Bull. Korean Math. Soc. **40** (2003), No. 4, pp. 703–713

INVERSE SHADOWING FOR EXPANSIVE FLOWS

Keonhee Lee and Zoonhee Lee

Abstract. We extend the notion of inverse shadowing defined for diffeomorphisms to flows, and show that an expansive flow on a compact manifold with the shadowing property has the inverse shadowing property with respect to the classes of continuous methods. As a corollary we obtain that a hyperbolic flow also has the inverse shadowing property with respect to the classes of continuous methods.

1. Introduction

Hyperbolic systems (both diffeomorphisms and flows) were the main objects of interests in the global qualitative theory of dynamical systems in the last 30 years. Two significant consequences of hyperbolicity are the shadowing property and the expansivity.

Various approaches were applied to show that a hyperbolic diffeomorphism has the shadowing property (for more details, see [9]). But the fact that a hyperbolic flow has the shadowing property was proved by Pilyugin recently (see [8]).

Corless and Pilyugin [2] introduced a concept of inverse shadowing for homeomorphisms "dual" to the shadowing, and Kloeden and Ombach [4] redefined this property using the concept of the method. Generally speaking, a homeomorphism has the inverse shadowing property with respect to a class of methods if any trajectory can be uniformly approximated with given accuracy by a δ-pseudotrajectory generated by a method from the chosen class if $\delta > 0$ is sufficiently small. An appropriate choice of the class of admissible pseudotrajectories is crucial here (see [2, 3, 4, 5, 6]). For example, Corless and Pilyugin [2] showed that a diffeomorphism satisfying the strong transversality condition do not

Received November 28, 2002.
2000 Mathematics Subject Classification: Primary 37C50; Secondary 37D20.
Key words and phrases: flow, expansive, method, shadowing, inverse shadowing.
This work was supported by grant No. R01-2000-000-00005 (2002) from the Korea Science and Engineering Foundation.

have the inverse shadowing property if the classes of methods are too large, while Kloeden and Ombach [4] have shown that a hyperbolic diffeomorphism on a compact manifold has the inverse shadowing property with respect to the classes of continuous methods.

Very recently, Pilyugin [10] proved that a structurally stable diffeomorphism has the inverse shadowing property with respect to classes of continuous methods.

In this paper we extend the notion of inverse shadowing defined for diffeomorphisms to flows, and prove that an expansive flow on a compact manifold with the shadowing property has the inverse shadowing property with respect to the classes of continuous methods. As a corollary we obtain that a hyperbolic flow also has the inverse shadowing property with respect to the classes of continuous methods.

2. Shadowing and inverse shadowing for flows

Let M be a compact smooth finite dimensional manifold with a Riemannian metric d.

Consider a C^1 vector field X on M and the system of differential equations

$$\dot{x} = X(x). \tag{1}$$

Let $f : M \times \mathbb{R} \longrightarrow M$ be the (solution) flow of system (1). We shall write xt instead of $f(x,t)$ for $x \in M$ and $t \in \mathbb{R}$.

For $\delta, \tau > 0$ we say that a mapping

$$\phi : \mathbb{R} \longrightarrow M$$

is a *(δ, τ)-pseudo solution* of system (1) if there exists an increasing sequence $\{t_k \in \mathbb{R} : k \in \mathbb{Z}\}$ such that

(i) $t_0 = 0$,
(ii) $t_{k+1} - t_k \geq \tau$,
(iii) $\lim_{t \to t_k^+} \phi(t) = \phi(t_k)$,
(iv) $\dot{\phi}(t) = X(\phi(t))$ for $t \in (t_k, t_{k+1})$,
(v) $d(\phi(t_k), \phi_-(t_k)) < \delta$,

where $\phi_-(t_k) = \lim_{t \to t_k^-} \phi(t)$ and $k \in \mathbb{Z}$.

REMARK. The definition of pseudo solution here is close to the standard definition of pseudo trajectory for flows in [11] and [12]. Note that we do not assume ϕ to be continuous.

