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GENERAL VARIATIONAL INCLUSIONS AND GENERAL RESOLVENT EQUATIONS

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ABSTRACT. In this paper, we introduce and study a new class of variational inclusions, called the general variational inclusion. We prove the equivalence between the general variational inclusions, the general resolvent equations, and the fixed-point problems, using the resolvent operator technique. This equivalence is used to suggest and analyze a few iterative algorithms for solving the general variational inclusions and the general resolvent equations. Under certain conditions, the convergence analyses are also studied. The results presented in this paper generalize, improve and unify a number of recent results.

1. Introduction

The variational inequality theory provides us a unified frame work for dealing with a wide class of problems arising in elasticity, oceanography, economics, transportation, operations research, structural analysis, and engineering science, etc. (see [1], [2], [4]-[10] and the references therein). One of the most interesting and important problems in the variational inequality theory is the development of an efficient and implementable iterative algorithm. In recent years, variational inequalities have been extended and generalized in different directions using novel and innovative techniques both for its own sake and for its applications. A useful and important generalization of variational inequalities is a variational inclusion.

Using the projection technique, Verma [9], [10] established the solvability of the generalized variational inequalities involving the relaxed

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Lipschitz and relaxed monotone operators in a Hilbert space setting. Noor [6], Noor-Noor [7], Noor-Noor-Rassias [8] introduced and studied some new classes of variational inclusions for set-valued mappings with compact valued in Hilbert spaces.

Inspired and motivated by the results in [6]-[10], in this paper, we introduce and study a new class of variational inclusions, which is called the general variational inclusion. Using essentially the general resolvent operator technique, we establish the equivalence between the general variational inclusions, the resolvent equations, and the fixed-point problems. This equivalence is used to suggest and analyze a few iterative algorithms for solving the general variational inclusions and the general resolvent equations. Under certain conditions, the convergence analyses are also studied. The results presented in this paper generalize, improve and unify a number of recent results due to Hassouni-Moudafi [2], Noor [3]-[6], Noor-Noor [7], Noor-Noor-Rassias [8] and Verma [9], [10].

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space endowed with the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, respectively, and I denotes the identity mapping on H. Let 2^H and $CB(H)$ stand for the families of all nonempty subsets and all nonempty closed bounded subsets of H, respectively. Let $A, B, C : H \to 2^H$ be multivalued mappings, $g : H \to H$ be a mapping and $N : H \times H \times H \to H$ be a nonlinear mapping. Suppose that $M : H \to 2^H$ is a maximal monotone mapping with $g(H) \cap dom(M) \neq \emptyset$. For each given $f \in H$, we consider the following problem:

Find $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$ such that $gu \in dom(M)$ and

$$f \in N(x,y,z) + M(gu). \tag{2.1}$$

Problem (2.1) is called the general variational inclusion.

Now we consider some special cases of problem (2.1):

(a) If $f = 0$ and $N(x,y,z) = N(x,y)$ for all $(x,y,z) \in H \times H \times H$, where $N : H \times H \to H$ is a nonlinear mapping, then problem (2.1) is equivalent to finding $u \in H$, $x \in Au$, $y \in Bu$ such that $gu \in dom(M)$ and

$$0 \in N(x,y) + M(gu). \tag{2.2}$$

This problem is called the general set-valued variational inclusion, a problem introduced and studied by Noor [6], using the resolvent equation technique.

(b) If $M = \partial\varphi$, where $\varphi : H \to R\bigcup\{+\infty\}$ is a proper convex lower semicontinuous function on H and $g(H)\bigcap \mathrm{dom}(\partial\varphi) \neq \emptyset$ and $\partial\varphi$ denotes the subdifferential of function φ, then problem (2.1) is equivalent to finding $u \in H$, $x \in Au$, $y \in Bu$ and $z \in Cu$ such that $gu \in \mathrm{dom}(\partial\varphi)$ and

$$(2.3) \qquad \langle N(x,y,z) - f, v - gu\rangle \geq \varphi(gu) - \varphi(v), \qquad \text{for all } v \in H.$$

This problem seems to be a new one.

