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PROPERTIES OF A kTH ROOT OF A HYPONORMAL OPERATOR

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ABSTRACT. In this paper, we study some properties of $(\sqrt[k]{H})$ (defined below). In particular we show that an operator $T \in (\sqrt[k]{H})$ satisfying the translation invariant property is hyponormal and an invertible operator $T \in (\sqrt[k]{H})$ and its inverse T^{-1} have a common nontrivial invariant closed set. Also we study some cases which have nontrivial invariant subspaces for an operator in $(\sqrt{H})$.

Let $\mathcal{H}$ and $\mathcal{K}$ be separable, complex Hilbert spaces and $\mathcal{L}(\mathcal{H},\mathcal{K})$ denote the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. If $\mathcal{H} = \mathcal{K}$, we write $\mathcal{L}(\mathcal{H})$ in place of $\mathcal{L}(\mathcal{H},\mathcal{K})$.

An operator T is called hyponormal if $T^*T \geq TT^*$, or equivalently, if $\|Th\| \geq \|T^*h\|$ for all $h \in \mathcal{H}$. Let (H) denote the class of hyponormal operators. We say that an operator $T \in \mathcal{L}(\mathcal{H})$ is *a kth root of a hyponormal operator* if T^k is hyponormal for some positive integer k (≥ 2). We denote this class by $(\sqrt[k]{H})$. In particular the class $(\sqrt{H})(= (\sqrt[2]{H}))$ consists of square roots of hyponormal operators.

In this paper, we study some properties of $(\sqrt[k]{H})$ (defined below). In particular we show that an operator $T \in (\sqrt[k]{H})$ satisfying the translation invariant property is hyponormal and an invertible operator $T \in (\sqrt[k]{H})$ and its inverse T^{-1} have a common nontrivial invariant closed set. Also we study some cases which have nontrivial invariant subspaces for an operator in $(\sqrt{H})$.

1. Some properties

We start this section with some examples of kth roots of hyponormal operators.

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EXAMPLE 1.1. If $T \in \mathcal{L}(\mathcal{H})$ is any nilpotent operator of order $k-1$, then by Halmos characterization T is unitarily equivalent to the following operator matrix

$$A = \begin{pmatrix} 0 & A_{12} & \cdots & \cdots & A_{1k} \\ & 0 & \cdots & \cdots & A_{2k} \\ & & \ddots & & \vdots \\ & & & & 0 \end{pmatrix}.$$

Since $A \in (\sqrt[k]{H})$ and kth roots of hyponormal operators are unitarily invariant, $T \in (\sqrt[k]{H})$.

The following are the straightway conclusions about shifts.

PROPOSITION 1.2. *Let T be a weighted shift with nonzero weights $\{\alpha_n\}_{n=0}^{\infty}$. Then $T \in (\sqrt[k]{H})$ if and only if $|\alpha_{n-k}| \cdots |\alpha_{n-1}| \leq |\alpha_n| \cdots |\alpha_{n+k-1}|$ for $n = k, k+1, \cdots$.*

Proof. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. Since $T^k e_n = \alpha_n \cdots \alpha_{n+k-1} e_{n+k}$ and $T^{*k} e_n = \bar{\alpha}_{n-1} \cdots \bar{\alpha}_{n-k} e_{n-k}$, it is easy to calculate that T^k is hyponormal if and only if $|\alpha_{n-k}| \cdots |\alpha_{n-1}| \leq |\alpha_n| \cdots |\alpha_{n+k-1}|$ for $n = k, k+1, \cdots$. □

COROLLARY 1.3. *Let T be a weighted shift with nonzero weights $\{\alpha_n\}_{n=0}^{\infty}$. If T is hyponormal, then $T \in (\sqrt[k]{H})$ for every $k \in \mathbf{N}$.*

Next we give another example of kth roots of hyponormal operators.

EXAMPLE 1.4. Let T_x be the weighted shift with nonzero weights $\alpha_0 = x$, $\alpha_1 = \sqrt{\frac{2}{3}}$, $\alpha_2 = \sqrt{\frac{3}{4}}, \cdots$. Then it is an easy calculation from Proposition 1.2 that $T_x \in (\sqrt[k]{H})$ if and only if $0 < x \leq \sqrt{\frac{(k+1)^2}{4k+2}}$.

We observe that T_x is a $(k+1)$th root of a hyponormal operator, but is not a kth root of a hyponormal operator if $\sqrt{\frac{(k+1)^2}{4k+2}} < x \leq \sqrt{\frac{(k+2)^2}{4k+6}}$. In particular, T_x is a kth root of a hyponormal operator, but is not a hyponormal operator if $\sqrt{\frac{2}{3}} < x \leq \sqrt{\frac{(k+1)^2}{4k+2}}$.

