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PROJECTIVELY FLAT FINSLER SPACES WITH CERTAIN (α,β)-METRICS

HONG-SUH PARK, HA-YONG PARK,
BYUNG-DOO KIM AND EUN-SEO CHOI

ABSTRACT. The (α,β)-metric is a Finsler metric which is constructed from a Riemannian metric α and a differential 1-form β. In this paper, we discuss the projective flatness of Finsler spaces with certain (α,β)-metrics ([5]) in a locally Minkowski space.

1. Introduction

A Finsler metric function $L(x,y)$ is called an (α,β)-metric if L is a positively homogeneous function of a Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ and a differential 1-form $\beta = b_iy^i$ of degree one. The specially interesting examples of (α,β)-metric are the Randers metric and Kropina metric.

A Finsler space $F^n = (M^n, L)$ is called a locally Minkowski space ([7]) if M^n is covered by coordinate neighborhood system (x^i) in each of which L is a function of y^i only. A Finsler space $F^n = (M^n, L)$ is called projectively flat if F^n is projective to a locally Minkowski space.

The condition for a Randers space to be projectively flat was given by Hashiguchi-Ichijyō ([4]) and Matsumoto ([6]). The projective flatness of Kropina space was investigated by Matsumoto ([6]) and of Matsumoto space was studied by Aikou-Hashiguchi-Yamauchi ([2]). The condition for a Finsler space with a generalized Randers metric L satisfying $L^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2$, where c's are constants, to be projectively flat was given by Park and Choi ([8]). A locally Minkowski space with (α,β)-metric is called flat-parallel ([1]) if α is locally flat and β is parallel with respect to α.

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It is well-known ([5]) that a locally Minkowski space $F^n = (M^n, L)$ with one of the following (α, β)-metrics is flat-parallel:

(1) $L = c_1\alpha + c_2\beta + \alpha^2/\beta$, $c_1 \neq 0$,
(2) $L = c_1\alpha + c_2\beta + \beta^2/\alpha$, $c_2 \neq 0$,
(3) $L = (c_1\alpha^2 + c_2\alpha\beta + c_3\beta^2)/(\alpha + \beta)$,

where c's are constants.

The purpose of the present paper is to consider the projective flatness of Finsler spaces with the above (α, β)-metrics (1), (2) and (3).

2. Preliminaries

In a Finsler space $F^n = (M^n, L)$ with an (α, β)-metric, let $\gamma_j{}^i{}_k(x)$ be the Christoffel symbols constructed from the Riemannian metric a_{ij}. We denote by $(;)$ the covariant differentiation with respect to $\gamma_j{}^i{}_k(x)$. In a Finsler space F^n with (α, β)-metric, we define

$$\text{(2.1)} \qquad \begin{aligned} &2r_{ij} = b_{i;j} + b_{j;i}, \quad 2s_{ij} = b_{i;j} - b_{j;i}, \quad s^i{}_j = a^{ir}s_{rj}, \\ &s_i = b^r s_{ri}, \quad \gamma_{jhk} = a_{hr}\gamma_j{}^r{}_k, \quad b^2 = a^{rs}b_r b_s. \end{aligned}$$

Then, by Theorem 1 of [6] a Finsler space F^n with an (α, β)-metric is projectively flat if and only if the space is covered by coordinate neighborhoods on which $\gamma_j{}^i{}_k(x)$ satisfies

$$\text{(2.2)} \qquad \begin{aligned} &(\gamma_0{}^i{}_0 - \gamma_{000}y^i/\alpha^2)/2 + (\alpha L_\beta/L_\alpha)s^i{}_0 \\ &\quad + (L_{\alpha\alpha}/L_\alpha)(C + \alpha r_{00}/2\beta)(\alpha^2 b^i/\beta - y^i) = 0, \end{aligned}$$

where a subscript 0 means a contraction by y^i, $L_\alpha = \partial L/\partial\alpha$, $L_\beta = \partial L/\partial\beta, L_{\alpha\alpha} = \partial L_\alpha/\partial\alpha, L_{\beta\beta} = \partial L_\beta/\partial\beta$ and C is given by

$$\text{(2.3)} \qquad \begin{aligned} &C + (\alpha^2 L_\beta/\beta L_\alpha)s_0 \\ &\quad + (\alpha L_{\alpha\alpha}/\beta^2 L_\alpha)(\alpha^2 b^2 - \beta^2)(C + \alpha r_{00}/2\beta) = 0. \end{aligned}$$

