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EXISTENCE OF A PRODUCT SUBMANIFOLD OF AN LP-SASAKIAN MANIFOLD WITH A COEFFICIENT α

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ABSTRACT. In this paper, we have proved that there exists a product submanifold of an LP-Sasakian manifold with a coefficient α and the manifold and its product submanifold possess a CR-structure respectively.

0. Introduction

In [1] Matsumoto introduced the notion of Lorentzian paracontact structure and studied its several properties. Then I. Mihai and R. Rosca [2] defined the same notion independently and obtained many results in this manifold. The notion of an LP-Sasakian manifold with a coefficient α is introduced by De et al. [3].

The purpose of this paper is to prove that for an LP-Sasakian manifold with a coefficient α there exists a product submanifold of this manifold. Also it is shown that the LP-Sasakian manifold with a coefficient α and its product submanifold both possess CR-structure.

1. Preliminaries

Let M^n be an n-dimensional real differentiable manifold of differentiability class C^∞ endowed with a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g of type $(0,2)$ such that for each point $x \in M^n$, the tensor $g_x : T_xM^n \times T_xM^n \to R$ is a non-degenerate inner product of signature $(-,+,+,\dots,+)$, where T_xM^n denotes the tangent vector space of M^n at x and R is the real

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number space, which satisfies

$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1 \tag{1.1}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \tag{1.2}$$

$$g(X, \xi) = \eta(X) \tag{1.3}$$

for all vector fields X, Y tangent to M^n. Such a structure (ϕ, ξ, η, g) is termed as Lorentzian paracontact structure [1] and the manifold M^n is known as Lorentzian paracontact manifold.

Also in a Lorentzian paracontact manifold, the following relations hold:

$$\phi\xi = 0, \ \eta \circ \phi = 0, \ \text{rank } \phi = n - 1.$$

If we put

$$\Phi(X, Y) = g(X, \phi Y), \tag{1.4}$$

then $\Phi(X, Y)$ is a symmetric (0,2)-tensor field [1], that is

$$\Phi(X, Y) = \Phi(Y, X). \tag{1.5}$$

A Lorentzian paracontact manifold M^n is called *Lorentzian para-Sasakian* (briefly, *LP-Sasakian*) *manifold with a coefficient* α [3] if the following relations hold:

$$\begin{aligned}(\nabla_Z \Phi)(X, Y) = \alpha\Big[&\{g(X, Z) + \eta(X)\eta(Z)\}\,\eta(Y) \\ &+ \{g(Y, Z) + \eta(Y)\eta(Z)\}\,\eta(X)\Big], \ (\alpha \neq 0)\end{aligned} \tag{1.6}$$

and

$$\Phi(X, Y) = \frac{1}{\alpha}(\nabla_X \eta)(Y), \tag{1.7}$$

for vector fields X, Y, Z tangent to M^n, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function.

From (1.6) and (1.7) we have [3]

$$(\nabla_X \phi)(Y) = \alpha\,[g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi] \tag{1.8}$$

and

$$(\nabla_X \eta)(Y) = \alpha\Phi(X, Y) = (\nabla_Y \eta)(X), \tag{1.9}$$

for vector fields X, Y tangent to M^n.

From (1.9), it follows that 1-form η is closed in M^n and

$$\nabla_X \xi = \alpha\phi X. \tag{1.10}$$

It is to be noted that for α =1, an LP-Sasakian manifold with a coefficient α reduces to an LP-Sasakian manifold [1].

In an LP-Sasakian manifold with a coefficient α, the torsion tensor N_{ij}^h defined by

$$\begin{aligned} N_{ij}^h &= \phi_i^k \nabla_k \phi_j^h - \phi_j^k \nabla_k \phi_i^h - \left(\nabla_i \phi_j^k - \nabla_j \phi_i^k \right) \phi_k^h \\ &\quad - \left(\nabla_i \eta_j - \nabla_j \eta_i \right) \xi^h \end{aligned}$$

vanishes [3].

2. Product submanifold of an LP-Sasakian manifold with a coefficient α

In this section, the integrability of certain distributions on an LP-Sasakian manifold with a coefficient α and properties of integral submanifolds are studied.

