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THE INVARIANT PROPERTIES OF INFINITE MATRIX ALGEBRAS

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Abstract. In this paper, we characterize the compact sets full-invariant properties, dual invariant properties and quasi full invariant properties in a class of infinite matrix algebras.

Let ω be the space of all scalar valued sequences, ϕ the space with only finitely many non-zero terms, m the space of bounded sequences, λ and μ the linear subspaces of ω. The β dual space of λ is defined by: $\lambda^\beta = \{x : \sum_i x_i y_i \text{ is convergent for each } y \in \lambda\}$.

We say a non-zero vector sequence $\{z^{(n)}\}$ in ω is a block sequence if there exists a strictly increasing sequence $\{k_j\}$ of integers with $k_0 = 0$ such that

$$z^{(n)} = (0, 0, \cdots, 0, z^{(n)}_{k_{n-1}+1}, \cdots, z^{(n)}_{k_n}, 0, \cdots).$$

λ is said to have the signed-weak gliding hump property (s-wghp) if given any $x = (x_i) \in \lambda$ and any block sequence $\{x^{(k)}\}$ with $x = \sum_{k=1}^\infty x^{(k)}$ (pointwise sum), then each index sequence $\{m_k\}$ has a further subsequence $\{n_k\}$ and a signed sequence $\{s_k\}$ with $s_k = 1$ or $s_k = -1 (k \in \mathbf{N})$, such that $\tilde{x} = \sum_{k=1}^\infty s_k x^{(n_k)} \in \lambda$ (pointwise sum) [1].

A sequence $\{t^{(n)}\}_{n=1}^\infty$ in ω is said to be coordinatewise convergent to $t^{(0)}$ if for each $i \in \mathbf{N}$, $\{t_i^{(n)}\}_{n=1}^\infty$ converges to $t_i^{(0)}$.

If $\phi \subseteq \lambda$, then λ and λ^β are in duality with respect to the bilinear map $[t, u] = \sum_i u_i t_i$ for $x = (x_i) \in \lambda, y = (y_i) \in \lambda^\beta$. The weak topology, Mackey topology and strong topology of λ from this pairing are denoted by $\sigma(\lambda, \lambda^\beta)$, $\tau(\lambda, \lambda^\beta)$ and $\beta(\lambda, \lambda^\beta)$, the similar notations are used for the weak topology, Mackey topology and strong topology of λ^β.

If $\phi \subseteq \mu \subseteq \lambda$, and λ, μ have the s-wghp, let (λ, μ) denote all scalar matrices A such that A transforms λ into μ. It is obviously that (λ, μ)

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is a vector space under the usually matrix addition and matrix scalar multiplication operations.

If $A = (a_{ij})$ and $B = (b_{ij}) \in (\lambda, \mu)$, it is clear that for each $i \in \mathbf{N}$, $\{a_{ij}\}_{j=1}^{\infty} \in \lambda^{\beta}$. For each $j \in \mathbf{N}$, let e_j be the scalar sequence with a 1 in the j^{th} coordinate and 0 elsewhere, it follows from $e_j \in \lambda$ and $Be_j \in \mu$ that $\{b_{ij}\}_{i=1}^{\infty} \in \mu$. Note that $\lambda^{\beta} \subseteq \mu^{\beta}$, so for $i, j \in \mathbf{N}$, the series $\sum_{k=1}^{\infty} a_{ik}b_{kj}$ is convergent. Thus, we can define the matrix multiplication of A and B by $(\sum_{k=1}^{\infty} a_{ik}b_{kj})$. Wu Junde and Lu Shijie in [10] proved that (λ, μ) is an algebra with respect to the matrix multiplication.