By the homogenity of L we know $\alpha^2 L_{\alpha\alpha} = \beta^2 L_{\beta\beta}$, so the formula (2.3) can be rewritten in the following form:

$$\text{(2.4)} \qquad \begin{aligned} &\{1 + (L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 b^2 - \beta^2)\}(C + \alpha r_{00}/2\beta) \\ &= (\alpha/2\beta)\{r_{00} - (2\alpha L_\beta/L_\alpha)s_0\} \end{aligned}$$

If $1+(L_{\beta\beta}/\alpha L_\alpha)(\alpha^2b^2-\beta^2)\neq 0$, then we can eliminate $(C+\alpha r_{00}/2\beta)$ in (2.2) and it is written as the form:

$$\begin{aligned}&\{1+L_{\beta\beta}(\alpha^2b^2-\beta^2)/(\alpha L_\alpha)\}\{(\gamma_0{}^i{}_0-\gamma_{000}y^i/\alpha^2)/2\\ &+(\alpha L_\beta/L_\alpha)s^i{}_0\}+(L_{\alpha\alpha}/L_\alpha)(\alpha/2\beta)\{r_{00}\\ &-(2\alpha L_\beta/L_\alpha)s_0\}(\alpha^2b^i/\beta-y^i)=0.\end{aligned}\tag{2.5}$$

Thus we have

THEOREM 2.1. *If $1+(L_{\beta\beta}/\alpha L_\alpha)(\alpha^2b^2-\beta^2)\neq 0$, then a Finsler space F^n with an (α,β)-metric is projectively flat if and only if* (2.5) *is satisfied.*

It is known ([3]) that if α^2 contains β as a factor, then the dimension is equal to two and $b^2=0$.

Throughout this paper, we assume that the dimension is more than two and $b^2\neq 0$, that is, $\alpha^2\not\equiv 0 \pmod{\beta}$.

3. A Finsler space with metric $L=c_1\alpha+c_2\beta+\alpha^2/\beta$

Let F^n be a Finsler space with an (α,β)-metric given by

$$L=c_1\alpha+c_2\beta+\alpha^2/\beta,\quad c_1\neq 0.\tag{3.1}$$

It is known ([5]) that a Finsler space with (α,β)-metric (3.1) is flat-parallel if it is locally Minkowski.

In this section, we find the condition for a Finsler space F^n with (3.1) to be projectively flat.

The partial derivatives with respect to α and β of a metric (3.1) are given by

$$\begin{aligned}L_\alpha=c_1+2\alpha/\beta,\quad L_\beta=c_2-\alpha^2/\beta^2,\\ L_{\alpha\alpha}=2/\beta,\quad L_{\beta\beta}=2\alpha^2/\beta^3.\end{aligned}\tag{3.2}$$

If $1+(L_{\beta\beta}/\alpha L_\alpha)(\alpha^2b^2-\beta^2)=0$, then we have $c_1\beta^3+2b^2\alpha^3=0$ which leads a contradiction. Thus Theorem 2.1 can be applied.

Substituting (3.2) into (2.5), we get

$$\begin{aligned}&(c_1\beta^3+2b^2\alpha^3)\{-2s^i{}_0\alpha^5+2\beta(\gamma_0{}^i{}_0+c_2s^i{}_0\beta)\alpha^3\\ &+c_1\gamma_0{}^i{}_0\beta^2\alpha^2-2\gamma_{000}\beta y^i\alpha-c_1\gamma_{000}\beta^2y^i\}+2\{2s_0\alpha^6\\ &+2\beta(r_{00}-c_2\beta s_0)\alpha^4+c_1r_{00}\beta^2\alpha^3\}(\alpha^2b^i-\beta y^i)=0.\end{aligned}\tag{3.3}$$

Then the above equation (3.3) can be rewritten as a polynomial of eighth degree in α as follows:

$$p_8\alpha^8 + p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0 + \alpha(p_5\alpha^4 + p_3\alpha^2 + p_1) = 0,$$

where

$$\begin{aligned}
p_8 &= 4(b^i s_0 - b^2 s^i{}_0),\\
p_6 &= 4\beta\{b^2(\gamma_0{}^i{}_0 + c_2 s^i{}_0\beta) - s_0 y^i + b^i(r_{00} - c_2\beta s_0)\},\\
p_5 &= 2c_1\beta^2(-s^i{}_0\beta + b^2\gamma_0{}^i{}_0 + r_{00}b^i),\\
p_4 &= -4\beta\{b^2\gamma_{000}y^i + \beta(r_{00} - c_2\beta s_0)y^i\},\\
p_3 &= 2c_1\beta^2(\gamma_0{}^i{}_0\beta^2 + c_2 s^i{}_0\beta^3 - b^2\gamma_{000}y^i - r_{00}\beta y^i),\\
p_2 &= c_1^2\gamma_0{}^i{}_0\beta^5, \quad p_1 = -2c_1\gamma_{000}\beta^4 y^i, \quad p_0 = -c_1^2\gamma_{000}\beta^5 y^i.
\end{aligned}$$