We say that a (δ,τ)-pseudo solution ϕ of (1) is *ε-shadowed* by a point $x \in M$ if there exists an increasing homeomorphism $h : \mathbb{R} \longrightarrow \mathbb{R}$ with $h(0) = 0$ such that

$$d(xh(t), \phi(t)) < \varepsilon \quad \text{for all} \quad t \in \mathbb{R}.$$

The flow f of system (1) has the *shadowing property* (or *pseudo orbit tracing property*) if for any $\varepsilon > 0$ and $\tau > 0$ there exists $\delta > 0$ such that any (δ,τ)-pseudo solution of system (1) is ε-shadowed by some point of M.

Now we extend the notion of inverse shadowing to flows. We first give the definition of a method for flows.

DEFINITION 2.1. For $\delta, \tau > 0$ we say that a mapping

$$\Phi : M \times \mathbb{R} \longrightarrow M$$

is a *(δ,τ)-method* for f if, for any $x \in M$, the map $\Phi_x : \mathbb{R} \longrightarrow M$ defined by

$$\Phi_x(t) = \Phi(x,t), \ t \in \mathbb{R},$$

is a (δ,τ)-pseudo solution of system (1).

A method Φ is said to be *complete* if $\Phi(x,0) = x$ for $x \in M$.

Note that a (δ,τ)-method for f can be considered as a family of (δ,τ)-pseudo solutions of system (1).

A method Φ for f is said to be *continuous* if the map

$$\tilde{\Phi} : M \longrightarrow M^{\mathbb{R}}$$

given by

$$\tilde{\Phi}(x)(t) = \Phi(x,t), \ x \in M, \ t \in \mathbb{R}$$

is continuous under the topology of compact convergence on $M^{\mathbb{R}}$, where $M^{\mathbb{R}}$ denotes the set of all functions from $\mathbb{R}$ into M.

The set of all complete (δ,τ)-methods [resp. complete continuous (δ,τ)-methods] for f will be denote by $\mathcal{T}_a(\delta,\tau,f)$ [resp. $\mathcal{T}_c(\delta,\tau,f)$].

It is clear that if Y is another vector field on M which is sufficiently close to X then the system

$$\dot{x} = Y(x)$$

induces a complete continuous method for f.

We denote by *Rep* the set of all increasing homeomorphisms h mapping $\mathbb{R}$ onto $\mathbb{R}$ with $h(0) = 0$.

DEFINITION 2.2. We say that a flow f has the *inverse shadowing property* with respect to the *class* $\mathcal{T}_\alpha$ [or *$\mathcal{T}_\alpha$-inverse shadowing property*],

$\alpha = a$ or c, if for any $\varepsilon > 0$ and $\tau > 0$ there exists $\delta > 0$ such that for any (δ, τ)-method $\Phi \in \mathcal{T}_\alpha(\delta, \tau, f)$ and a point $x \in M$ there are $y \in M$ and $h \in Rep$ for which

$$d(xh(t), \Phi_y(t)) < \varepsilon. \tag{2}$$

for all $t \in \mathbb{R}$.

REMARK. Say that the flow f has the *shadowing property* with respect to the *class* $\mathcal{T}_\alpha$ [or $\mathcal{T}_\alpha$*-shadowing property*], $\alpha = a, c$, if for any $\varepsilon > 0$ and $\tau > 0$ there exists $\delta > 0$ such that for any (δ, τ)-method $\Phi \in \mathcal{T}_\alpha(\delta, \tau, f)$ and a point $y \in M$ there are $x \in M$ and $h \in Rep$ for which

$$d(xt, \Phi_y(h(t))) < \varepsilon \tag{3}$$

for all $t \in \mathbb{R}$.

It is easy to show that the flow f has the shadowing property with respect to the class $\mathcal{T}_a$ if and only if it has the shadowing property in the original sense.

Clearly we see that the $\mathcal{T}_a$-shadowing property [resp. $\mathcal{T}_c$-inverse shadowing property] implies the $\mathcal{T}_c$-shadowing property [resp. $\mathcal{T}_c$-inverse shadowing property].