(c) If $N(x,y,z) = N(x,y)$ for all $(x,y,z) \in H \times H \times H$ and $f = 0$, then problem (2.3) collapses to finding $u \in H$, $x \in Au$, $y \in Bu$ such that $gu \in \mathrm{dom}(\partial\varphi)$ and

$$(2.4) \qquad \langle N(x,y), v - gu\rangle \geq \varphi(gu) - \varphi(v), \qquad \text{for all } v \in H.$$

Problem (2.4) is called the generalized set-valued variational inequality, a problem introduced and studied by Noor-Noor-Rassias [8], using the resolvent equation technique.

(d) If $N(x,y,z) = gx - N(y,z)$ for all $(x,y,z) \in H \times H \times H$ and $f = 0$, then problem (2.3) collapses to finding $u \in H$, $x \in Au$, $y \in Bu$ such that $gu \in \mathrm{dom}(\partial\varphi)$ and

$$(2.5) \qquad \langle gu - N(x,y), v - gu\rangle \geq \varphi(gu) - \varphi(v), \qquad \text{for all } v \in H.$$

Problem (2.5) is called the multivalued mixed variational inequality, a problem introduced and studied by Noor-Noor [7].

(e) If $N(x,y,z) = Ax - By$ for all $(x,y,z) \in H \times H \times H$ and $f = 0$, where $A, B : H \to H$ are single-valued mappings, then problem (2.3) is equivalent to finding $u \in H$ such that $gu \in \mathrm{dom}(\partial\varphi)$ and

$$(2.6) \qquad \langle Au - Bu, v - gu\rangle \geq \varphi(gu) - \varphi(v), \qquad \text{for all } v \in H,$$

a problem introduced and studied by Hassouni-Moudafi [2].

(f) If $N(x,y,z) = gx - (y - z)$ for all $(x,y,z) \in H \times H \times H$, $f = 0$ and the function φ is the indicator function of a closed convex set K in H, that is,

$$\varphi(u) = I_K(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{otherwise,} \end{cases}$$

then problem (2.3) is equivalent to finding $u \in H$, $x \in Au$, $y \in Bu$ such that gu is in K and

$$\langle gu - (x - y), v - gu \rangle \geq 0, \quad \text{for all } v \in K, \tag{2.7}$$

a problem studied by Verma [9], using the projection method.

(g) If $g = I$, $N(x, y, z) = Ax + y$ for all $(x, y, z) \in H \times H \times H$ and $f = 0$, where $A : H \to H$ is a single-valued mapping and φ is the indicator function of a closed convex set K in H, then problem (2.3) is equivalent to finding $u \in K$, $y \in Bu$ such that

$$\langle Au + y, v - u \rangle \geq 0, \quad \text{for all } v \in K, \tag{2.8}$$

a problem studied by Verma [10], using the projection method.

DEFINITION 2.1 [1]. If M is a maximal monotone mapping from H into 2^H, then for a constant $\rho > 0$, the resolvent operator associate with M is defined by

$$J_M(u) = (I + \rho M)^{-1}(u), \quad \text{for all } u \in H.$$

It is known that the resolvent operator J_M is single-valued and non-expansive.

In relation to problem (2.1), we consider the problem of finding $w, u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$ such that

$$N(x, y, z) + \rho^{-1} R_M w = f, \tag{2.9}$$

where $\rho > 0$ is a constant, $R_M = I - J_M$ and J_M is the resolvent operator. The equations of the type (2.9) are called the general resolvent equations. Moreover, if $M(u) = I_K(u)$ is the indicator function of K, the resolvent operator $J_M \equiv P_K$, the projection of H onto K. Consequently, problem (2.9) is equivalent to finding $w, u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$ such that

$$N(x, y, z) + \rho^{-1} Q_K w = f, \tag{2.10}$$

where $Q_K = I - P_K$ and $\rho > 0$ is a constant. The equations (2.10) are called the Wiener-Hopf equations. For the formulations and applications of the resolvent equations and Wiener-Hopf equations, see [4]-[8].

It is well known that there exist maximal monotone mappings, which are not subdifferentials of lower semicontinuous proper convex functions. For a suitable choice of the mappings g, A, B, C, N, M, the element f, and the space H, one can obtain a number of known and new classes of variational inequalities, variational inclusions, and related optimization problems from the general variational inclusions (2.1). Furthermore, these types of variational inclusions enable us to study many important problems arising in mechanics, physics, optimization and control, nonlinear programming, economics, finance, regional, structural, transportation, elasticity, and applied sciences in a general and unified framework.