Since $p_8\alpha^8 + p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0$ and $p_5\alpha^4 + p_3\alpha^2 + p_1$ are rational and α is irrational in y^i, we have

$$p_8\alpha^8 + p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0 = 0, \tag{3.4}$$

$$p_5\alpha^4 + p_3\alpha^2 + p_1 = 0. \tag{3.5}$$

It follows from (3.4) that the term $4(-b^2 s^i{}_0 + s_0 b^i)\alpha^8$ must have a factor β. Since $\alpha^2 \not\equiv 0 \pmod{\beta}$, we have a vector $\lambda^i = \lambda^i(x)$ satisfying

$$s_0 b^i - b^2 s^i{}_0 = \lambda^i\beta.$$

Transvecting this by $y_i = a_{ij}y^j$, we get $s_0 = \lambda^i y_i$, so that $\lambda_i = s_i$. Therefore we have $b^2 s^i{}_0 = s_0 b^i - s^i\beta$, that is,

$$b^2 s_{ij} = b_i s_j - b_j s_i. \tag{3.6}$$

Secondly, we observe in (3.5) that the term $-2c_1\gamma_{000}\beta^4 y^i$ must have a factor α^2. Hence we have 1-form $\nu_0 = \nu_i(x)y^i$ such that

$$\gamma_{000} = \nu_0\alpha^2. \tag{3.7}$$

From (3.4) and (3.7), the term $c_1^2(\gamma_0{}^i{}_0 - \nu_0 y^i)\beta^5$ must have a factor α^2. Hence we have $\mu^i = \mu^i(x)$ satisfying

$$\gamma_0{}^i{}_0 - \nu_0 y^i = \mu^i\alpha^2. \tag{3.8}$$

Transvecting (3.8) by y_i, we have from (3.7), $\mu^i y_i = 0$, which implies $\mu^i = 0$. Thus we have

$$\gamma_0{}^i{}_0 = \nu_0 y^i, \tag{3.9}$$

that is,

$$2\gamma_j{}^i{}_k = \nu_k \delta^i_j + \nu_j \delta^i_k, \tag{3.10}$$

which shows that the associated Riemannian space is projectively flat.

Next, substituting (3.7) and (3.9) into (3.3), we have

$$\begin{aligned}(c_1\beta^3 + 2b^2\alpha^3)(c_2\beta^2 - \alpha^2)s^i{}_0 + \{2s_0\alpha^3 \\ + 2\beta(r_{00} - c_2 s_0\beta)\alpha + c_1 r_{00}\beta^2\}(\alpha^2 b^i - \beta y^i) = 0.\end{aligned} \tag{3.11}$$

Transvecting (3.11) by b_i, we get

$$\begin{aligned}2\{(b^2 r_{00} - s_0\beta)\alpha^2 + (c_2 s_0\beta - r_{00})\beta^2\}\alpha \\ + c_1(b^2 r_{00} - s_0\beta)\beta\alpha^2 + c_1(c_2 s_0\beta - r_{00})\beta^3 = 0,\end{aligned} \tag{3.12}$$

which implies

$$(b^2 r_{00} - s_0\beta)\alpha^2 + (c_2 s_0\beta - r_{00})\beta^2 = 0. \tag{3.13}$$

Therefore there exists a function $k = k(x)$ such that

$$r_{00} - c_2 s_0\beta = k\alpha^2, \qquad b^2 r_{00} - s_0\beta = k\beta^2. \tag{3.14}$$

Eliminating r_{00} from (3.14), we have

$$(c_2 b^2 - 1)s_0\beta = k(\beta^2 - b^2\alpha^2), \tag{3.15}$$

that is,

$$(c_2 b^2 - 1)(s_i b_j + s_j b_i) = 2k(b_i b_j - b^2 a_{ij}). \tag{3.16}$$

Transvecting (3.16) by a^{ij}, we have $(1-n)b^2 k = 0$, which implies $k = 0$. We assume that $b^2 \neq 1/c_2$. Then from (3.15), we have $s_0 = 0$, and hence from (3.14) we obtain $r_{00} = 0$, that is, $r_{ij} = 0$.