Let $\mathcal{M}$ and $\mathcal{D}$ be the bounded subsets families of $(\lambda, \sigma(\lambda, \lambda^{\beta}))$ and $(\mu^{\beta}, \sigma(\mu^{\beta}, \mu))$, respectively, and satisfy that $\bigcup_{M \in \mathcal{M}} M = \lambda$, $\bigcup_{D \in \mathcal{D}} D = \mu^{\beta}$. Wu Junde and Lu Shijie in [10] proved also that (λ, μ) with respect to the topology is determined by the following neighborhood basis $\{V(0, M, D) : M \in \mathcal{M}, D \in \mathcal{D}\}$ of 0 in (λ, μ) is a topological algebra, where

$$V(0, M, D) = \{A \in (\lambda, \mu) : \sup_{u \in D, x \in M} |u(Ax)| \leq 1\}.$$

The topology defined by $\mathcal{M}$ and $\mathcal{D}$ is said to be a polar topology of (λ, μ) and denoted by $T_{\mathcal{MD}}$. If $\mathcal{M}$ and $\mathcal{D}$ are the all finitely subsets of $(\lambda, \sigma(\lambda, \lambda^{\beta}))$ and $(\mu^{\beta}, \sigma(\mu^{\beta}, \mu))$, respectively, then the topology of (λ, μ) defined by $\mathcal{M}$ and $\mathcal{D}$ is said to be weak topology, and denote by T_{σ}, similar, T_{τ} and T_{β} denote the Mackey topology and strong topology of infinite matrix algebra (λ, μ).

As is known, studying the invariant property is a crucial problem in mathematics. In the locally convex spaces case, the boundedness, closed convexity and subseries convergent are all duality-invariant property [4, 7].

In [2, 8], Li Ronglu, Cui Chengri and Wu Junde showed that c_0 or $l^p (0 < p < \infty)$-multiplier convergent series is full invariant, i.e., the c_0-multiplier convergent and the l^p-multiplier convergent $(0 < p < \infty)$ of series are invariant with respect to all polar topologies. In [9], Wu Junde and Lu Shijie obtained a nice full invariant property characterization in compact operator spaces. Note that invariant property is rare and rare in the locally convex space theory. In this paper, we study the compact sets invariant property in infinite matrix algebra (λ, μ).

We say that (λ, μ) has the compact sets full invariant property if all polar topology between T_{σ} and T_{β} have the same compact sets, (λ, μ) has the compact sets dual invariant property if all polar topology between T_{σ} and T_{τ} have the same compact sets, (λ, μ) has the compact sets quasi full invariant property if all polar topology between T_τ and T_β have the same compact sets.

LEMMA L ([3]). *Let $\phi \subseteq \mu \subseteq \lambda$, λ and μ have the s-wghp. If T is a polar topology of (λ, μ), then $W \subseteq (\lambda, \mu)$ is a T-compact set if and only if W is T-bounded and if $\{A^{(n)}\} \subseteq W$ is coordinatewise convergent to $A^{(0)}$, then $\{A^{(n)}\}$ is also T-convergent to $A^{(0)}$, and $A^{(0)} \in W$.*

LEMMA 2 ([10]). *If $\phi \subseteq \mu \subseteq \lambda$ and λ and μ have both the s-wghp, then for each $A \in (\lambda, \mu)$, A^* transforms μ^β into λ^β, and if $x \in \lambda, u \in \mu^\beta$, we have:*

$$[Ax, u] = [x, A^*u],$$

where A^* is the transpose matrix of A.

LEMMA 3 ([5, 6]). *If $\phi \subseteq \lambda$ and λ has the s-wghp, then λ^β is weak sequentially complete.*

LEMMA 4 ([11]). *If (X, T_0) is a sequentially complete locally convex space, and $\{x_n\}$ is a convergent sequence, then the absolutely convex closure of $\{x_n\}$ is a T_0-compact set.*

We say that a sequence space λ has the compact sets full invariant property, if for all topologies between the weak topology and strong topology λ has the same compact sets; and λ has the compact sets dual invariant property, if for all topologies between the weak topology and Mackey topology λ has the same compact sets; similarly, we can define the compact sets quasi full invariant properties of λ.

The sequence space (λ, T_1) is said to be an AK-space, if for each $u = (u_i) \in \lambda$, then $(u_1, u_2, \cdots, u_n, 0, 0, \cdots)$ converges to u with respect to the topology T_1.