DEFINITION 2.2. A multivalued mapping $A : H \to 2^H$ is said to be strongly monotone with respect to the first argument of $N(\cdot,\cdot,\cdot) : H \times H \times H \to H$, if there exists a constant $r > 0$ such that

$$\langle N(x,\cdot,\cdot) - N(y,\cdot,\cdot), u - v\rangle \geq r\|u-v\|^2 \quad \text{for all } x \in Au, y \in Av.$$

DEFINITION 2.3. A multivalued mapping $B : H \to 2^H$ is said to be relaxed Lipschitz with respect to the second argument of $N(\cdot,\cdot,\cdot) : H \times H \times H \to H$, if there exists a constant $r > 0$ such that

$$\langle N(\cdot,x,\cdot) - N(\cdot,y,\cdot), u - v\rangle \leq -r\|u-v\|^2 \quad \text{for all } x \in Bu, y \in Bv.$$

DEFINITION 2.4. A multivalued mapping $C : H \to 2^H$ is said to be relaxed monotone with respect to the third argument of $N(\cdot,\cdot,\cdot) : H \times H \times H \to H$, if there exists a constant $r > 0$ such that

$$\langle N(\cdot,\cdot,x) - N(\cdot,\cdot,y), u - v\rangle \geq -r\|u-v\|^2 \quad \text{for all } x \in Cu, y \in Cv.$$

DEFINITION 2.5. A mapping $N : H \times H \times H \to H$ is said to be Lipschitz continuous with respect to the first argument if there exists a constant $t > 0$ such that

$$\|N(x,\cdot,\cdot) - N(y,\cdot,\cdot)\| \leq t\|x-y\| \quad \text{for all } x, y \in H.$$

In a similar way, we can define Lipschitz continuity of the mapping $N(\cdot,\cdot,\cdot)$ with respect to the second or third argument.

DEFINITION 2.6. A multivalued mapping $A : H \to CB(H)$ is said to be H-Lipschitz continuous if there exists a constant $r > 0$ such that

$$H(Ax, Ay) \leq r\|x-y\| \quad \text{for all } x, y \in H,$$

where $H(\cdot,\cdot)$ is the Hausdorff metric on $CB(H)$.

3. Iterative algorithms

LEMMA 3.1. *Let f be a given element in H. Then the following conditions are equivalent to each other:*

(i) *the general variational inclusion* (2.1) *has a solution $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$ with $gu \in dom(M)$;*

(ii) *there exists $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$ satisfy the relation*

$$gu = J_M(gu + \rho f - \rho N(x, y, z)), \tag{3.1}$$

where $\rho > 0$ is a constant and J_M is the resolvent operator;

(iii) *the general resolvent equation* (2.9) *has a solution $w, u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$ with*

$$w = gu + \rho f - \rho N(x, y, z). \tag{3.2}$$

Proof. It is evident that the general variational inclusion (2.1) has a solution $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$ with $gu \in dom(M)$ if and only if

$$\rho f \in \rho N(x, y, z) + \rho M(gu) = -gu + \rho N(x, y, z) + (I + \rho M)(gu),$$

which is equivalent to

$$gu = J_M(gu + \rho f - \rho N(x, y, z)),$$

where $u \in H, x \in Au$, $y \in Bu$, $z \in Cu$. That is, (i) and (ii) is equivalent.

Suppose that (ii) holds. It follows from (3.1) that

$$\begin{aligned} &R_M(gu + \rho f - \rho N(x, y, z)) \\ &= gu + \rho f - \rho N(x, y, z) - J_M(gu + \rho f - \rho N(x, y, z)) \\ &= \rho f - \rho N(x, y, z), \end{aligned}$$

which means that

$$N(x, y, z) + \rho^{-1} R_M(w) = f,$$

where $w = gu + \rho f - \rho N(x, y, z)$. That is, (iii) is satisfied.