3. Inverse shadowing for expansive flows

DEFINITION 3.1. We say that a flow f on a compact manifold M is *expansive* if for any $\varepsilon > 0$, there exists $\delta > 0$ with the property that if $d(xt, ys(t)) < \delta$ for all $t \in \mathbb{R}$, for a pair of points $x, y \in M$ and a continuous map $s : \mathbb{R} \longrightarrow \mathbb{R}$ with $s(0) = 0$, then $y = xt$, where $|t| < \varepsilon$.

The constant $\delta > 0$ is said to be an *expansive constant* of f corresponding to ε.

It is clear from the definition that there are only a finite number of fixed points for an expansive flow and each is an isolated point of M. This reduces the study of expansive flows to those without fixed points, and so we assume that all the expansive flows on M do not have fixed points throughout the section.

It is well known that hyperbolic flows are expansive and Smale's Axiom A flows are also expansive when restricted to their nonwandering sets. Further examples of expansive flows are provided by suspensions of expansive homeomorphisms.

The main result of this paper is the following.

THEOREM 3.2. *If a flow f on a compact manifold M is expansive and has the shadowing property then it has the inverse shadowing property with respect to the class $\mathcal{T}_c$.*

Very recently, Pilyugin [10] proved that a structurally stable diffeomorphism on a compact manifold has the inverse shadowing property with respect to classes of continuous methods. We know that structural stability implies the shadowing property, but the expansivity plus the shadowing property do not imply the structural stability in general.

It is well known that a hyperbolic toral automorphism is expansive and has the shadowing property, and the suspension flow of expansive homeomorphisms with the shadowing property are also expansive and has the shadowing property.

We say that a flow f (or system (1)) is *structurally stable* if there exists a neighborhood $\mathcal{U}$ of the vector field X in $\mathcal{X}^1(M)$ such that for any $Y \in \mathcal{U}$ there is a homeomorphism of M mapping oriented trajectories of system (1) onto oriented trajectories of system $\dot{x} = Y(x)$.

The following example shows that a structurally stable flow does not have the inverse shadowing property with respect to the class $\mathcal{T}_a$. Similarly we can show that the suspension flow of a hyperbolic toral automorphism does not have the inverse shadowing property with respect to the class $\mathcal{T}_a$.

EXAMPLE 3.3. Let X be a vector field on the torus $M = [0,1] \times [0,1]/\sim$ given by

$$X(\theta_1, \theta_2) = (r \sin 2\pi\theta_1, r \sin 2\pi\theta_2),$$

where $\theta_1, \theta_2 \in [0,1]$ and $0 < r < \frac{1}{2\pi}$. Let f be the flow on M generated by X. Then it is clear that the flow f is structurally stable.

To show that f does not have the inverse shadowing property with respect to the class $\mathcal{T}_a$, we let $\varepsilon = \frac{1}{2}$, and let $0 < \delta < \frac{1}{2}$ be arbitrary. Put $p = (\frac{1}{2}, 0)$. Choose a natural number $N \geq 1$ such that

$$d\Big(f((\frac{1}{2}, \frac{\delta}{N}), 1), (\frac{1}{2}, \frac{\delta}{N})\Big) < \frac{(N-1)\delta}{N}.$$

Define a map $\Phi : M \times \mathbb{R} \longrightarrow M$ by

$$\Phi(x,t) = \begin{cases} f(x,t) & \text{if } x \neq p, \\ p & \text{if } x = p \text{ and } t \leq 0, \\ (\frac{1}{2}, \frac{\delta}{N}t) & \text{if } x = p \text{ and } 0 < t < 1, \\ f((\frac{1}{2}, \frac{\delta}{N}), t-1) & \text{if } x = p \text{ and } t \geq 1. \end{cases}$$

Then it is clear that Φ is a $(\delta,1)$-method for f such that the inequality (2) does not hold for all $y\in M$ and $h\in Rep$, where $x=p$.