On the other hand, from $s_i = 0$ and (3.6) we have $s_{ij} = 0$. So we get $b_{i;j} = 0$.

Conversely it is easy to see that (3.3) is a consequence of (3.9) and $b_{i;j} = 0$. Thus we have

THEOREM 3.1. *A Finsler space $F^n (n > 2)$ with an (α,β)-metric (3.1) provided $b^2 \neq 1/c_2$ is projectively flat if and only if the associated Riemannian space (M^n, α) is projectively flat and $b_{i;j} = 0$.*

4. A Finsler space with metric $L = c_1\alpha + c_2\beta + \beta^2/\alpha$

Let F^n be a Finsler space with an (α, β)-metric given by

$$L = c_1\alpha + c_2\beta + \beta^2/\alpha, c_2 \neq 0. \tag{4.1}$$

It is known ([5]) also that a Finsler space F^n with (α, β)-metric (4.1) is flat-parallel if it is locally Mincowski. The partial derivatives with respect to α and β of (4.1) are given by

$$\begin{aligned} L_\alpha &= c_1 - \beta^2/\alpha^2, \quad L_\beta = c_2 + 2\beta/\alpha, \\ L_{\alpha\alpha} &= 2\beta^2/\alpha^3, \quad L_{\beta\beta} = 2/\alpha. \end{aligned} \tag{4.2}$$

If $1 + (L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 b^2 - \beta^2) = 0$, then we have $\alpha^2(c_1 + 2b^2) = 3\beta^2$ which leads a contradiction. Thus $1 + (L_{\alpha\alpha}/\alpha L_\alpha)(\alpha^2 b^2 - \beta^2) \neq 0$ and Theorem 2.1 can be applied.

Substituting (4.2) into (2.5), we get

$$\begin{aligned} &(c_1\alpha^2 - 3\beta^2 + 2b^2\alpha^2)(c_1\gamma_0{}^i{}_0\alpha^4 - \gamma_0{}^i{}_0\alpha^2\beta^2 \\ &- c_1\gamma_{000}\alpha^2 y^i + \gamma_{000}\beta^2 y^i + 2c_2\alpha^5 s^i{}_0 + 4\beta\alpha^4 s^i{}_0) \\ &+ 2\alpha^2(c_1 r_{00}\alpha^2 - r_{00}\beta^2 - 2c_2\alpha^3 s_0 \\ &- 4\alpha^2\beta s_0)(\alpha^2 b^i - \beta y^i) = 0. \end{aligned} \tag{4.3}$$

This equation (4.3) can be rewritten as a polynomial of seventh degree in α as follows:

$$(q_7\alpha^6 + q_5\alpha^4)\alpha + q_6\alpha^6 + q_4\alpha^4 + q_2\alpha^2 + q_0 = 0,$$

where

$$\begin{aligned} q_7 &= 2c_2\{(c_1 + 2b^2)s^i{}_0 - 2b^i s_0\}, \\ q_6 &= c_1(c_1 + 2b^2)\gamma_0{}^i{}_0 + 4(c_1 + 2b^2)\beta s^i{}_0 + 2c_1 r_{00} b^i - 8\beta s_0 b^i, \\ q_5 &= 2c_2\beta(-3\beta s^i{}_0 + 2s_0 y^i), \\ q_4 &= -2(2c_1 + b^2)\beta^2\gamma_0{}^i{}_0 - c_1(c_1 + 2b^2)\gamma_{000} y^i - 12\beta^3 s^i{}_0 \\ &\quad - 2(b^i\beta + c_1 y^i) r_{00}\beta + 8\beta^2 s_0 y^i, \\ q_2 &= \beta^2\{2(2c_1 + b^2)\gamma_{000} y^i + 3\beta^2\gamma_0{}^i{}_0 + 2r_{00}\beta y^i\}, \\ q_0 &= -3\beta^4\gamma_{000} y^i. \end{aligned}$$