Our main results are:

THEOREM 1. *If λ, μ have the s-wghp and $\phi \subseteq \mu \subseteq \lambda$, $\lambda^{\beta\beta} = \lambda$, then the following states are equivalent:*

(1) *$(\lambda^\beta, \tau(\lambda^\beta, \lambda))$ and $(\mu, \tau(\mu, \mu^\beta))$ are both barrelled spaces;*

(2) *(λ, μ) has the compact sets quasi full invariant property.*

(3) *$(\lambda^\beta, \beta(\lambda^\beta, \lambda))$ and $(\mu, \beta(\mu, \mu^\beta))$ are both AK-spaces;*

(4) *$(\lambda^\beta, \beta(\lambda^\beta, \lambda))$ and $(\mu, \beta(\mu, \mu^\beta))$ are both separable spaces;*

(5) *Each weakly bounded and coordinate convergent sequence in λ or μ^β must be weakly convergent sequence.*

Proof. Suppose (1) is true. Let $M \subseteq \lambda$ be a weakly bounded set. Without loss of generality, we may assume M is an absolutely convex weakly closed bounded set. Since $(\lambda^\beta, \tau(\lambda^\beta, \lambda))$ is a barrelled space, so each pointwise bounded continuous linear functional family M of

$(\lambda^\beta, \tau(\lambda^{\beta,\lambda}))$ must be equicontinuous, thus M must be also relatively weakly compact set [7]. It follows from M is a weakly closed set that M is a weakly compact set. Similarly, we can prove that each absolutely convex weakly bounded closed sets of μ^β is also weakly compact set. So, $T_\tau = T_\beta$. So $((\lambda, \mu), T_\tau)$ and $((\lambda, \mu), T_\beta)$ have the same compact sets. (1) $\Rightarrow$ (2) is proved.

Suppose (2) is true. Since μ and λ^β have both the s-wghp, so $(\lambda^\beta, \tau(\lambda^\beta, \lambda))$ and $(\mu, \tau(\mu, \mu^\beta))$ are both AK-spaces [1]. Let $u = (u_i) \in \lambda^\beta$, then $(u_1, u_2, \cdots, u_n, 0, 0, \cdots)$ converges to u with respect to the topology $\tau(\lambda^\beta, \lambda)$. Denote

$$A^{[n]} = \begin{pmatrix} u_1 & u_2 & \cdots & u_n & 0 & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & & & \end{pmatrix}, A = \begin{pmatrix} u_1 & u_2 & \cdots & u_n & \cdots \\ 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \cdots & 0 & \cdots \end{pmatrix}.$$

It is easily to know that $\{A^{[n]}\} \subseteq (\lambda, \mu)$ converges to A with respect to the topology T_τ.

By (2) that $\{A^{[n]}\}$ convergent to A with respect to T_β. Thus we follow that $(u_1, u_2, \cdots, u_n, 0, \cdots)$ converges to u with respect to the topology $\beta(\lambda^\beta, \lambda)$, i.e., $(\lambda^\beta, \beta(\lambda^\beta, \lambda))$ is an AK-space. Similar, we may prove another conclusion. (2) $\Rightarrow$ (3) is true.

(3) $\Rightarrow$ (4) is clear.

Let (4) hold, then there exists a countable set Q such that Q is a dense subset of λ^β with respect to the topology $\beta(\lambda^\beta, \lambda)$. Let $\{x^{(n)}\} \subseteq \lambda$ be weakly bounded and coordinate convergent, now we show that $\{x^{(n)}\}$ is weakly convergence. Since λ^β has the s-wghp and $\lambda^{\beta\beta} = \lambda$, it follows from Lemma 3 that $(\lambda, \sigma(\lambda, \lambda^\beta))$ is a sequentially complete space, thus, we only need to prove that for each $u \in \lambda^\beta$, $\{[x^{(n)}, u]\}$ is convergent.