To prove the theorem, we need several lemmas.

LEMMA 3.4 ([1], Theorem 3). *A flow f on M is expansive if and only if for all $\varepsilon>0$, there exists $r>0$ such that if $t=(t_i)_{-\infty}^{\infty}$, $u=(u_i)_{-\infty}^{\infty}$ are doubly infinite sequences of real numbers with $u_0=t_0=0$, $0<t_{i+1}-t_i\le r$, $|u_{i+1}-u_i|\le r$, $t_i\longrightarrow\infty$, $t_{-i}\longrightarrow-\infty$, as $i\longrightarrow\infty$, and if $x,y\in M$ satisfy $d(xt_i,yu_i)\le r$ for all $i\in\mathbb{Z}$, then there exists t such that $|t|<\varepsilon$ and $y=xt$.*

LEMMA 3.5 ([1], Lemma 2). *Let f be an expansive flow. Then there is $T_0>0$ such that for every T satisfying $0<T<T_0$, there exists $\gamma_0>0$ with $d(xT,x)\ge\gamma_0$ for every $x\in M$.*

LEMMA 3.6 ([11], Lemma 3.6). *Let $\{\alpha_j\}_{j=1}^{\infty}$ be a family of continuous increasing functions from $[0,a]$ into $\mathbb{R}$ with $\alpha_j(0)=0$ for all j, and assume $\alpha_j(a)\longrightarrow\infty$ as $j\longrightarrow\infty$. Then for every $\lambda,\beta>0$ there are j and $s_1,s_2\in[0,a]$ with $s_1<s_2$ such that $s_2-s_1<\lambda$ and $\alpha_j(s_2)-\alpha_j(s_1)=\beta$, where $[0,a]$ is a closed interval in $\mathbb{R}$.*

By a slight modification of Lemma 3.6, we have the following lemma.

LEMMA 3.7. *Let $\{\alpha_j\}_{j=1}^{\infty}$ be a family of continuous increasing functions from $[a,0]$ into $\mathbb{R}$ with $\alpha_j(0)=0$ for all j, and assume $\alpha_j(a)\longrightarrow-\infty$ as $j\longrightarrow\infty$. Then for every $\lambda,\beta>0$ there are j and $s_1,s_2\in[a,0]$ with $s_1<s_2$ such that $s_2-s_1<\lambda$ and $\alpha_j(s_2)-\alpha_j(s_1)=\beta$.*

Proof of Theorem 3.2. Let $T_0>0$ be a constant as in Lemma 3.5, and $\varepsilon>0$ and $\tau>0$ be constants with $\varepsilon<\frac{1}{2}T_0$. Given $\varepsilon>0$, choose $r>0$ as in Lemma 3.4. Applying Lemma 3.5, we can select $\gamma_0>0$ with

$$d(y(\frac{1}{2}r),y)\ge\gamma_0$$

for all $y\in M$. Let $\varepsilon'>0$ be an expansive constant corresponding to r with $\varepsilon'<\gamma_0$. Since f has the shadowing property, given $\varepsilon'>0$ and $\tau>0$, there exists $\delta>0$ such that any (δ,τ)-pseudo solution of system (1) is $\frac{\varepsilon'}{12}$-shadowed by some point of M.

To show that f has the inverse shadowing property with respect to the class $\mathcal{T}_c$, choose a continuous (δ,τ)-method Φ for f. For each $y\in M$, the map

$$\Phi_y:\mathbb{R}\longrightarrow M,\quad \Phi_y(t)=\Phi(y,t),$$

is an (δ,τ)-pseudo solution of system (1). Since f is expansive and has the shadowing property, we can easily check that the (δ,τ)-pseudo solution Φ_y is $\frac{\varepsilon'}{6}$-shadowed by a unique orbit of f, say $O(f,z)$. Let $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ be a homeomorphism such that

$$\alpha(0) = 0 \text{ and } d(\Phi_y(t), z\alpha(t)) < \frac{\varepsilon'}{6}$$

for all $t \in \mathbb{R}$. Define a set A_y by

$$\begin{aligned} A_y = \ & \{x \in M \mid \text{for any } \eta, T > 0 \text{ there is a homeomorphism} \\ & \alpha : \mathbb{R} \longrightarrow \mathbb{R} \text{ with } \alpha(0) = 0 \text{ such that} \\ & d(x\alpha(t), \Phi_y(t)) < \tfrac{\varepsilon'}{6} + \eta \text{ for all } t \in [-T, T]\}. \end{aligned}$$