Since $q_7\alpha^6 + q_5\alpha^4$ and $q_6\alpha^6 + q_4\alpha^4 + q_2\alpha^2 + q_0$ are rational and α is irrational in y^i, we have

$$q_7\alpha^2 + q_5 = 0, \tag{4.4}$$

$$q_6\alpha^6 + q_4\alpha^4 + q_2\alpha^2 + q_0 = 0. \tag{4.5}$$

The equation (4.4) is rewritten as follows:

$$-3\beta^2 s^i{}_0 + 2\beta s_0 y^i + (c_1 s^i{}_0 + 2b^2 s^i{}_0 - 2b^i s_0)\alpha^2 = 0. \tag{4.6}$$

Transvecting (4.6) by b_i, we have $s_0(c_1\alpha^2 - \beta^2) = 0$. Since $c_1\alpha^2 - \beta^2 \neq 0$, we get $s_0 = 0$. Substituting this equation into (4.6), we get

$$s^i{}_0(-3\beta^2 + c_1\alpha^2 + 2b^2\alpha^2) = 0,$$

from which $s^i{}_0 = 0$ by virtue of $(-3\beta^2 + c_1\alpha^2 + 2b^2\alpha^2) \neq 0$, that is, $s_{ij} = 0$.

On the other hand, from (4.5), q_0 must have a factor α^2. Therefore there exists 1-form $\mu_0 = \mu_i(x)y^i$ such that

$$\gamma_{000} = \mu_0\alpha^2. \tag{4.7}$$

Substituting $s^i{}_0 = 0$, $s_0 = 0$ and (4.7) into (4.3), we have

$$(c_1\alpha^2 - 3\beta^2 + 2b^2\alpha^2)(\gamma_0{}^i{}_0 - \mu_0 y^i) + 2r_{00}(\alpha^2 b^i - \beta y^i) = 0 \tag{4.8}$$

by virtue of $c_1\alpha^2 - \beta^2 \neq 0$.

The terms $-3\beta^2(\gamma_0{}^i{}_0 - \mu_0 y^i) - 2r_{00}\beta y^i$ of (4.8) seemingly does not contain α^2. Hence we must have 1-form $\nu^i{}_0 = \nu^i{}_j(x)y^j$ such that

$$3\beta(\gamma_0{}^i{}_0 - \mu_0 y^i) + 2r_{00}y^i = \nu^i{}_0\alpha^2. \tag{4.9}$$

Transvecting (4.9) by y_i and using (4.7), we have

$$2r_{00} = \nu^i{}_0 y_i. \tag{4.10}$$

On the other hand, (4.8) is rewritten as the form

$$\alpha^2\{(c_1 + 2b^2)(\gamma_0{}^i{}_0 - \mu_0 y^i) + 2r_{00}b^i\} = \beta\{3\beta(\gamma_0{}^i{}_0 - \mu_0 y^i) + 2r_{00}y^i\}.$$

Therefore, from (4.9) this equation is reduced to

$$(c_1 + 2b^2)(\gamma_0{}^i{}_0 - \mu_0 y^i) + 2r_{00}b^i = \beta\nu^i{}_0. \tag{4.11}$$

Substituting (4.10) into (4.11), we get

$$(c_1 + 2b^2)(\gamma_0{}^i{}_0 - \mu_0 y^i) = \beta\nu^i{}_0 - \nu_{00}b^i, \tag{4.12}$$

where $\nu_{ij} = a_{ir}\nu^r{}_j$ and $\nu_{ij} = \nu_{ji}$. Eliminating $(\gamma_0{}^i{}_0 - \mu_0 y^i)$ from (4.8) and (4.12), we have

$$\nu_{i0}(c_1\alpha^2 - 3\beta^2 + 2b^2\alpha^2) = \nu_{00}(c_1 y_i - 3\beta b_i + 2b^2 y_i). \tag{4.13}$$

If we define the tensor $E_{ij} = (c_1 + 2b^2)a_{ij} - 3b_i b_j$, then (4.13) is written in the form $\nu_{i0}E_{00} = \nu_{00}E_{i0}$, which implies

$$E_{hj}\nu_{ik} + E_{jk}\nu_{ih} + E_{kh}\nu_{ij} = \nu_{hj}E_{ik} + \nu_{jk}E_{ih} + \nu_{kh}E_{ij}. \tag{4.14}$$

