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SUBORDINATION CHAINS AND UNIVALENCE CRITERIA

Nicolae N. Pascu, Dorina Răducanu and Shigeyoshi Owa

Abstract. The object of the present paper is to give an univalence condition for analytic functions in the open unit disk $\mathbb{U}$ by using the properties for subordination chains.

1. Introduction

Let $\mathbb{U}$ be the open unit disk in the complex plane $\mathbb{C}$, i.e. $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We denote by $\mathcal{A}$ the class of functions $f(z)$ which are analytic in $\mathbb{U}$ with $f(0) = 0$, $f'(0) = 1$ and by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are univalent in $\mathbb{U}$.

Let $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{S}$. Then $f(z)$ is said to be subordinate to $g(z)$ (written by $f(z) \prec g(z)$) in $\mathbb{U}$ if $f(\mathbb{U}) \subset g(\mathbb{U})$.

A function $L : \mathbb{U} \times [0,\infty) \to \mathbb{C}$ is said to be a subordination chain if $L(\cdot, t)$ is analytic and univalent in $\mathbb{U}$ for all $t \in [0,\infty)$ and $L(z,s) \prec L(z,t)$ whenever $0 \le s \le t < \infty$.

The following result concerning subordination chains is due to Pommerenke [3].

Theorem 1. *Let $L(z,t) = a_1(t)\, z + \cdots$ be a function from $\mathbb{U} \times [0,\infty)$ into $\mathbb{C}$, such that:*

(i) *$L(\cdot, t)$ is analytic in $\mathbb{U}$ for all $t \in [0,\infty)$.*

(ii) *$L(z,t)$ is a locally absolutely continuous function of t, locally uniformly with respect to $z \in \mathbb{U}$.*

(iii) *$a_1(t) \neq 0$ for all $t \in [0,\infty)$ and $\lim_{t\to\infty} |a_1(t)| = \infty$.*

(iv) *The family of functions $\left\{ \dfrac{L(z,t)}{a_1(t)} \right\}_{t \ge 0}$ forms a normal family in $\mathbb{U}$.*

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Let $p : \mathbb{U}\times[0,\infty)\to\mathbb{C}$ *be an analytic function in* $\mathbb{U}$ *with* $\operatorname{Re} p(z,t) > 0$ *for all* $(z,t)\in\mathbb{U}\times[0,\infty)$ *and such that:*

$$\frac{\partial L(z,t)}{\partial t} = zp(z,t)\frac{\partial L(z,t)}{\partial z} \tag{1.1}$$

a.e. $t\in[0,\infty)$ *and for all* $z\in\mathbb{U}$.

Then the function $L(z,t)$ *is a subordination chain in* $\mathbb{U}$.

2. Sufficient conditions for univalence

By using Theorem 1, we obtain an univalence condition which generalize some known univalence criteria for analytic functions in the open unit disk $\mathbb{U}$.

Let $a(t)$ be a complex valued function on $[0,\infty)$ satisfying:

$$a\in C^1[0,\infty),\quad a(0)=1,\quad a(t)\neq 0 \tag{2.1}$$

and

$$a(t)+a'(t)\neq 0,\ t\in[0,\infty),$$

and

(2.2) the modulus of $a(t)$ is increasing to ∞.

Definition. Let $F = F(u,v)$ be a function from $\mathbb{U}\times\mathbb{C}$ into $\mathbb{C}$ and let $L(z,t) = F\left(e^{-t}z, a(t)z\right)$ for all $(z,t)\in\mathbb{U}\times[0,\infty)$. We say that the function F satisfies (PA) conditions if:

(i) $L(\cdot,t)$ is analytic in $\mathbb{U}$ for all $t\in[0,\infty)$.

(ii) $L(z,t)$ is a locally absolutely continuous function of t, locally uniformly with respect to $z\in\mathbb{U}$.

(iii) The function

$$\frac{\dfrac{\partial L(z,t)}{\partial t}}{z\dfrac{\partial L(z,t)}{\partial z}}$$

is analytic in $\bar{\mathbb{U}}$ for all $t>0$ and is analytic in $\mathbb{U}$ for $t=0$.

(iv) $\dfrac{\partial F(0,0)}{\partial \nu}\neq 0$ and $\dfrac{\dfrac{\partial F(0,0)}{\partial u}}{\dfrac{\partial F(0,0)}{\partial \nu}}\notin(-\infty,-1]$.

(v) The family of functions

$$\left\{\frac{F\left(e^{-t}z, a(t)z\right)}{e^{-t}\frac{\partial F(0,0)}{\partial u}+a(t)\frac{\partial F(0,0)}{\partial v}}\right\}_{t\geq 0}$$

is a normal family in $\mathbb{U}$.

Now, we derive

THEOREM 2. *Let $a : [0,\infty) \to \mathbb{C}$ be a function satisfying* (2.1) *and* (2.2). *Further, suppose $F : \mathbb{U} \times \mathbb{C}$ is a function which satisfies (PA) conditions. If*

$$\left|G(z,z) + \frac{a(t) - a'(t)}{2a(t)}\right| < \frac{|a(t) + a'(t)|}{2|a(t)|}, \quad z \in \mathbb{U}, \quad t \geq 0 \tag{2.3}$$

and

$$\max_{|z|=e^{-t}} \left|G\left(z, a(t)\frac{z}{|z|}\right) + \frac{a(t) - a'(t)}{2a(t)}\right| \leq \frac{|a(t) + a'(t)|}{2|a(t)|}, \quad z \in \mathbb{U}\backslash\{0\}, \quad t \geq 0, \tag{2.4}$$

where

$$G(u,v) = \frac{u}{v} \cdot \frac{\dfrac{\partial F(u,v)}{\partial u}}{\dfrac{\partial F(u,v)}{\partial v}}, \tag{2.5}$$

then $F(z,z)$ is an univalent function in $\mathbb{U}$.