If not, there exist $u^{(0)} \in \lambda^\beta, \epsilon_0 > 0$, and a subsequence $\{x^{(n_k)}\}$ of $\{x^{(n)}\}$, without loss of generality, we may assume $\{x^{(n)}\}$ is just $\{x^{(n_k)}\}$, such that

$$(1) \qquad |[x^{(n+1)} - x^{(n)}, u^{(0)}]| \geq \epsilon_0, n = 1, 2, \cdots .$$

Since Q is a countable set, $\{x^{(n)}\}$ is a weakly bounded set, by the diagonal process, we can obtain a subsequence $\{x^{(n_i)}\}$ of $\{x^{(n)}\}$, we may also assume that $\{x^{(n_i)}\}$ is just $\{x^{(n)}\}$, such that for each $u \in Q$, $\{[u, x^{(n)}]\}$ is convergent. Note that Q is a dense set of $(\lambda^\beta, \beta(\lambda^\beta, \lambda))$ and $\{x^{(n+1)} - x^{(n)}\}$ is a weakly bounded set, so there exists $u^{(1)} \in Q$ such that

$$|[u^{(0)} - u^{(1)}, x^{(n+1)} - x^{(n)}]| \leq \frac{\epsilon_0}{4}, n = 1, 2, \cdots .$$

On the other hand, since $\{x^{(n)}\}$ is pointwise convergent on Q, so there exists $n_1 \in \mathbf{N}$ such that

$$|[u^{(1)}, (x^{(n+1)} - x^{(n)})]| \leq \frac{\epsilon_0}{4}, n \geq n_1.$$

Thus we follows that

$$|[u^{(0)}, (x^{(n+1)} - x^{(n)})]| \leq \frac{\epsilon_0}{2}, n \geq n_1.$$

This contradicts inequality (1) and so $\{x^{(n)}\}$ is weakly convergent. Similar, we can prove another conclusion, too. (4) $\Rightarrow$ (5) is proved.

If (5) hold, in order to prove $(\lambda^\beta, \tau(\lambda^\beta, \lambda))$ is a barrelled space, we only need to prove each weakly bounded closed set in λ must be weakly compact set. This can follows from hypothesis (5) and Lemma 1 easily. So $(\lambda^\beta, \tau(\lambda^\beta, \lambda))$ is a barrelled space. Similarly, $(\mu, \tau(\mu, \mu^\beta))$ is also a barrelled space. (5) $\Rightarrow$ (1) is true. The theorem is proved. □

THEOREM 2. *If $\phi \subseteq \mu \subseteq \lambda$, λ, μ and λ^β all have the s-wghp, $\lambda^{\beta\beta} = \lambda$, then (λ, μ) has the compact sets full invariant property if and only if λ^β and μ have also the compact sets full invariant properties.*

Proof. Necessity. Let $\{u^{(n)}\} \subseteq \lambda^\beta$ be weakly convergent to $u^{(0)}$. Denote

$$A^{(n)} = \begin{pmatrix} u_1^{(n)} & u_2^{(n)} & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & & & \end{pmatrix},$$

$$A^{(0)} = \begin{pmatrix} u_1^{(0)} & u_2^{(0)} & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & & & \end{pmatrix}.$$

It is clear that $\{A^{(n)}\}$ converges to $A^{(0)}$ with respect to the weakly topology T_σ. By the hypothesis of theorem that $\{A^{(n)}\}$ must also converge to $\{A^{(0)}\}$ with respect to the topology T_β. Thus, we may prove that λ^β has the compact sets full invariant property. Similar, μ has also the compact sets full invariant property. The necessity is proved.