Then it is clear that $A_y \subset O(f,z)$. Furthermore we have the following two properties;

1) the length of the interval $\{t \in \mathbb{R} : f(z,t) \in A_y\}$ is less than ε,
2) the set A_y is closed in M.

To show 1), we let $\{\eta_i\}_{i=1}^{\infty}$, $\{T_i\}_{i=1}^{\infty}$ be sequences of positive real numbers such that $\eta_i \longrightarrow 0$ and $T_i \rightarrow \infty$ as $i \rightarrow \infty$, and let $x \in A_y$. Choose homeomorphisms $\alpha_i : \mathbb{R} \longrightarrow \mathbb{R}$ with $\alpha_i(0) = 0$ such that

$$d(x\alpha_i(t), \Phi_y(t)) < \frac{1}{6}\varepsilon' + \eta_i,$$

for all $t \in [-T_i, T_i]$ and $i \in \mathbb{N}$. Then we have

$$\begin{aligned} d(x\alpha_i(t), z\alpha(t)) &\le d(x\alpha_i(t), \Phi_y(t)) + d(\Phi_y(t), z\alpha(t)) \\ &< \tfrac{\varepsilon'}{3} + \eta_i, \end{aligned}$$

for all $t \in [-T_i, T_i]$. Let $T_i' = \min\{|\alpha(T_i)|, |\alpha(-T_i)|\}$. Then we have

$$d(x\alpha_i\alpha^{-1}(u), zu) < \frac{\varepsilon'}{3} + \eta_i$$

for all $u \in [-T_i', T_i']$. Put $\beta_i = \alpha_i\alpha^{-1}$ for $i \in \mathbb{N}$. Then each map $\beta_i : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing homeomorphism with $\beta_i(0) = 0$. For each i, choose $0 < s_i < r$ such that if $|u - u'| < s_i$ then $|\beta_i(u) - \beta_i(u')| < \frac{r}{2}$. Since $\eta_i \longrightarrow 0$, we may assume that $\eta_i < \frac{1}{6}\varepsilon'$ for all i. Then we get

$$\begin{aligned} d((x\beta_i(u))(\beta_{i+1}(u) - \beta_i(u)), x\beta_i(u)) &= d(x\beta_{i+1}(u), x\beta_i(u)) \\ &\le d(x\beta_{i+1}(u), zu) + d(zu, x\beta_i(u)) < \frac{2}{3}\varepsilon' + 2\eta_i < \varepsilon', \end{aligned}$$

for all $u \in [-T_i', T_i']$. Since β_i, β_{i+1} are continuous, $\beta_{i+1}(0) = \beta_i(0) = 0$ and $\varepsilon' < \gamma_0$, we get that

$$|\beta_{i+1}(u) - \beta_i(u)| < \frac{r}{2}.$$

Choose ξ_i with $0 < \xi_i < r$ such that if $u \le T'_i \le u'$ and $|u - u'| < \xi_i$, then $|\beta_{i+1}(u) - \beta_{i+1}(u')| < \frac{r}{2}$. Then we have

$$|\beta_{i+1}(u) - \beta_i(u')| \le r.$$

Fix i_0, and choose a strictly increasing sequence $\{u_j\}_{j=-\infty}^{\infty}$ of real numbers with $u_0 = 0$ such that;

- if $u_j \in [0, T'_{i_0}]$, then $u_{j+1} - u_j < \min\{\xi_{i_0}, s_{i_0}\}$ for $j \ge 1$,
- if $u_{j+1} \in [-T'_{i_0}, 0]$, then $u_j - u_{j+1} < \min\{\xi_{i_0}, s_{i_0}\}$ for $j \le 1$,
- for each $i \ge i_0$, if $u_j \in [T'_i, T'_{i+1}]$ then $u_{j+1} - u_j < \min\{\xi_i, s_i\}$,
- for each $i \ge i_0$, if $u_{j+1} \in [-T'_{i+1}, -T'_i]$ then $u_{j+1} - u_j < \min\{\xi_i, s_i\}$.