Next we state some properties of an operator in $(\sqrt[k]{H})$.

PROPOSITION 1.5. *Let $T \in (\sqrt[k]{H})$. Then*
(a) *$\alpha T \in (\sqrt[k]{H})$ for all scalar α.*
(b) *If T is invertible, then T^{-1} is a kth root of a hyponormal operator.*
(c) *If $\mathcal{M} \in Lat(T)$, then $T|_{\mathcal{M}}$ is a kth root of a hyponormal operator.*

(d) *The set of all kth roots of hyponormal operators is closed in the norm topology.*

Proof. (a) It is obvious.

(b) If T is invertible, then T^k is invertible and hyponormal. Hence $T^{-k} = (T^{-1})^k$ is hyponormal. Thus $T^{-1} \in (\sqrt[k]{H})$.

(c) If $\mathcal{M} \in Lat(T)$, then $(T|_{\mathcal{M}})^k = T^k|_{\mathcal{M}}$. Since $T^k|_{\mathcal{M}}$ is hyponormal, $T|_{\mathcal{M}} \in (\sqrt[k]{H})$.

(d) If $T_n \to T$, then $T_n^k \to T^k$. Since the set of all hyponormal operators is closed in the norm topology and T_n^k are hyponormal, T^k is hyponormal. Thus $T \in (\sqrt[k]{H})$. □

PROPOSITION 1.6. *$(\sqrt[k]{H})$ is a proper subclass of $\mathcal{L}(\mathcal{H})$.*

Proof. Since T^k is hyponormal, ker T^k = ker T^{2k}. Hence ker T^k = ker T^{k+1}. Let U^* be any unilateral backward shift on $l^2(\mathbf{N})$. Since $\ker(U^*)^k \neq \ker(U^*)^{k+1}$ for any $k \in \mathbf{N}$, $U^* \notin (\sqrt[k]{H})$. □

Next we characterize a matrix on 2-dimensional complex Hilbert space which is in $(\sqrt[k]{H})$. Since every matrix on a finite dimensional complex Hilbert space is unitarily equivalent to a upper triangular matrix and a kth root of a hyponormal operator is unitarily invariant, it suffices to characterize a upper triangular matrix T. From the direct calculation, we get the following characterization.

PROPOSITION 1.7. *For $k \geq 2$ we have*

$$T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in (\sqrt[k]{H}) \Longleftrightarrow b(a^{k-1} + a^{k-2}c + \cdots + c^{k-1}) = 0.$$

We remark here that Proposition 1.7 offers the convenient criterion to find some examples of operators in $(\sqrt[k]{H})$. Also we observe that $(\sqrt[k]{H})$ is not necessarily normal on a finite dimensional space.

EXAMPLE 1.8. If $k = 3$ in Proposition 1.7, then $T \in (\sqrt[3]{H})$ if and only if $b(a^2 + ac + c^2) = 0$. Take $a = 2$, $b = 1$, and $c = -1 + \sqrt{3}i$. Then

$$T = \begin{pmatrix} 2 & 1 \\ 0 & -1+\sqrt{3}i \end{pmatrix} \in (\sqrt[3]{H}),$$

but T is not a normal operator.

It is known that hyponormal operators have translation-invariant property. On the other hand, the class of square roots of hyponormal operators may not have the translation-invariant property. For example, if $T \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ is defined as

$$T = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$

then T is a square root of a hyponormal operator. But

$$[(T-\lambda)^{*2}, (T-\lambda)^2] = \begin{pmatrix} -4|\lambda|^2 AA^* & 0 \\ 0 & 4|\lambda|^2 A^*A \end{pmatrix},$$

which is not positive. Hence $(T-\lambda)^2$ is not necessarily hyponormal.

In light of the above statement, it is natural to ask the following question: What is the class of operators in $(\sqrt[k]{H})$ satisfying the translation invariant property?