Let M^n be an LP-Sasakian manifold with a coefficient α with the structure (ϕ, ξ, η, g). The tensor ϕ has constant eigenvalues 1,-1 and 0. Let s, t be the multiplicities of the eigenvalues 1 and -1 respectively. The eigenvalue 0 has multiplicity 1. $M^n (n = s + t + 1)$ has an orthonormal frame $\{e_1, \ldots, e_s, e_{s+1}, \ldots, e_{s+t}, e_n = \xi\}$ such that $\phi(e_i) = e_i, \phi(e_{s+v}) = -e_{s+v}$ and $\phi(e_n) = 0$, for $i = 1, 2, \ldots, s$ and $v = 1, 2, \ldots, t$.

We define the following distributions on an LP-Sasakian manifold M^n with a coefficient α:

$$(2.1)_1 \qquad D^+ = \{X \in T(M^n) : \phi X = X\} \text{ with } \dim(D^+) = s$$

$$(2.1)_2 \qquad D^- = \{X \in T(M^n) : \phi X = -X\} \text{ with } \dim(D^-) = t$$

$$(2.1)_3 \qquad D^0 = \{X \in T(M^n) : \phi X = 0\} \text{ with } \dim(D^0) = 1$$

$$(2.1)_4 \qquad D = \{X \in T(M^n) : \eta(X) = 0\}.$$

Here

$$(2.2) \qquad D = D^+ \oplus D^- \text{ and } T(M^n) = D \oplus D^0.$$

THEOREM 2.1. *In an LP-Sasakian manifold with a coefficient α, the distributions D^+, D^- and D are integrable.*

Proof. For $X, Y \in D^+$, by using the definition of D^+ and (1.8) we have by virtue of $\eta \circ \phi = 0$

$$\begin{aligned} \phi[X,Y] &= (\nabla_Y \phi)(X) - (\nabla_X \phi)(Y) + [X,Y] \\ &= \alpha[\eta(X)Y - \eta(Y)X] + [X,Y] \\ &= [X,Y]. \end{aligned} \tag{2.3}$$

Similarly, for $X, Y \in D^-$, by using the definition of D^- and (1.8) we have

$$\phi[X,Y] = -[X,Y]. \tag{2.4}$$

Next, let $X, Y \in D$. Then as η is closed, using the definition of D we get

$$\eta[X,Y] = -\{X\eta(Y) - Y\eta(X) - \eta[X,Y]\} = -2d\eta(X,Y) = 0. \tag{2.5}$$

Hence from (2.3), (2.4) and (2.5) the theorem follows. □

On the other hand we give a definition.

DEFINITION 2.1. A distribution D' on an LP-Sasakian manifold M^n with a coefficient α is said to be *parallel* if for $X \in D'$ and $Y \in T(M^n)$, $\nabla_Y X \in D'$.

THEOREM 2.2. *In an LP-Sasakian manifold M^n with a coefficient α, if N, N^+ and N^- be the maximal integral submanifolds of the distributions D, D^+ and D^- respectively, then N is locally a Riemannian direct product $N^+ \times N^-$.*

Proof. From Theorems 2.1 and (2.2), it follows that D, D^+ and D^- are integrable and D is the direct sum of D^+ and D^-. We will now prove that both D^+ and D^- are parallel.

For $X \in D$, we put

$$\phi X = PX + FX, \text{where } PX \in D \text{ and } FX \in D^0. \tag{2.6}$$

Using $\eta \circ \phi = 0$ and the definition of D, we have from (2.6) $\eta(FX) = 0$, which shows that $FX = 0$, for $X \in D$.

Then from (2.6), we get

$$\phi X = PX, \text{ for } X \in D. \tag{2.7}$$

Now from (2.7) it follows from (1.1) that

$$P^2 X = P(\phi X) = \phi^2 X = X, \text{ for } X \in D. \tag{2.8}$$

Thus P is an almost product structure on N [4].

Let $\bar{\nabla}$ be the operator of covariant differentiation on N and h be the second fundamental tensor of N.

Then Gauss formula is given by

$$\nabla_X Y = \bar{\nabla}_X Y + h(X,Y), \text{ for } X, Y \in D \text{ and } h(X,Y) \in D^0. \tag{2.9}$$

As $h(X,Y) \in D^0$, we have

$$\phi h(X,Y) = 0, \text{for } X, Y \in D. \tag{2.10}$$

Now for $X \in D^+$ and $Y \in D$, using (1.8), (2.7), (2.8), (2.9) and (2.10) and the definition of D^+, we get

$$\begin{aligned} P(\bar{\nabla}_Y X) &= \phi(\bar{\nabla}_Y X) = \phi(\nabla_Y X) = \phi(\nabla_Y(\phi X)) \\ &= \phi^2(\nabla_Y X) = \phi^2(\bar{\nabla}_Y X) = P^2(\bar{\nabla}_Y X) = \bar{\nabla}_Y X, \end{aligned}$$

which shows that D^+ is parallel. Similarly it can be proved that D^- is also parallel. Hence the theorem follows. □