For $u_j \in [-T'_{i_0}, T'_{i_0}]$, define t_j by

$$t_j = \begin{cases} \beta_{i_0}(u_j) & \text{if } \ u_j \in [-T'_{i_0}, T'_{i_0}], \\ \beta_{i+1}(u_j) & \text{if } \ i \ge i_0 \ \text{ and } \ u_j \in [-T'_{i+1}, -T'_i] \bigcup [T'_i, T'_{i+1}]. \end{cases}$$

Then we get

$$t_0 = u_0 = 0, \ |t_{j+1} - t_j| < r \ \text{ and } \ d(xt_j, zu_j) < r$$

for all $j \in \mathbb{Z}$. If we applying Lemma 3.4, we have $x = zt$ with $|t| < \varepsilon$. This completes the proof of 1).

To prove 2) it is enough to show that A_y is closed in the orbit $O(f, z)$ under the relative topology. Let z' be a limit point of A_y and assume $z' \in O(f, z)$. Take a sequence $\{z_i\}_{i=1}^{\infty}$ in A_y with $z_i \longrightarrow z'$. Let $\eta, T > 0$ be arbitrary. Since $z_i \in A_y$ for each $i \in \mathbb{N}$, there is a homeomorphism $\alpha_i : \mathbb{R} \longrightarrow \mathbb{R}$ with $\alpha_i(0) = 0$ such that

$$d(z_i\alpha_i(t), \Phi_y(t)) < \frac{1}{6}\varepsilon' + \frac{1}{2}\eta$$

for all $t \in [-T, T]$. Since z_i and z' are in the same orbit segment whose time length is less than ε, there exists an integer N such that

$$d(z_i t, z' t) < \frac{1}{2}\eta$$

for $t \in \mathbb{R}$ and $i \ge N$. Hence we have

$$d(z_i\alpha_i(t), z'\alpha_i(t)) < \frac{1}{2}\eta$$

for all $t \in \mathbb{R}$. Consequently we get

$$d(z'\alpha_i(t), \Phi_y(t)) < \frac{1}{6}\varepsilon' + \eta$$

for $t \in [-T, T]$ and $i \ge N$. It follows that $z' \in A_y$, and so A_y is closed in $O(f, z)$.

Now we define a map $h : M \longrightarrow M$ by

$$h(y) = L.L.A_y, \quad y \in M,$$

where $L.L.A_y$ is the largest limit of the set A_y, i.e. $x = L.L.A_y$ if and only if $x = x'w$ with $w \geq 0$ for $x' \in A_y$. Then the map h is well defined. Moreover if $\eta < \frac{1}{2}\varepsilon'$ then we get

$$d_0(h, id) < \varepsilon,$$

where $d_0(h, id) = \sup\{d(hx, x) : x \in M\}$.

Since h is a map on a compact manifold which is sufficiently close to the identity map, we can see that if the map h is continuous then it is surjective. Then we can easily show that the flow f has the inverse shadowing property with respect to the class $\mathcal{T}_c$. In fact, for any $x \in M$, take $y \in M$ satisfying $h(y) = x$. Given (δ, τ)-method Φ for f, $\{\Phi_y(t) : t \in \mathbb{R}\}$ is an (δ, τ)-pseudo solution of f. By the definition of the map h and the shadowing property of f, we can find $\alpha \in Rep$ such that

$$d(x\alpha(t), \Phi_y(t)) = d(h(y)\alpha(t), \Phi_y(t)) < \varepsilon$$

for all $t \in \mathbb{R}$.

