \int_0^1 \{A'(x_0 + t\rho_1 + (1-t)\rho_2) - A'(x_0)\}(\rho_1-\rho_2)dt. \end{aligned}$$

Hence

$$(3.2) \quad \begin{aligned} & ||T\rho_2 - T\rho_1|| \\ \leq \; & \int ||[A'(x_0)]^{-1}|| ||A'(x_0 + t\rho_1 + (1-t)\rho_2) - A'(x_0)|| ||\rho_1 - \rho_2|| dt. \end{aligned}$$

Since A is a C^1 mapping, the middle term of the last integral can be made arbitrarily small by choosing $||\rho_1||$, $||\rho_2||$ sufficiently small; and hence for some constant $0 \leq \alpha < 1$ (and independent of $y - y_0$) and sufficiently small $\epsilon > 0$, $||T\rho_2 - T\rho_1|| \leq \alpha||\rho_2 - \rho_1||$ for all $\rho_1, \rho_2 \in S(0,\epsilon)$. Furthermore, T maps $S(0,\epsilon)$ into itself. For, $||T\rho|| = ||T\rho - T(0)|| + ||T(0)|| \leq \alpha||\rho|| + ||T(0)||$ and $||T(0)|| = ||[A'(x_0)]^{-1}(y - y_0)|| < (1-\alpha)\epsilon$ provided $||y - y_0|| < (1-\alpha)\epsilon||[A'(x_0)]^{-1}||^{-1}$. Hence T is a contraction map of $S(0,\epsilon)$ into itself. By the contraction mapping theorem, T has a unique fixed point ρ^* in $S(0,\delta)$ where $\delta \leq \epsilon$ is chosen so small that $A(S(0,\delta) \subset S(y_0, (1-\alpha)\epsilon||[A'(x_0)]^{-1}||^{-1})$. Reversing the steps in the argument, one finds that $A(x_0 + \rho) = y$ has one and only one solution when $||y - y_0||$ and $||\rho||$ are sufficiently small. Also, $A^{-1}(y) = x$ is a well-defined and continuous mapping from a sphere $S(y_0, \eta)$ in Y to X. □

COROLLARY 3.2. *Under the conditions of Theorem 3.1, the iteration sequence*

$$x_{n+1} = x_n + [A'(x_0)]^{-1}r_n, \; r_n = [y - A(x_n)],$$

converges to the unique solution of $A(x) = y$ *in* $U(x_0)$.

Proof. Since the operator T in the proof of Theorem 3.1 is a contraction map, the sequence $\rho_n = T\rho_{n-1}$ converges to the unique fixed point of T. From Theorem 3.1, for $||y - A(x_0)||$ sufficiently small, $A(x) = y$ has a unique solution $x = x_0 + \rho^*$, where ρ^* is the limit of the sequence $\rho_0 = 0$, $\rho_{n+1} = T\rho_n$. It then follows that the sequence $x_n = x_0 + \rho_n$

converges to $x_0 + \rho^*$, the unique solution of $A(x) = y$ in $U(x_0)$. Now,

$$\begin{aligned} x_n = x_0 + \rho_n &= x_0 + T\rho_{n-1} \\ &= x_0 + [A'(x_0)]^{-1}[y - A(x_0)] - R(x_0, \rho_{n-1})] \\ &= x_0 + [A'(x_0)]^{-1}[y + A'(x_0)\rho_{n-1} - A(x_0 + \rho_{n-1})] \\ &= x_0 + \rho_{n-1} + [A'(x_0)]^{-1}[y - A(x_{n-1})] \\ &= x_{n-1} + [A'(x_0)]^{-1}[y - A(x_{n-1}]. \end{aligned}$$

□

Henceforth, an operator A defined on a dense domain $D(A)$ of a real Banach space will be called K-positive definite if A is Fréchet differentiable and there exist a continuously $D(A)-$invertible closed linear operator K with $D(A) \subseteq D(K)$, and a constant $c > 0$ such that for $j \in J(Ku)$, we have

$$\langle Au, j\rangle \geq ||Ku||^2, \qquad u \in D(A).$$

COROLLARY 3.3. *Suppose A is a Kpd operator defined on a dense domain $D(A)$ of a real Banach space, X with range $R(T)$ in X. If for some $x_0 \in X$, $A'(x_0)$ is a linear homeomorphism of X onto Y, then A is a linear homeomorphism of a neighborhood $U(x_0)$ of x_0 to a neighborhood of $A(x_0)$. Furthermore, if $||y - A(x_0)||$ is sufficiently small, the sequence $x_{n+1} = x_n + K^{-1}r_n$, where $r_n = [y - A(x_n)]$ converges to the unique solution of $A(x) = y$ in $U(x_0)$.*