Conversely, suppose that (iii) holds. Then the general resolvent equation (2.9) has a solution $w, u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$ and (3.2) holds. Substituting (3.2) into (2.9), we infer that

$$\begin{aligned} f &= N(x,y,z) + \rho^{-1} R_M(w) \\ &= N(x,y,z) \\ &\quad + \rho^{-1}[gu + \rho f - \rho N(x,y,z) - J_M(gu + \rho f - \rho N(x,y,z))] \\ &= \rho^{-1} gu + f - \rho^{-1} J_M(gu + \rho f - \rho N(x,y,z)), \end{aligned}$$

which yields that

$$gu = J_M(gu + \rho f - \rho N(x,y,z)) = J_M w. \tag{3.3}$$

That is, (ii) is fulfilled. This completes the proof. □

REMARK 3.1. Lemma 3.1 extends Lemma 2.1 in [2], Lemma 3.1 and Theorem 3.1 in [4] and [6], Lemma 3.1 in [5], Lemma 3.1 and Theorem 5.1 in [7], Lemma 3.1 and Theorem 3.2 in [8], and Lemma 3.2 in [9] and [10].

Now we invoke Lemma 3.1, (3.3) and Nadler's result [3] to suggest a number of iterative algorithms for solving the general variational inclusion (2.1) and the general resolvent equation (2.9).

ALGORITHM 3.1. Let $M : H \to 2^H, g : H \to H, N : H \times H \times H \to H$, $A, B, C : H \to CB(H)$. Let f be a given element in H and $\rho > 0$ be a constant and $g(H) \supseteq J_M(H)$. For given $w_0, u_0 \in H$, $x_0 \in Au_0$, $y_0 \in Bu_0$, $z_0 \in Cu_0$, compute $\{w_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, and $\{z_n\}_{n=0}^{\infty}$ by the iterative schemes

$$gu_n = J_M w_n, \tag{3.4}$$

$$\begin{aligned} \|x_n - x_{n+1}\| &\le (1 + (n+1)^{-1}) H(Au_n, Au_{n+1}), \ x_n \in Au_n, \\ \|y_n - y_{n+1}\| &\le (1 + (n+1)^{-1}) H(Bu_n, Bu_{n+1}), \ y_n \in Bu_n, \\ \|z_n - z_{n+1}\| &\le (1 + (n+1)^{-1}) H(Cu_n, Cu_{n+1}), \ z_n \in Cu_n, \end{aligned} \tag{3.5}$$

$$w_{n+1} = (1 - \lambda) w_n + \lambda(gu_n + \rho f - \rho N(x_n, y_n, z_n)), \tag{3.6}$$

for all $n \ge 0$, where $\lambda \in (0, 1]$ is a parameter.

ALGORITHM 3.2. Let $M : H \to 2^H, g : H \to H, N : H\times H\times H \to H$, $A, B, C : H \to CB(H)$. Let f be a given element in H and $\rho > 0$ be a constant and $g(H) \supseteq J_M(H)$. For given $w_0, u_0 \in H$, $x_0 \in Au_0$, $y_0 \in Bu_0$, $z_0 \in Cu_0$, compute $\{w_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, and $\{z_n\}_{n=0}^{\infty}$ by the iterative schemes

$$gu_n = J_M w_n,$$

$$\begin{aligned} \|x_n - x_{n+1}\| &\le (1 + (n+1)^{-1})H(Au_n, Au_{n+1}),\ x_n \in Au_n,\\ \|y_n - y_{n+1}\| &\le (1 + (n+1)^{-1})H(Bu_n, Bu_{n+1}),\ y_n \in Bu_n,\\ \|z_n - z_{n+1}\| &\le (1 + (n+1)^{-1})H(Cu_n, Cu_{n+1}),\ z_n \in Cu_n, \end{aligned}$$

$$w_{n+1} = (1-\lambda)w_n + \lambda(gu_n + f - N(x_n, y_n, z_n) + (1-\rho^{-1})R_M w_n),$$

for all $n \ge 0$, where $\lambda \in (0, 1]$ is a parameter.