Proof. We wish to show that the function $L(z,t) = F\left(e^{-t}z, a(t)z\right)$ satisfies the conditions of Theorem 1 and hence $L(\cdot,t)$ is univalent in $\mathbb{U}$, for all $t \in [0,\infty)$.

If $F\left(e^{-t}z, a(t)z\right) = a_1(t)z + \cdots$, then

$$a_1(t) = e^{-t}\frac{\partial F(0,0)}{\partial u} + a(t)\frac{\partial F(0,0)}{\partial v}.$$

By using the conditions (iv) and (v) of the Definition 1 we have $a_1(t) \neq 0$ for all $t \geq 0$, $\lim_{t\to\infty} |a_1(t)| = \infty$ and the family of functions $\left\{\dfrac{L(z,t)}{a_1(t)}\right\}_{t\geq 0}$ is a normal family in $\mathbb{U}$. Let

$$p(z,t) = \frac{\dfrac{\partial L(z,t)}{\partial t}}{z\dfrac{\partial L(z,t)}{\partial z}}, \quad (z,t) \in \mathbb{U} \times [0,\infty). \tag{2.6}$$

Then the condition (1.1) of Theorem 1 is satisfied for all $z \in \mathbb{U}$ and for all $t \in [0,\infty)$. It remains to prove that the function $p(z,t)$ has a positive real part in $\mathbb{U}$ for all $t \in [0, \infty)$. If

$$w(z,t) = \frac{1 - p(z,t)}{1 + p(z,t)}, \quad (z,t) \in \mathbb{U} \times [0, \infty), \tag{2.7}$$

then $\mathrm{Re}\, p(z,t) > 0$ if and only if $|w(z,t)| < 1$. According with (2.5), (2.6) and (2.7), we have

$$\begin{aligned} w(z,t) &= \frac{2a(t)}{a(t) + a'(t)} G\left(e^{-t}z, a(t)z\right) \\ &\quad + \frac{a(t) - a'(t)}{a(t) + a'(t)}, \quad (z,t) \in \mathbb{U} \times [0, \infty). \end{aligned} \tag{2.8}$$

By using the inequality (2.3) we obtain $|w(z,0)| < 1$ for all $z \in \mathbb{U}$. For $t > 0$ the function $p(z,t)$ is analytic in $\bar{\mathbb{U}}$ and it follows

$$\begin{aligned} |w(z,t)| &< \max_{|\zeta|=1} |w(\varsigma,t)| \\ &= \max_{|\zeta|=1} \left| \frac{2a(t)}{a(t) + a'(t)} G\left(e^{-t}\varsigma, a(t)\varsigma\right) + \frac{a(t) - a'(t)}{a(t) + a'(t)} \right|. \end{aligned}$$

If we let $z = e^{-t}\varsigma$ with $|\zeta| = 1$, then $|z| = e^{-t}$ and by using (2.4) we have

$$|w(z,t)| < \max_{|z|=e^{-t}} \left| \frac{2a(t)}{a(t) + a'(t)} G\left(z, a(t)\frac{z}{|z|}\right) + \frac{a(t) - a'(t)}{a(t) + a'(t)} \right| \leq 1.$$

Since $L(z,t)$ satisfies all the conditions of Theorem 1, it follows that $L(z,t)$ is a subordination chain in $\mathbb{U}$ and $F(z,z) = L(z,0)$ is an univalent function in $\mathbb{U}$.

Remark 1. If we take $a(t) = e^t$, then the ineqaulity (2.3) becomes $|G(z.z)| < 1$, and the inequality (2.4) becomes $|G(z, \frac{1}{\bar{z}})| \leq 1$. This is the result due to Pascu [1].

Remark 2. Let us consider the function

$$L(z,t) = F(u,v) = f(u) + \frac{(v-u)R(u)}{1 - (v-u)Q(u)}$$

with $u = e^{-t}z$ and $v = a(t)z$. Taking some analytic functions for $R(u)$ and $Q(u)$ which satisfy the conditions in Theorem 2, we obtain the result concerning univalence criteria by Pfaltzgraff [Theorem 1, 2]. For example, if we take $R(u) = f'(u) = 1 + \cdots$ and $Q(u) = 0$, then we have

$$L(z,t) = F(u,v) = f(u) + (v-u)f'(u).$$

This function $L(z,t)$ satisfies the conditions in Theorem 2.

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NICOLAE N. PASCU AND DORINA RĂDUCANU, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES, TRANSILVANIA UNIVERSITY, IULIU MANIU, 50 BRAŞOV 2200, ROMÂNIA
E-mail: n.pascu@unitbv.ro

SHIGEYOSHI OWA, DEPARTMENT OF MATHEMATICS, KINKI UNIVERSITY, HIGASHI-OSAKA, OSAKA 577-8502, JAPAN
E-mail: owa@math.kindai.ac.jp