It is easy to show that the tensor E_{ij} has the reciprocal

$$E^{ij} = \frac{1}{c_1 + 2b^2}\left(a^{ij} + \frac{3b^i b^j}{c_1 - b^2}\right),$$

where $b^2 \neq c_1, -c_1/2$. Transvecting (4.14) by E^{hj}, we get $\nu_{ik} = EE_{ik}$, where we put $E = (E^{hj}\nu_{hj})/n$. Therefore we have

$$\nu_{ij} = E\{(c_1 + 2b^2)a_{ij} - 3b_i b_j\} \tag{4.15}$$

and (4.10) is written as $r_{00} = E\{(c_1 + 2b^2)\alpha^2 - 3\beta^2\}/2$, that is,

$$r_{ij} = \frac{1}{2}E\{(c_1 + 2b^2)a_{ij} - 3b_i b_j\}.$$

Hence, from this equation and $s_{ij} = 0$, we have

$$b_{i;j} = \frac{1}{2}E\{(c_1 + 2b^2)a_{ij} - 3b_i b_j\}. \tag{4.16}$$

Next, from (4.15) the equation (4.12) is reduced to

$$\gamma_0{}^i{}_0 = \mu_0 y^i + E(\beta y^i - \alpha^2 b^i), \tag{4.17}$$

that is,

$$2\gamma_j{}^i{}_k = \mu_j\delta^i_k + \mu_k\delta^i_j + E(b_j\delta^i_k + b_k\delta^i_j - 2a_{jk}b^i). \tag{4.18}$$

Conversely, it can be easily verified that (4.3) is a consequence of (4.16) and (4.17). Thus we have

THEOREM 4.1. *A Finsler space $F^n (n > 2)$ with an (α, β)-metric (4.1) provided $b^2 \neq c_1, -c_1/2$ is projectively flat if and only if $b_{i;j}$ is written in the form (4.16) and F^n is covered by coordinate neighborhoods on which the Christoffel symbols of the associated Riemannian space with the metric α are written in the form (4.18).*

5. A Finsler space with metric $L = (c_1\alpha^2 + c_2\alpha\beta + c_3\beta^2)/(\alpha+\beta)$

If a Finsler space $F^n = (M^n, L)$ with metric

$$L = \frac{c_1\alpha^2 + c_2\alpha\beta + c_3\beta^2}{(\alpha+\beta)}, \tag{5.1}$$

where c's are constants, is a locally Minkowski space, then F^n is flat-parallel ([5]).

In this section, we shall find the condition for F^n with metric (5.1) to be projectively flat. From (5.1) we have

$$\begin{aligned} L_\alpha &= \frac{c_1\alpha^2 + 2c_1\alpha\beta + (c_2 - c_3)\beta^2}{(\alpha+\beta)^2}, \\ L_\beta &= \frac{(c_2 - c_1)\alpha^2 + 2c_3\alpha\beta + c_3\beta^2}{(\alpha+\beta)^2}, \\ L_{\alpha\alpha} &= \frac{2(c_1 - c_2 + c_3)\beta^2}{(\alpha+\beta)^3}, \quad L_{\beta\beta} = \frac{2(c_1 - c_2 + c_3)\alpha^2}{(\alpha+\beta)^3}. \end{aligned} \tag{5.2}$$

If $1 + (L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 b^2 - \beta^2) = 0$, then we have

$$\{c_1 + 2(c_1 - c_2 + c_3)b^2\}\alpha^3 + 3c_1\alpha^2\beta + 3(c_2 - c_3)\alpha\beta^2 + (c_2 - c_3)\beta^3 = 0$$

which leads a contradiction. Therefore we can apply Theorem 2.1.

Substituting (5.2) into (2.5), we get

$$\begin{aligned} &\{(c_1 + 2db^2)\alpha^3 + 3c_1\alpha^2\beta + 3(c_2 - c_3)\alpha\beta^2 \\ &+ (c_2 - c_3)\beta^3\}[\gamma_0{}^i{}_0\alpha^2\{c_1\alpha^2 + 2c_1\alpha\beta + (c_2 - c_3)\beta^2\} \\ &- \gamma_{000}y^i\{c_1\alpha^2 + 2c_1\alpha\beta + (c_2 - c_3)\beta^2\} \\ &+ 2s^i{}_0\alpha^3\{(c_2 - c_1)\alpha^2 + 2c_3\alpha\beta + c_3\beta^2\}] \\ &+ 2d\alpha^3[r_{00}\{c_1\alpha^2 + 2c_1\alpha\beta + (c_2 - c_3)\beta^2\} \\ &- 2s_0\alpha\{(c_2 - c_1)\alpha^2 + 2c_3\alpha\beta + c_3\beta^2\}](\alpha^2 b^i - \beta y^i) = 0, \end{aligned} \tag{5.3}$$

where we put $d = c_1 - c_2 + c_3$.