ALGORITHM 3.3. Let $M : H \to 2^H, g : H \to H, N : H\times H\times H \to H$, $A, B, C : H \to CB(H)$. Let f be a given element in H and $\rho > 0$ be a constant. For given $u_0 \in H$, $x_0 \in Au_0$, $y_0 \in Bu_0$, $z_0 \in Cu_0$, compute $\{u_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, and $\{z_n\}_{n=0}^{\infty}$ by the iterative schemes

$$u_{n+1} = (1-\lambda)u_n + \lambda\{u_n - gu_n + J_M(gu_n + \rho f - \rho N(x_n, y_n, z_n))\},$$

$$\begin{aligned} \|x_n - x_{n+1}\| &\le (1 + (n+1)^{-1})H(Au_n, Au_{n+1}),\ x_n \in Au_n,\\ \|y_n - y_{n+1}\| &\le (1 + (n+1)^{-1})H(Bu_n, Bu_{n+1}),\ y_n \in Bu_n,\\ \|z_n - z_{n+1}\| &\le (1 + (n+1)^{-1})H(Cu_n, Cu_{n+1}),\ z_n \in Cu_n \end{aligned}$$

for all $n \ge 0$, where $\lambda \in (0, 1]$ is a parameter.

REMARK 3.2. Algorithm 3.1, Algorithm 3.2 and Algorithm 3.3 include a few known algorithms in [2] and [4]-[10] as special cases.

4. Existence and convergence theorems

Now we study those conditions under which the approximate solution w_n obtained from Algorithm 3.1 converges to the exact solution $w \in H$ of the general resolvent equation (2.9). In a similar way, one can study the convergence analysis of Algorithms 3.2 and 3.3.

THEOREM 4.1. *Let $g : H \to H$ be strongly monotone and Lipschitz continuous with constants σ and δ, respectively, $M : H \to 2^H$ be a maximal monotone mapping with $g(H) \supseteq J_M(H)$. Let N be Lipschitz continuous with respect to the first, second and third arguments with constants β, η and a, respectively, $A, B, C : H \to CB(H)$ be H-Lipschitz continuous with constants μ, ξ and b, respectively, and A be strongly monotone with respect to the first argument of N with constant α. Let*

$$k = 2\sqrt{1 - 2\sigma + \delta^2}, \ \beta\mu \geq \alpha. \tag{4.1}$$

If there exists a constant $\rho > 0$ satisfying

$$k + \rho(\xi\eta + ab) < 1, \tag{4.2}$$

and one of the following conditions

$$\begin{aligned} &\beta\mu > \eta\xi + ab, \ |\alpha - (1-k)(\eta\xi + ab)| \\ &\quad > \sqrt{k(2-k)(\beta^2\mu^2 - (\eta\xi + ab)^2)}, \\ &\quad \left|\rho - \frac{\alpha - (1-k)(\eta\xi + ab)}{\beta^2\mu^2 - (\eta\xi + ab)^2}\right| \\ &\quad < \frac{\sqrt{(\alpha - (1-k)(\eta\xi + ab))^2 - k(2-k)(\beta^2\mu^2 - (\eta\xi + ab)^2)}}{\beta^2\mu^2 - (\eta\xi + ab)^2}; \end{aligned} \tag{4.3}$$

$$\beta\mu = \eta\xi + ab, \ \alpha > (1-k)\beta\mu, \ \rho > \frac{k(2-k)}{2[\alpha - (1-k)\beta\mu]}; \tag{4.4}$$

$$\begin{aligned} &\beta\mu < \eta\xi + ab, \\ &\quad \left|\rho - \frac{(1-k)(\eta\xi + ab) - \alpha}{(\eta\xi + ab)^2 - \beta^2\mu^2}\right| \\ &\quad > \frac{\sqrt{[(1-k)(\eta\xi + ab) - \alpha^2]^2 + k(2-k)((\eta\xi + ab)^2 - \beta^2\mu^2)}}{(\eta\xi + ab)^2 - \beta^2\mu^2}, \end{aligned} \tag{4.5}$$

then for each given $f \in H$, there exist $w, u \in H$, $x \in Au$, $y \in Bu$ and $z \in Cu$ with $w = gu + \rho f - \rho N(x, y, z)$ satisfying the general resolvent equation (2.9) and the sequences $\{w_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ generated by Algorithm 3.1 converge, respectively, to w, u, x, y and z strongly in H.