Sufficiency. If $u = (u_i) \in \lambda^\beta$, it is clear that for each $x \in \lambda$, $\sum_{i=n+1}^{\infty} u_i x_i \to 0$, i.e., $(u_1, u_2, \cdots, u_n, 0, \cdots)$ converges to u with respect to $\sigma(\lambda^\beta, \lambda)$, it follows from hypothesis condition of theorem that $(u_1, u_2, \cdots, u_n, 0, \cdots)$ converges to u with respect to $\beta(\lambda^\beta, \lambda)$. Thus, $(\lambda^\beta, \beta(\lambda^\beta, \lambda))$ is an AK-space. It follows from Theorem 1 that each weakly bounded and coordinate convergent sequence of λ must be $(\lambda, \beta(\lambda, \lambda^\beta))$ convergent. It follows from Lemma 1 that each weakly bounded closed set in

λ must be also $\beta(\lambda, \lambda^{\beta})$) compact set. Similar, for μ^{β} we may prove the same conclusion.

Now, by Lemma 1, we only need to prove $((\lambda, \mu), T_{\sigma})$ and $((\lambda, \mu), T_{\beta})$ have the same convergent sequences.

If $\{A^{(n)}\} \subseteq (\lambda, \mu)$ and $\{A^{(n)}\}$ is T_{σ}-convergent to 0, but $\{A^{(n)}\}$ is not T_{β}-convergent to 0, then there exist a weakly bounded sequence $\{x^{(n)}\}$ in λ and a weakly bounded sequence $\{u^{(n)}\}$ in μ^{β}, such that $\{[A^{(n)}x^{(n)}, u^{(n)}]\}$ does not convergent to 0. From the above proof and Lemma 1, we may assume that $\{x^{(n)}\}$ is $\beta(\lambda, \lambda^{\beta})$-convergent to $x^{(0)}$, $\{u^{(n)}\}$ is $\beta(\mu^{\beta}, \mu)$-convergent to $u^{(0)}$, and for each $n \geq n_0$,

$$|[x^{(n)} - x^{(0)}, u^{(n)}A^{(n)}]| \leq \frac{\epsilon_0}{4}, \tag{2}$$

$$|[(u^{(n)} - u^{(0)}), (A^{(n)}x^{(0)})]| \leq \frac{\epsilon_0}{4}. \tag{3}$$

Note that $\{A^{(n)}\}$ is T_{σ} convergent to 0, therefore, there exists $n_1 \in \mathbf{N}$ such that whenever $n \geq n_1$, we have

$$|[u^{(0)}, (A^{(n)}x^{(0)})]| \leq \frac{\epsilon_0}{4}. \tag{4}$$

It follows from (2), (3) and (4) that whenever $n \geq \max(n_0, n_1)$,

$$|[u^{(n)}, (A^{(n)}x^{(n)})]| \leq \frac{3\epsilon_0}{4}.$$

This is a contradict. The sufficiency is true. The theorem is proved. □

THEOREM 3. *If $\phi \subseteq \mu \subseteq \lambda$, $\lambda, \mu, \lambda^{\beta}, \mu^{\beta}$ all have the s-wghp, $\lambda^{\beta\beta} = \lambda$, $\mu^{\beta\beta} = \mu$, then the following states are equivalent:*

(1) (λ, μ) has the compact sets dual invariant property;

(2) λ^{β} and μ have both the compact sets dual invariant property.

(3) λ and μ^{β} have both the compact sets dual invariant property.

Proof. (1) $\to$ (2) is the same as the necessity proof of the Theorem 2.

(2) $\to$ (3). If $\{x^{(n)}\} \subseteq \lambda$ is weak convergent to 0 but is not Mackey convergent to 0, then there exists an absolutely convex weak compact subset M of λ^{β} such that $\{x^{(n)}\}$ does not convergent to 0 uniformly on M. Without loss generality, we may assume that there exists a sequence $\{u^{(n)}\} \subseteq M$ and $\varepsilon_0 > 0$ such that

$$|[u^{(n)}, x^{(n)}]| \geq \varepsilon, n \in \mathbf{N}. \tag{5}$$

By Lemma 1, we may assume that $\{u^{(n)}\}$ is weak convergent to $u^{(0)}$.