Consequently the proof is completed by showing that the map h is continuous. Let $\eta, T > 0$ be arbitrary and choose $y \in M$. Define a set $A_{y.\eta.T}$ by

$$A_{y.\eta.T} = \{x \in M| \text{ there exists a homeomorphism } \alpha : \mathbb{R} \longrightarrow \mathbb{R} \text{ such that } \alpha(0) = 0, \ d(x\alpha(t), \Phi_y(t)) < \tfrac{1}{6}\varepsilon' + \eta \text{ for all } t \in [-T, T]\}.$$

Then we can see that the set $A_{y.\eta.T}$ has the following properties;

- if $\eta_1 \geq \eta_2 > 0$, then $A_{y.\eta_1.T} \supseteq A_{y.\eta_2.T}$ for all $y \in M$ and $T \in \mathbb{R}$,
- if $0 < T_1 \leq T_2$, then $A_{y.\eta.T_1} \supseteq A_{y.\eta.T_2}$ for all $y \in M$ and $\eta \in \mathbb{R}$,
- if $0 > \eta_1 > \eta_2$ and $0 < T_1 \leq T_2$ then $A_{y.\eta_1.T_1} \supseteq A_{y.\eta_2.T_2}$ for all $y \in M$,
- if $\{\eta_i\}$, $\{T_i\}$ are sequences of positive real numbers with $\eta_i \longrightarrow 0$ and $T_i \longrightarrow \infty$ as $i \longrightarrow \infty$, then $A_y = \bigcap_{i=1}^{\infty} A_{y.\eta_i.T_i}$.

Moreover we have the following three facts whose proofs are the same as those of Lemmas 3.8, 3.9, 3.10 in [11], respectively.

(a) for any $\lambda > 0$ and $y \in M$, there are $\eta, T > 0$ such that $d(x, A_y) < \lambda$ for all $x \in A_{y.\eta.T}$,
(b) for every $\lambda > 0$, there are $\eta, T > 0$ such that for every $y \in M$, $d(x, A_y) < \lambda$ for all $x \in A_{y.\eta.T}$,

(c) let $\{y_i\}$, $\{z_i\}$ be sequences of points in M. Then if $z_i \in A_{y_i}$ for all i, and $z_i \longrightarrow z$, and $y_i \longrightarrow y$ we have $z \in A_y$.

Let $\{y_i\}$ be a sequence in M with $y_i \longrightarrow y$. To show that the map h is continuous, it is enough to show that $h(y_i) = L.L.A_{y_i}$ converges to $h(y) = L.L.A_y$. Let $z_i = L.L.A_{y_i}$ for each $i \in \mathbb{N}$, and assume that $\{z_i\}$ converges to a point, say $z' \in M$. It is clear that $z' \in A_y$ by the above fact (c). Let $\{\lambda_i\}$ be a sequence of positive real numbers which converges to 0. For each λ_i, choose η_i, $T_i > 0$ such that $d(x, A_{y_i}) < \lambda_i$ for all $x \in A_{y_i \cdot \eta_i \cdot T_i}$, where $i \in \mathbb{N}$. Since $y_i \longrightarrow y$, there exists a subsequence $\{y_{k_i}\}$ such that

$$d(\Phi_{y_{k_i}}(t), \Phi_y(t)) < \frac{1}{2}\eta_i,$$

for $t \in [-T_i, T_i]$ and $i \in \mathbb{Z}$. Since $x \in A_y$, we have

$$d(x\alpha_i(t), \Phi_{y_{k_i}}(t)) < \frac{1}{6}\varepsilon' + \eta_i,$$

for $t \in [-T_i, T_i]$ and $i \in \mathbb{N}$. This implies that $x \in A_{y_{k_i} \cdot \eta_i \cdot T_i}$. By the above fact (b), we get $d(x, A_{y_{k_i}}) < \lambda_i$ for all $i \in \mathbb{N}$. Since each $A_{y_{k_i}}$ is closed, we can choose $x_{k_i} \in A_{y_{k_i}}$ such that