Proof. In view of Algorithm 3.1, we obtain that

$$
\begin{aligned}
&\|w_{n+1} - w_n\| \\
&= \|(1-\lambda)w_n + \lambda(gu_n + \rho f - \rho N(x_n, y_n z_n)) \\
&\quad - (1-\lambda)w_{n-1} - \lambda(gu_{n-1} + \rho f - \rho N(x_{n-1}, y_{n-1}, z_{n-1}))\| \\
&\le (1-\lambda)\|w_n - w_{n-1}\| \\
&\quad + \lambda\|gu_n - gu_{n-1} - \rho(N(x_n, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_{n-1}))\| \\
&\le (1-\lambda)\|w_n - w_{n-1}\| + \lambda\|u_n - u_{n-1} - (gu_n - gu_{n-1})\| \\
&\quad + \lambda\|u_n - u_{n-1} - \rho(N(x_n, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_{n-1}))\| \\
&\le (1-\lambda)\|w_n - w_{n-1}\| + \lambda\|u_n - u_{n-1} - (gu_n - gu_{n-1})\| \\
&\quad + \lambda\|u_n - u_{n-1} - \rho(N(x_n, y_n, z_n) - N(x_{n-1}, y_n, z_n))\| \\
&\quad + \lambda\rho\|N(x_{n-1}, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_n)\| \\
&\quad + \lambda\rho\|N(x_{n-1}, y_{n-1}, z_n) - N(x_{n-1}, y_{n-1}, z_{n-1})\|.
\end{aligned} \tag{4.6}
$$

Note that g is Lipschitz continuous and strongly monotone. Hence we have

$$
\begin{aligned}
&\|u_n - u_{n-1} - (gu_n - gu_{n-1})\|^2 \\
&= \|u_n - u_{n-1}\|^2 - 2\langle gu_n - gu_{n-1}, u_n - u_{n-1}\rangle + \|gu_n - gu_{n-1}\|^2 \\
&\le (1 - 2\sigma + \delta^2)\|u_n - u_{n-1}\|^2.
\end{aligned} \tag{4.7}
$$

Since A is H-Lipschitz continuous and strongly monotone with respect to the first argument of N, and N is Lipschitz continuous with respect to the first argument, we conclude that

$$
\begin{aligned}
&\|u_n - u_{n-1} - \rho(N(x_n, y_n, z_n) - N(x_{n-1}, y_n, z_n))\|^2 \\
&= \|u_n - u_{n-1}\|^2 - 2\rho\langle N(x_n, y_n, z_n) - N(x_{n-1}, y_n, z_n), u_n - u_{n-1}\rangle \\
&\quad + \rho^2\|N(x_n, y_n, z_n) - N(x_{n-1}, y_n, z_n)\|^2 \\
&\le (1 - 2\rho\alpha)\|u_n - u_{n-1}\|^2 + \rho^2\beta^2\|x_n - x_{n-1}\|^2 \\
&\le (1 - 2\rho\alpha + \rho^2\beta^2\mu^2(1 + n^{-1})^2)\|u_n - u_{n-1}\|^2.
\end{aligned} \tag{4.8}
$$

Since N is Lipschitz continuous with respect to the second and third arguments, respectively, and B and C are H-Lipschitz continuous, we know that

$$
\|N(x_{n-1}, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_n)\| \le \xi\eta(1 + n^{-1})\|u_n - u_{n-1}\|, \tag{4.9}
$$

(4.10)
$$\|N(x_{n-1}, y_{n-1}, z_n) - N(x_{n-1}, y_{n-1}, z_{n-1})\| \le ab(1+n^{-1})\|u_n - u_{n-1}\|.$$

Using (3.4),(4.1) and (4.7), we get that

$$\begin{aligned}\|u_n - u_{n-1}\| &\le \|u_n - u_{n-1} - (gu_n - gu_{n-1})\| + \|J_M w_n - J_M w_{n-1}\| \\ &\le \sqrt{1-2\sigma+\delta^2}\|u_n - u_{n-1}\| + \|w_n - w_{n-1}\| \\ &= 2^{-1}k\|u_n - u_{n-1}\| + \|w_n - w_{n-1}\|,\end{aligned}$$

which means that

$$\|u_n - u_{n-1}\| \le \left(1 - 2^{-1}k\right)^{-1} \|w_n - w_{n-1}\|. \tag{4.11}$$

Substituting (4.7)-(4.11) into (4.6), we infer that

$$\|w_{n+1} - w_n\| \le \theta_n \|w_n - w_{n-1}\|, \tag{4.12}$$

where

$$\begin{aligned}\theta_n = &(1-\lambda) + \lambda\left(1-2^{-1}k\right)^{-1} \\ &\left[2^{-1}k + \sqrt{1-2\rho\alpha+\rho^2\beta^2\mu^2(1+n^{-1})^2} + \rho(1+n^{-1})(\xi\eta+ab)\right].\end{aligned}$$