On the other hand, it follows from Lemma 3 and Lemma 4 that the absolutely convex weak closure of $\{x^{(n)}\}$ is also weak compact set. Note that $\{u^{(n)}\}$ is weak convergent to $u^{(0)}$, by the hypothesis (2) that $\{u^{(n)}\}$ is also Mackey topology convergent to $u^{(0)}$. It follows from the construction of Mackey topology that

$$[u^{(n)} - u^{(0)}, x^{(n)}] \to 0. \tag{6}$$

Note that

$$[u^{(0)} x^{(n)}] \to 0 \tag{7}$$

is also true. Combine (5), (6) and (7) we obtain a constracdict. So λ has the compact sets dual invariant property. Similar, me may prove the another case.

(3)$\to$ (1). If not, it follows from Lemma 1 that there exist a sequence $\{A^{(n)}\}$ of (λ, μ) which is T_σ-convergent to 0, but $\{A^{(n)}\}$ is not T_τ-convergent to 0. Without loss of generality, we may assume that there exist a sequence $\{x^{(n)}\}$ of λ which is weakly convergent to $x^{(0)}$, and a sequence $\{u^{(n)}\}$ of μ^β which is weakly convergent to $u^{(0)}$, such that for each $n \in \mathbf{N}$,

$$|[u^{(n)}, A^{(n)} x^{(n)}]| \geq \varepsilon_0. \tag{8}$$

Now, we prove that $\{u^{(n)} A^{(n)}\} \subseteq \lambda^\beta$ is weakly convergent to 0. At first, as the above paragraph, we may prove that $(\mu, \sigma(\mu, \mu^\beta))$ and $(\mu, \tau(\mu, \mu^\beta))$ have the same convergent sequences.

For each $x \in \lambda$, it follows from Lemma 2 that $\{[u^{(n)} A^{(n)}, x]\} = \{[u^{(n)}, A^{(n)}(x)]\}$, since $\{A^{(n)}\}$ is T_σ-convergent to 0, so $\{A^{(n)} x\}$ must be also $\tau(\mu, \mu^\beta)$-convergent to 0.

Note that μ has the s-wghp and $\{u^{(n)}\}$ is weakly convergent to $u^{(0)}$, it follows from Lemma 3, 4 that the absolutely convex weakly closure of $\{u^{(n)}\}$ is a weakly compact set. Thus, by the structure of $\tau(\mu, \mu^\beta)$ that$\{[u^{(n)}, A^{(n)}(x)]\}$ convergent to 0. That is, $\{u^{(n)} A^{(n)}\}$ is weakly convergent to 0.

Since $\{x^{(n)}\}$ is $\sigma(\lambda, \lambda^\beta)$-convergent to $x^{(0)}$, it follows from the hypothesis 3 that $\{x^{(n)}\}$ must be $\tau(\lambda, \lambda^\beta)$ convergent to $x^{(0)}$. Similar, $\{u^{(n)}\}$ is also $\tau(\mu^\beta, \mu)$-convergent to $\mu^{(0)}$. Note that $\{A^{(n)} x^{(0)}\} \subseteq \mu$ is also weakly convergent to 0, use the same method we may prove that the absolutely convex weakly closure of $\{u^{(n)} A^{(n)}\}$ and $\{A^{(n)} x^{(0)}\}$ are both weakly compact sets. Thus, there exists $n_0 \in \mathbf{N}$, whenever $n \geq n_0$, we have

$$|[u^{(n)} A^{(n)}, x^{(n)} - x^{(0)}]| < \frac{\epsilon_0}{4}, \tag{9}$$

$$(10) \qquad |[u^{(n)} - u^{(0)}, A^{(n)}x^{(0)}]| < \frac{\epsilon_0}{4}.$$

Since $\{A^{(n)}\}$ is T_σ-convergent to 0, so there exists $n_1 \in \mathbf{N}$, whenever $n \geq n_1$, we have

$$(11) \qquad |[A^{(n)}x^{(0)}, u^{(0)}]| < \frac{\epsilon_0}{4}.$$

Thus, it follows from (9), (10) and (11) that the inequality (8) is not true. This is a contradiction. The theorem is proved. □

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