THEOREM 1.9. *If $T-\lambda$ is in $(\sqrt[k]{H})$ for every $\lambda \in \mathbf{C}$, then T is hyponormal.*

Proof. If $(T-\lambda)^k$ is hyponormal for every $\lambda \in \mathbf{C}$, then

$$[(T^*-\bar{\lambda})^k, (T-\lambda)^k] \geq 0.$$

Therefore, we have

$$\begin{aligned} 0 &\leq [(T^*-\bar{\lambda})^k, (T-\lambda)^k] \\ &= (T^*-\bar{\lambda})^k(T-\lambda)^k - (T-\lambda)^k(T^*-\lambda)^k \\ &= \Big[\sum_{r=0}^{k} \begin{pmatrix} k \\ r \end{pmatrix} (T^*)^{k-r}(-\bar{\lambda})^r\Big]\Big[\sum_{s=0}^{k} \begin{pmatrix} k \\ s \end{pmatrix} T^{k-s}(-\lambda)^k\Big] \\ &\quad -\Big[\sum_{s=0}^{k} \begin{pmatrix} k \\ s \end{pmatrix} T^{k-s}(-\lambda)^k\Big]\Big[\sum_{r=0}^{k} \begin{pmatrix} k \\ r \end{pmatrix} (T^*)^{k-r}(-\bar{\lambda})^r\Big]. \qquad (1) \end{aligned}$$

Set $\lambda = \rho e^{\imath\theta}$ for every $0 \leq \theta < 2\pi$ and $\rho > 0$. Then we get

$$\begin{aligned} (1) &= \sum_{r=0}^{k}\sum_{s=0}^{k}(-1)^{r+s} \begin{pmatrix} k \\ r \end{pmatrix}\begin{pmatrix} k \\ s \end{pmatrix} \rho^{r+s}e^{i(s-r)\theta}(T^*)^{k-r}T^{k-s} \\ &\quad -\sum_{r=0}^{k}\sum_{s=0}^{k}(-1)^{r+s} \begin{pmatrix} k \\ r \end{pmatrix}\begin{pmatrix} k \\ s \end{pmatrix} \rho^{r+s}e^{i(s-r)\theta}T^{k-s}(T^*)^{k-r}. \end{aligned}$$

Since terms in (1) are eliminated when $r=s=k$, $r=k$, and $s=k$, we do eliminate these terms and then divide by ρ^{2k-2}. Then we obtain

$$0 \leq \begin{pmatrix} k \\ k-1 \end{pmatrix}\begin{pmatrix} k \\ k-1 \end{pmatrix}[T^*T - TT^*] + \frac{1}{\rho}(\text{the other terms}).$$

Letting $\rho \to \infty$, we get $T^*T \geq TT^*$. □

We remark that the converse of Theorem 1.9 may not hold. For example, let $U \in \mathcal{L}(l^2(\mathbf{N}))$ denote the unlateral shift. Then it is known that $T = 2U+U^*$ is hyponormal, but T^2 is no longer hyponormal. Hence the class of operators in $(\sqrt[k]{H})$ with the translation-invariant property forms a proper subclass of hyponormal operators.

Recall that if $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$, then $\{T^n x\}_{n=0}^{\infty}$ is called the orbit of x under T, and is denoted by $orb(T, x)$. If $orb(T, x)$ is dense in $\mathcal{H}$, then x is called a *hypercyclic* vector for T.

THEOREM 1.10. *If $T \in (\sqrt[k]{H})$ is invertible, then T and T^{-1} have a common nontrivial invariant closed set.*

Proof. Since T^k is hyponormal, it follows from [6] that T^k has no hypercyclic vector. Then T has no hypercyclic vector from [1]. [6, Theorem 2.15] implies that T and T^{-1} have a common nontrivial invariant closed set. □

COROLLARY 1.11. *If $T \in (\sqrt[k]{H})$ is invertible, then T^{-1} has no hypercyclic vector.*

Proof. Since T^{-1} is hyponormal by Proposition 1.5, it follows from the proof of Theorem 1.10 that T^{-1} has no hypercyclic vector. □

LEMMA 1.12. ([6, Theorem 2.1]) *Let $\mathcal{L}(\mathcal{H})$. Then T has a hypercyclic vector if and only if for any non-empty open subsets V and W of $\mathcal{H}$ there exists a non-negative integer n with $T^{-n}(V) \cap W \neq \phi$.*

THEOREM 1.13. *Let $T = U|T|$ (polar decomposition) be invertible in $(\sqrt[k]{H})$. Then the Aluthge transform of T, $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ has no hypercyclic vector.*