This equation (5.3) is rewritten as a polynomial of eighth degree in α as follows:

$$\begin{aligned} &g_8\alpha^8 + g_6\alpha^6 + g_4\alpha^4 + g_2\alpha^2 + g_0 \\ &+ \alpha(g_7\alpha^6 + g_5\alpha^4 + g_3\alpha^2 + g_1) = 0, \end{aligned}$$

where

$$
\begin{aligned}
g_8 &= 2(c_2 - c_1)\{(c_1 + 2db^2)s^i{}_0 - 2db^i s_0\},\\
g_7 &= c_1(c_1 + 2db^2)\gamma_0{}^i{}_0 + 2\{3c_1(c_2 - c_1) + 2c_3(c_1 + 2db^2)\}s^i{}_0\beta\\
&\quad + 2c_1 db^i r_{00} - 8c_3 db^i s_0\beta,\\
g_6 &= \beta[c_1(5c_1 + 4db^2)\gamma_0{}^i{}_0 + 2\{7c_1c_3 - 3(c_1 - c_2)(c_2 - c_3)\\
&\quad + 2c_3db^2\}s^i{}_0\beta + 4c_1db^i r_{00} - 4d\{c_3b^i\beta + (c_1 - c_2)y^i\}s_0],\\
g_5 &= 2\{c_1(3c_1 + 2c_2 - 2c_3) + (c_2 - c_3)db^2\}\gamma_0{}^i{}_0\beta^2\\
&\quad - c_1(c_1 + 2db^2)\gamma_{000}y^i + 2\{3c_1c_3 + 6c_3(c_2 - c_3)\\
&\quad + (c_2 - c_1)(c_2 - c_3)\}s^i{}_0\beta^3 + 2d\{(c_2 - c_3)b^i\beta - c_1y^i\}r_{00}\beta\\
&\quad + 8c_3ds_0\beta^2y^i,\\
g_4 &= \beta\{-c_1(5c_1 + 4db^2)\gamma_{000}y^i + 10c_1(c_2 - c_3)\gamma_0{}^i{}_0\beta^2\\
&\quad + 10c_3(c_2 - c_3)s^i{}_0\beta^3 - 4c_1d\beta y^i r_{00} + 4c_3d\beta^2y^i s_0\},\\
g_3 &= \beta^2[(c_2 - c_3)(2c_1 + 3c_2 - 3c_3)\gamma_0{}^i{}_0\beta^2\\
&\quad - \{6c_1^2 + 2(c_2 - c_3)(2c_1 + db^2)\}\gamma_{000}y^i\\
&\quad + 2c_3(c_2 - c_3)s^i{}_0\beta^3 - 2(c_2 - c_3)dr_{00}\beta y^i],\\
g_2 &= (c_2 - c_3)\beta^3\{(c_2 - c_3)\gamma_0{}^i{}_0\beta^2 - 10c_1\gamma_{000}y^i\},\\
g_1 &= -\{2c_1 + 3(c_2 - c_3)\}(c_2 - c_3)\beta^4\gamma_{000}y^i,\\
g_0 &= -(c_2 - c_3)^2\beta^5\gamma_{000}y^i.
\end{aligned}
$$

Since $g_8\alpha^8 + g_6\alpha^6 + g_4\alpha^4 + g_2\alpha^2 + g_0$ and $g_7\alpha^6 + g_5\alpha^4 + g_3\alpha^2 + g_1$ are rational and α is irrational in y^i, we have

$$g_8\alpha^8 + g_6\alpha^6 + g_4\alpha^4 + g_2\alpha^2 + g_0 = 0, \tag{5.4}$$

$$g_7\alpha^6 + g_5\alpha^4 + g_3\alpha^2 + g_1 = 0. \tag{5.5}$$

The term which does not contain β in (5.4) is $g_8\alpha^8$. Therefore there exists a homogeneous polynomial V_8 of degree eight in y^i such that

$$2(c_2 - c_1)\{(c_1 + 2db^2)s^i{}_0 - 2db^i s_0\}\alpha^8 = \beta V_8.$$

Since $\alpha^2 \not\equiv 0 \pmod{\beta}$, we must have a function $u^i = u^i(x)$ satisfying