$$d(x, x_{k_i}) = d(x, A_{y_{k_i}}) = \lambda_i.$$

Thus we get $x_{k_i} \longrightarrow x$. Since $z_{k_i} = L.L.A_{y_{k_i}}$, there are $w_{k_i} \geq 0$ with $z_{k_i} = x_{k_i} w_{k_i}$. Consequently we have $z' = xw$ with $w \geq 0$ for every $x \in A_y$. This means that $z' = L.L.A_y$, and so completes the proof. □

Let us recall the notion of hyperbolicity for flows. We say that the flow f for system (1) is *hyperbolic* if there exist numbers $C > 1$, $0 < \lambda < 1$, and continuous families of subspaces $S(p), U(p)$ of $T_p(M)$ such that

(i) the families S, U are Df-invariant,

(ii) for $p \in M$,

$$S(p) \oplus U(p) = T_p(M) \quad \text{if} \quad X(p) = 0,$$
$$S(p) \oplus U(p) \oplus < X(p) >= T_p(M) \quad \text{if} \quad X(p) \neq 0,$$

where $< X(p) >$ is the span of $X(p)$.

(iii)

$$| D\Phi(t,p)v | \leq C\lambda^t \mid v \mid \quad \text{for} \quad v \in S(p), \ t \geq 0,$$
$$| D\Phi(t,p)v | \leq C\lambda^{-t} \mid v \mid \quad \text{for} \quad v \in U(p), \ t \leq 0.$$

It is well known that if f is hyperbolic then it is both expansive and structurally stable. If we apply the Pilyugin's result in [8] which says that every structurally stable flow has the shadowing property, then we can obtain the following corollary from our theorem.

COROLLARY 3.8. *If a flow f on a compact manifold M is hyperbolic then it has the inverse shadowing property with respect to the class $\mathcal{T}_c$.*

References

[1] R. Bowen and P. Walters, *Expansive one-parameter flows*, J. Differential Equations **12** (1972), 180–193.

[2] R. Corless and S. Pilyugin, *Approximate and real trajectories for generic dynamical systems*, J. Math. Anal. Appl. **189** (1995), 409–423.

[3] P. Diamond, Y. Han and K. Lee, *Bishadowing and hyperbolicity*, International J. of Bifurcation and Chaos **12** (2002), 1779–1788.

[4] P. Kloeden and J. Ombach, *Hyperbolic homeomorphisms and bishadowing*, Ann. Polon. Math. **65** (1997), 171–177.

[5] P. Kloeden, J. Ombach and A. Pokrovskii, *Continuous and inverse shadowing*, Functional Differential Equations **6** (1999), 137–153.

[6] K. Lee, *Continuous inverse shadowing and hyperbolicity*, Bull. Austral. Math. Soc. **67** (2003), 15–26.

[7] J. Ombach, *Sinks, sources and saddles for expansive flows with the pseudo orbit tracing property*, Ann. Pol. Math. **53.3** (1991), 237–252.

[8] S. Pilyugin, *Shadowing in structurally stable flows*, J. Differential Equations **140** (1997), 238–265.

[9] ———, *Shadowing in dynamical systms*, Lecture Notes in Math. **1706**, Springer-Verlag, 1999.

[10] ———, *Inverse shadowing by continuous methods*, Discrete and Continuous Dynamical Systems **8** (2002), 29–38.

[11] R. Thomas, *Stability properties of one-parameter flows*, Proc. London Math. Soc. **45** (1982), 479–505.

[12] ———, *Cannonical coordinates and the pseudo orbit tracing property*, J. Differential Equations **90** (1991), 316–343.

DEPARTMENT OF MATHEMATICS, CHUNGNAM NATIONAL UNIVERSITY, DAEJEON 305-764, KOREA
E-mail: khlee@math.cnu.ac.kr
zhlee@math.cnu.ac.kr