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REMARKS ON ABSOLUTELY STAR-LINDELÖF SPACES

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ABSTRACT. Vaughan proved that if X is countably compact, then the Alexandroff duplicate $A(X)$ is acc. In this note, we give an example to show that the result can not be generalized to star-Lindelöf spaces. Moreover, we give a positive result.

1. Introduction

By a space, we mean a topological space. Let us recall that a space X is *countably compact* if every countable open cover of X has a finite subcover. Matveev defined in [5] a space X to be *absolutely countably compact* (=*acc*) if for every open cover $\mathcal{U}$ of X and every dense subspace D of X, there exists a finite subset F of D such that $St(F,\mathcal{U}) = X$, where $St(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\}$. He proved that every Hausdorff acc space is countably compact.

Matveev defined in [6] a space X to be *star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there exists a countable subset F of X such that $St(F,\mathcal{U}) = X$. It is clear that every separable space is star-Lindelöf.

In [2], a star-Lindelöf space is called * Lindelöf; In [3], a star-Lindelöf space is called strongly star-Lindelöf.

Bonanzinga defined in [1] a space X to be *absolutely star-Lindelöf* if for every open cover $\mathcal{U}$ of X and every dense subset D of X, there exists a countable subset F of D such that $St(F,\mathcal{U}) = X$.

From the above definition, it is not difficult to see that every acc space is absolutely star-Lindelöf and every absolutely star-Lindelöf space is star-Lindelöf.

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Throughout the paper, the extent $e(X)$ of a space X is the smallest infinite cardinal κ such that every discrete closed subset of X has cardinality at most κ. The cardinality of a set A is denoted by $|A|$. Let $\mathfrak{c}$ denote the cardinality of the continuum, ω_1 the first uncountable cardinal and ω the first infinite cardinal. Other terms and symbols that we do not define will be used as in [4].

2. Some results on absolutely star-Lindelöf spaces

For a space X, recall that the Alexandroff duplicate $A(X)$ of a space X, denoted by $A(X)$, is constructed in the following way: The underlying set of $A(X)$ is $X \times \{0,1\}$ and each point of $X \times \{1\}$ is isolated; a basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is a set of the form $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x of X. It is well-known that $A(X)$ is countably compact if X is countably compact and $A(X)$ is compact if X is compact. Moreover, Vaughan [9] proved that if X is countably compact, then $A(X)$ is acc. In this section, we give an example to show that the result can not be generalized to star-Lindelöf.

EXAMPLE 2.1. There exists a star-Lindelöf space X such that $A(X)$ is not absolutely star-Lindelöf.

Proof. Let $X = \omega \cup \mathcal{R}$ be the Isbell-Mrówka space [8], where $\mathcal{R}$ be a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$. Then, X is star-Lindelöf, since ω is a countable dense subspace of X. To show that $A(X)$ is not absolutely star-Lindelöf. Let us consider the open cover

$$\mathcal{U} = \{\langle r, 0\rangle \cup (\omega \times \{0,1\}) : r \in \mathcal{R}\} \cup \{\langle r, 1\rangle : r \in \mathcal{R}\}$$

and the dense subset

$$D = (\omega \times \{0\}) \cup \{\langle x, 1\rangle : x \in X\}$$

of $A(X)$. Then, for any countable subset F of D, there exists a $r \in \mathcal{R}$ such that $\langle r, 1\rangle \notin F$, since $|\mathcal{R}| = \mathfrak{c}$. Hence, $\langle r, 1\rangle \notin St(F, \mathcal{U})$, since $\{\langle r, 1\rangle : r \in \mathcal{R}\}$ is open and closed in $A(X)$ and $\langle r, 1\rangle$ is isolated for every $r \in \mathcal{R}$, which completes the proof. □

In the rest of this section, we give a positive result. First we give a lemma:

LEMMA 2.2. *Every T_1 space of countable extent is star-Lindelöf.*

Proof. If X is not star-Lindelöf, then there exists an open cover $\mathcal{U}$ of X such that $St(F,\mathcal{U}) \neq X$ for any countable subset F of X. Thus we may define a sequence of points $x_\alpha, \alpha < \omega_1$ such that $x_\alpha \notin St(\{x_\gamma : \gamma < \alpha\},\mathcal{U})$ for each $\alpha < \omega_1$. Then the set $\{x_\alpha : \alpha < \omega_1\}$ is an uncountable discrete and closed subset of X. Thus we get a contraction, which completes the proof. □

REMARK. The converse of Lemma 2.2 need not be true. The Niemytzki plane and The the Isbell-Mrówka space are star-Lindelöf, since they are separable, but their extent equal $\mathfrak{c}$. Recently, Matveev [7] gave a stronger example than a Tychonoff star-Lindelöf space with arbitrary large extent.