$$2(c_2 - c_1)\{(c_1 + 2db^2)s^i{}_0 - 2db^i s_0\} = u^i\beta. \tag{5.6}$$

Transvecting (5.6) by b_i, we have

$$2c_1(c_2-c_1)s_0 = u^i b_i \beta, \tag{5.7}$$

that is, $2c_1(c_2-c_1)s_j = u^i b_i b_j$. Furthermore transvecting this equation by b^j, we have $u^i b_i b^2 = 0$, that is, $u^i b_i = 0$. Substituting this equation into (5.7), we have $s_0 = 0$ provided $c_1(c_2-c_1) \neq 0$. Therefore, from (5.6), we get

$$2(c_2-c_1)(c_1+2db^2)s_{ij} = u_i b_j, \tag{5.8}$$

which implies $u_i b_j + u_j b_i = 0$. Transvecting this equation by b^j, we have $u_i b^2 = 0$ by virtue of $u_j b^j = 0$. Therefore we get $u_i = 0$. Hence, from (5.8), we have $s_{ij} = 0$, provided $(c_1+2db^2) \neq 0$.

On the other hand, from (5.5) we have 1-form $v_0 = v_i(x)y^i$ such that

$$\gamma_{000} = v_0\alpha^2. \tag{5.9}$$

Substituting $s_0 = 0$, $s^i{}_0 = 0$ and (5.9) into (5.3), we have

$$\begin{aligned}&\{(c_1+2db^2)\alpha^3 + 3c_1\alpha^2\beta + 3(c_2-c_3)\alpha\beta^2\\ &\quad + (c_2-c_3)\beta^3\}(\gamma_0{}^i{}_0 - v_0y^i) + 2dr_{00}\alpha(\alpha^2b^i - \beta y^i) = 0\end{aligned} \tag{5.10}$$

by virtue of $c_1\alpha^2 + 2c_1\alpha\beta + (c_2-c_3)\beta^2 \neq 0$. Then the equation (5.10) is written in the form $P\alpha + Q = 0$, where

$$\begin{aligned}P &= \{(c_1+3db^2)\alpha^2 + 3(c_2-c_3)\beta^2\}(\gamma_0{}^i{}_0 - v_0y^i)\\ &\quad + 2dr_{00}(\alpha^2b^i - \beta y^i),\\ Q &= \beta\{3c_1\alpha^2 + (c_2-c_3)\beta^2\}(\gamma_0{}^i{}_0 - v_0y^i).\end{aligned}$$

Since P and Q are rational and α is irrational in y^i, we have $P = 0$ and $Q = 0$.

First, it follows from $Q = 0$ that

$$\gamma_0{}^i{}_0 - v_0y^i = 0, \tag{5.11}$$

that is,

$$2\gamma_j{}^i{}_k = v_j\delta^i_k + v_k\delta^i_j, \tag{5.12}$$

which shows that the associated Riemannian space (M, α) is projectively flat.

Next, from $P = 0$ and (5.11) we have

$$dr_{00}(\alpha^2 b^i - \beta y^i) = 0. \tag{5.13}$$

Transvecting (5.13) by b_i, we have $dr_{00}(\alpha^2 b^2 - \beta^2) = 0$, from which $r_{00} = 0$ provided $d \neq 0$, that is, $r_{ij} = 0$. From $s_{ij} = 0$ and $r_{ij} = 0$ we have $b_{i;j} = 0$.

Conversely, it is easily verified that (5.3) is a consequence of (5.11) and $b_{i;j} = 0$. Thus we have

THEOREM 5.1. *A Finsler space $F^n (n > 2)$ with an (α, β)-metric (5.1), provided $(c_1 - c_2 + c_3) \neq 0$ and $c_1(c_2 - c_1)\{c_1 + 2(c_1 - c_2 + c_3)b^2\} \neq 0$, is projectively flat if and only if the associated Riemannian space (M^n, α) is projectively flat and $b_{i;j} = 0$.*

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HONG-SUH PARK, EMERITUS PROFESSOR OF YEUNGNAM UNIVERSITY, KYONGSAN 712-749, KOREA
E-mail: phs1230@unitel.co.kr

HA-YONG PARK AND BYUNG-DOO KIM, DEPARTMENT OF MATHEMATICS, KYUNGIL UNIVERSITY, KYONGSAN 712-701, KOREA
E-mail: hypark@kiu.ac.kr
bdkim@kiu.ac.kr

EUN-SEO CHOI, DEPARTMENT OF MATHEMATICS, YEUNGNAM UNIVERSITY, KYONGSAN 712-749, KOREA
E-mail: eschoi@yu.ac.kr