Proof. $A'(x_0)$ satisfies the condition for K in the definition of a *Kpd* operator. Hence setting $K = A'(x_0)$ in Theorem 3.1, we are done. □

REMARK 3.4. If X is a *separable* Banach space, with a *strictly convex dual* and the operator A is linear, a global existence result was obtained in the domain of A, $D(A)$ in Theorem 1 of [8].

REMARK 3.5. The iteration scheme $\{x_n\}$ in Corollary 3.3 above corresponds to the one of Theorem 2 in [8] by setting $t_n \equiv 1$. In Theorem 2 of [8], the scheme $x_{n+1} = x_n + t_nK^{-1}$ converges globally to the unique solution of $A(x) = y$ in Lp (*or* lp), $p \geq 2$, while in Corollary 3.2 above the corresponding scheme converges locally to the unique solution of $A(x) = y$ in some neighborhood of a point x_0 in a real Banach space X. Furthermore, under this setting, the operator A need not be linear but Fréchet differentiable.

By writing our iteration scheme in the form of Theorem CO [10], we prove the following Theorem for asymptotically K-positive definite operators in a *uniformly convex* Banach space.

THEOREM 3.6. *Suppose X is a real uniformly smooth Banach space. Suppose A is an asymptotically K-positive definite operator defined in a neighborhood $U(x_0)$ of a real uniformly smooth Banach space, X. Define the sequence $\{x_n\}$ by $x_0 \in U(x_0)$, $x_{n+1} = x_n + r_n$, $n \geq 0$, $r_n = K^{-1}y - K^{-1}A(x_n)$, $y \in R(A)$. Then $\{x_n\}$ converges strongly to the unique solution of $A(x) = y \in U(x_0)$.*

Proof. By the linearity of K we obtain $Kr_{n+1} = Kr_n - Ar_n$. Using Lemma 2.1 and Definition 1.1, we obtain the following estimates:

$$\begin{aligned} & ||K^n r_{n+1}||^2 \\ \leq\ & ||K^n r_n - K^{n-1}Ar_n||^2 \\ \leq\ & ||K^n r_n||^2 - 2\langle K^{n-1}Ar_n, j(K^n r_n)\rangle \\ & + \max\{||K^n r_n||, 1\}||K^{n-1}Ar_n||b(||K^{n-1}Ar_n||) \\ \leq\ & ||K^n r_n||^2 - 2ck_n||K^n r_n||^2 \\ & + \max\{||K^n r_n||, 1\}||K^{n-1}Ar_n||b(||K^{n-1}Ar_n||) \\ \leq\ & ||K^n r_n||^2 - 2ck_n||K^n r_n||^2 \\ & + (||K^n r_n|| + 1)||K^{n-1}Ar_n||b(||K^{n-1}Ar_n||). \end{aligned} \tag{3.3}$$

Since A is Fréchet differentiable and by the properties of the function b, the quantity $||K^{n-1}Ar_n||b(||K^{n-1}Ar_n||)$ can be made as small as possible in a small neighborhood $U(x_0)$ of X. Infact there exists c such that

$$||K^{n-1}Ar_n||b(||K^{n-1}Ar_n||) \leq ck_n||K^n r_n||^2. \tag{3.4}$$

Inequality (3.4) implies that the sequence $||K^n r_n||_{n=0}^{\infty}$ is monotone decreasing and hence converges to some real number $\beta \geq 0$. Inequalities (3.3) and (3.4) imply that

$$\lim_{n\to\infty} ||K^n r_n|| = 0.$$

Since K is continuously $D(A)$−invertible, this implies that $r_n \to 0$. Since A has a bounded inverse, this implies $x_n \to A^{-1}y$, the unique solution of $Ax = y$ in $U(x_0)$. □

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