THEOREM 2.3. *If X is a T_1 space X with $e(X) < \omega_1$, then $A(X)$ is absolutely star-Lindelöf.*

Proof. We prove that $A(X)$ is absolutely star-Lindelöf. To this end, let $\mathcal{U}$ be an open cover of $A(X)$. Obviously every point of $X \times \{1\}$ is isolated. Let B be the set of all isolated points of X, and let

$$D = (X \times \{1\}) \cup (B \times \{0\}).$$

Then, D is a dense subspace of $A(X)$ and every dense subset of $A(X)$ includes D. Thus, it is suffices to show that there exists a countable subset $E \subseteq D$ such that $St(E,\mathcal{U}) = A(X)$. For each $x \in X$, choose an open neighborhood $W_x = (V_x \times \{0,1\}) \setminus \{\langle x, 1\rangle\}$ of $\langle x, 0\rangle$ satisfying that there exists a $U \in \mathcal{U}$ such that $W_x \subseteq U$, where V_x is an open subset of X containing x. Put $\mathcal{V} = \{V_x : x \in X\}$. Then, $\mathcal{V}$ is an open cover of X. Since X is T_1 and $e(X) < \omega_1$, then X is star-Lindelöf by Lemma 2.2. Thus, there exists a countable subset $E_0 \subseteq X$ such that $X = St(E_0, \mathcal{V})$. Put $E_1 = E_0 \times \{1\}$. Let

$$E_1' = \{x \in E_0 : x \text{ is not isolated in X }\}.$$

For every $x \in E_1'$, pick $y_x \in V_x$ such that $x \neq y_x$. Then, $\langle y_x, 1\rangle \in W_x$ and $\langle x, 0\rangle \in W_x$.

For every $x \in X \setminus (E_0 \cup \{V_x : x \in E_1'\})$, there exists $x' \in X$ such that $x \in V_{x'}$ and $V_{x'} \cap E_0 \neq \emptyset$, hence $W_{x'} \cap E_1 \neq \emptyset$. Let

$$E_2 = E_1 \cup \{\langle y_x, 1\rangle : x \in E_1'\} \cup ((E_0 \setminus E_1') \times \{0\}).$$

Then, E_2 is a countable subset of D and $X \times \{0\} \subseteq St(E_2,\mathcal{U})$. Let $E_3 = A(X) \setminus St(E_2,\mathcal{U})$. Then, E_3 is a discrete and closed subset of $A(X)$. Since $e(X) < \omega_1$, then $e(A(X)) < \omega_1$. Thus we have E_3 is countable. If we put $E = E_2 \cup E_3$, then E is a countable subset of D and $A(X) = St(E,\mathcal{U})$, which completes the proof. □

COROLLARY 2.4. *Every T_1 space X with $e(X) < \omega_1$ can be embedded in an absolutely star-Lindelöf space as a closed subspace.*

From the proof of Example 2.1, it is not difficult to find that the converse of Theorem 2.3 is true.

THEOREM 2.5. *If X is a T_1 space X and $A(X)$ is absolutely star-Lindelöf, then $e(X) < \omega_1$.*

Proof. Suppose that $e(X) \geqq \omega_1$. Then, there exists a closed and discrete subset B of X such that $|B| \geqq \omega_1$. Hence, $B \times \{1\}$ is a closed and open subset of $A(X)$ and every point of $B \times \{1\}$ is an isolated point of $A(X)$. To show that $A(X)$ is not absolutely star-Lindelöf. Let C be the set of all isolated points of X. Let us consider the open cover

$$\mathcal{U} = \{A(X) \setminus (B \times \{1\})\} \cup \{\langle x, 1\rangle : x \in B\}$$

and the dense subset

$$D = (C \times \{0\}) \cup \{\langle x, 1\rangle : x \in X\}$$

of $A(X)$. Then, for any countable subset E of D, there exists a $x \in B$ such that $\langle x, 1\rangle \notin E$, since $|B| \geqq \omega_1$. Hence, $\langle x, 1\rangle \notin St(E,\mathcal{U})$, since $\{\langle x, 1\rangle\}$ is isolated and the only element of $\mathcal{U}$ containing $\langle x, 1\rangle$ for every $x \in B$, which completes the proof. □

We have the following corollary from Theorems 2.3 and 2.5.

COROLLARY 2.6. *Let X be a T_1 space. Then, $e(X) < \omega_1$ if and only if $A(X)$ is absolutely star-Lindelöf.*

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