Proof. Assume $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ has a hypercyclic vector. Since T has no hypercyclic vector from the proof of Theorem 1.10, by Lemma 1.12 there exist non-empty open subsets V and W of $\mathcal{H}$ such that $T^{-n}(V) \cap W = \phi$ for all non-negative integer n. Hence for all non-negative integer n, $T^{-n}(V) \subset W^c$ where $W^c = \mathcal{H} \backslash W$. Thus $V \subset T^n(W^c)$ for all non-negative integer n. Since $T^n = U|T|^{1/2}\tilde{T}^{n-1}|T|^{1/2}$, we get that for all non-negative integer n

$$V \subset U|T|^{1/2}\tilde{T}^{n-1}|T|^{1/2}(W^c),$$

i.e.,

$$|T|^{1/2}(V) \subset \tilde{T}^n|T|^{1/2}(W^c).$$

Hence we have

$$\tilde{T}^{-n}[|T|^{1/2}(V)] \cap [|T|^{1/2}(W^c)]^c = \phi$$

for all non-negative integer n. Since $[|T|^{1/2}(W^c)]^c = |T|^{1/2}(W)$, we obtain

$$\tilde{T}^{-n}[|T|^{1/2}(V)] \cap [|T|^{1/2}(W)] = \phi$$

for all non-negative integer n. Since $|T|^{1/2}(V)$ and $|T|^{1/2}(W)$ are open, we have the contradiction, because $\tilde{T}$ has a hypercyclic vector. □

2. Subscalarity

A bounded linear operator S on $\mathcal{H}$ is called scalar of order m if it possesses a spectral distribution of order m, i.e., if there is a continuous unital morphism,

$$\Phi : C_0^m(\mathbf{C}) \longrightarrow \mathcal{L}(\mathcal{H})$$

such that $\Phi(z) = S$, where z stands for the identity function on $\mathbf{C}$ and $C_0^m(\mathbf{C})$ for the space of compactly supported functions on $\mathbf{C}$, continuously differentiable of order m, $0 \leq m \leq \infty$. An operator is called subscalar if it is similar to the restriction of a scalar operator to an invariant subspace.

Next we study some cases with subscalarity.

THEOREM 2.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be a square root of a hyponormal operator. If one of the following conditions holds;*

(1) T is compact,

(2) T^{2n} is normal for some integer n,

(3) T^* is a square root of a hyponormal operator, and

(4) $m(\sigma(T)) = 0$ where m is the planar Lebesgue measure, then T is subscalar.

Proof. (1) If T is compact, then T^2 is compact and hyponormal. By [3, Corollary 4.9], T^2 is normal. (2) If $(T^2)^n$ is normal for some integer n, T^2 is normal from [14]. (3) If T^* is a square root of a hyponormal operator, T^2 is normal. Also (4) if $m(\sigma(T)) = 0$ where m is the planar Lebesgue measure, then T^2 is normal by [12].

Since T^2 is normal in any cases, by [13, Theorem 1]

$$T = A \oplus \begin{pmatrix} B & C \\ 0 & -B \end{pmatrix},$$

where A and B are normal and C is a positive one-to-one operator commuting with B. By [8, Theorem 4.5], T is subscalar. □

COROLLARY 2.2. *Let T be a square root of a hyponormal operator. Suppose that T is compact, or T^{2n} is normal for some integer n, or T^* is a square root of a hyponormal operator. If $\sigma(T)$ has the property that there exists some non-empty open set U such that $\sigma(T) \cap U$ is dominating for U, then T has a nontrivial invariant subspace.*

Proof. The proof follows from Theorem 2.1 and [4]. □

It is known that a hyponormal and compact operator is normal. But we observe from Theorem 2.1 that a square root of a hyponormal operator, which is compact, is not necessary a normal operator. For example,

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is a square root of a hyponormal operator and is a compact operator, but is not necessary a normal operator.

THEOREM 2.3. *Let T be in $(\sqrt[k]{H})$. If T is quasinilpotent, then T is subscalar.*

Proof. Since $\sigma(T) = \{0\}$, by the spectral mapping theorem $\sigma(T^k) = \sigma(T)^k = \{0\}$. Since T^k is quasinilpotent and hyponormal, $T^k = 0$. Since T is nilpotent, T is subscalar by [8]. □

Recall that an $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is called a quasiaffinity if it has trivial kernel and dense range. An operator $A \in \mathcal{L}(\mathcal{H})$ is said to be a quasiaffine transform of an operator $T \in \mathcal{L}(\mathcal{K})$ there exists a quasiaffinity $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ such that $XA = TX$.

COROLLARY 2.4. *Let T be a square root of a hyponormal operator. Suppose that T is compact, quasinilpotent, or T^{2n} is normal for some integer n. If A is any quasiaffine transform of T, then $\sigma(T) \subset \sigma(A)$.*

Proof. It is clear from Theorem 2.1, Theorem 2.3, and [9]. □

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