COROLLARY 2.1. *In an LP-Sasakian manifold M^n with a coefficient α, there exists a submanifold N which is locally a product Riemannian manifold.*

COROLLARY 2.2. *In an LP-Sasakian manifold, there exists a product submanifold.*

3. CR-structure on an LP-Sasakian manifold with a coefficient α

It is known that a differentiable manifold with a Riemannian metric admits a CR-structure if and only if there is a differentiable distribution D and a (1,1)-tensor field J on M such that for all vector fields X and Y in D [5],

$$\begin{gathered} J^2 X = -X, \\ [J,J](X,Y) \equiv [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY] = 0, \\ [JX,JY] - [X,Y] \in D. \end{gathered}$$

In an analogous way, a Lorentzian paracontact manifold M is said to admit a *CR-structure* if and only if there is a differentiable distribution D and a (1,1)-tensor field ϕ on M such that for all vector fields X and Y in D,

$$\phi^2 X = X, \tag{3.1}$$

$$[\phi,\phi](X,Y) \equiv [\phi X,\phi Y] + [X,Y] - \phi[\phi X,Y] - \phi[X,\phi Y] = 0, \tag{3.2}$$

$$[\phi X,\phi Y] + [X,Y] \in D. \tag{3.3}$$

PROPOSITION 3.1. *An LP-Sasakian manifold M^n with a coefficient α possesses a CR-structure.*

Proof. Let us consider the distribution D given by (2.1) and (2.2) on M^n. Then for $X \in D$, we have from (2.8), $\phi^2 X = X$.

As the torsion tensor $[\phi, \phi]$ vanishes identically in an LP-Sasakian manifold with coefficient α [3], we have

$$[\phi, \phi](X, Y) = 0, \text{for } X, Y \in T(M^n)$$

which gives

$$[\phi X, \phi Y] + [X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \eta([X, Y])\xi = 0, \tag{3.4}$$

where we have used (1.1).

By the definition of D and using the integrability of D, we have

$$\eta([X, Y]) = 0, \text{for } X, Y \in D. \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$[\phi X, \phi Y] + [X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y] = 0, \text{ for all } X, Y \in D. \tag{3.6}$$

Also as D is integrable, for all $X, Y \in D$, by (2.7) we have

$$[\phi X, \phi Y] + [X, Y] = [PX, PY] + [X, Y] \in D.$$

Hence the proposition is proved. □

Owing to corollary 2.1, we have the following:

PROPOSITION 3.2. *The product submanifold N of an LP-Sasakian manifold M^n with a coefficient α possesses CR-structure.*

Proof. For $X \in D^+$, from (2.8) we have

$$P^2 X = X.$$

Also for $X, Y \in D^+$, we have from (3.6) by using (2.7)

$$[PX, PY] + [X, Y] - P[PX, Y] - P[X, PY] = 0.$$

As D^+ is parallel, for $X, Y \in D^+$, we have

$$\phi[X, Y] = \phi\nabla_X Y - \phi\nabla_Y X = \nabla_X Y - \nabla_Y X = [X, Y],$$

which shows that

$$[X, Y] \in D^+, \text{for } X, Y \in D^+.$$

Now by definition of D^+, we have

$$[PX, PY] + [X, Y] = [X, Y] + [X, Y] \in D^+, \text{for } X, Y \in D^+.$$

This proves the proposition. □

References

[1] K. Matsumoto, *On Lorentzian paracontact manifolds*, Bull. Yamagata Univ., Natur. Sci. **12** (1989), 151–156.

[2] I. Mihai and R. Rosca, *On Lorentzian P-Sasakian manifolds*, Classical Analysis, World Sci. Publishing, Singapore, (1992), 155–169.

[3] U. C. De, A. A. Shaikh and A. Sengupta, *On LP-Sasakian Manifolds with a coefficient* α, Kyungpook Math. J. **42** (2002), 177–186.

[4] K. Yano, *Differential Geometry on complex and almost complex spaces*, Pergamon Press, New York, 219.

[5] A. Bejancu, *Geometry of CR-submanifolds*, D. Reidel Publishing Co., Holland, 1986, pp.128–129.

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