Put
(4.13)
$$\theta = (1-\lambda)+\lambda\left(1-2^{-1}k\right)^{-1}\left[2^{-1}k + \sqrt{1-2\rho\alpha+\rho^2\beta^2\mu^2} + \rho(\xi\eta+ab)\right].$$

Clearly, $\theta_n \downarrow \theta$ as $n \to \infty$. It follows from (4.1), (4.2) and (4.13) that
(4.14)
$$\begin{aligned}&\theta < 1 \Leftrightarrow \sqrt{1-2\rho\alpha+\rho^2\beta^2\mu^2} < 1 - k - (\xi\eta+ab)\rho \\ &\Leftrightarrow [\beta^2\mu^2 - (\xi\eta+ab)^2]\rho^2 - 2[\alpha - (1-k)(\xi\eta+ab)]\rho < -k(2-k).\end{aligned}$$

It is easy to verify that (4.14) and one of (4.3)$-$(4.5) yield that $\theta < 1$. Thus $\theta_n < 1$ for n sufficiently large. Thus (4.12) means that $\{w_n\}_{n=0}^{\infty}$ is a Cauchy sequence in H. Consequently, there exists $w \in H$ such that $\lim_{n\to\infty} w_n = w$. By virtue of (4.11), we know that the sequence $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence in H, that is, there exists $u \in H$ with

$\lim_{n\to\infty} u_n = u$. Note that A, B, C are H-Lipschitz continuous. In view of (3.5), we have

$$\begin{aligned}\|x_n - x_{n-1}\| &\le \mu(1+n^{-1})\|u_n - u_{n-1}\|,\\ \|y_n - y_{n-1}\| &\le \xi(1+n^{-1})\|u_n - u_{n-1}\|,\\ \|z_n - z_{n-1}\| &\le b(1+n^{-1})\|u_n - u_{n-1}\|,\end{aligned}$$

which imply that $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{z_n\}_{n=0}^{\infty}$ are Cauchy sequences in H. Hence there exist $x, y, z \in H$ such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} z_n = z$. Observe that

$$\begin{aligned}d(x, Au) = &\ \inf\{\|x - t\| : t \in Au\} \le \|x_n - x\| + H(Au_n, Au)\\ &\le \|x_n - x\| + \mu\|u_n - u\| \to 0\end{aligned}$$

as $n \to \infty$. This means that $x \in Au$. Similarly, we have $y \in Bu$, $z \in Cu$. It follows from the continuity of the mappings A, B, C, g, J_M, N and (3.4) and (3.6) that

$$gu = J_M w$$

and

$$w = (1-\lambda)w + \lambda(gu + \rho f - \rho N(x, y, z)) \in H,$$

which imply that

$$w = gu + \rho f - \rho N(x, y, z)$$

and

$$gu = J_M(gu + \rho f - \rho N(x, y, z)).$$

Lemma 3.1 ensures that $w, u \in H$, $x \in Au$, $y \in Bu$ and $z \in Cu$ with $w = gu + \rho f - \rho N(x, y, z)$ is a solution of the resolvent equation (2.9). This completes the proof. □

REMARK 4.1. Theorem 4.1 extends, improves and unifies Theorem 4.1 in [4] , Theorem 3.2 in [6] and Theorem 5.2-5.4 in [7].

THEOREM 4.2. *Let $g : H \to H$ be strongly monotone and Lipschitz continuous with constants σ and δ, respectively, $M : H \to 2^H$ be a maximal monotone mapping with $g(H) \supseteq J_M(H)$. Let N be Lipschitz continuous with respect to the first, second and third arguments with constants β, η and a, respectively, $A, B, C : H \to CB(H)$ be H-Lipschitz continuous with constants μ, ξ and b, respectively. Suppose that A is* *strongly monotone with respect to the first argument of N with constant α, B is relaxed Lipschitz with respect to the second argument of N with constant c, and C is relaxed monotone with respect to the third argument of N with constant d. Let*

$$\beta\mu \geq \alpha,\ \eta\xi \geq c,\ k = 2\sqrt{1-2\sigma+\delta^2}; \tag{4.15}$$

$$m = \sqrt{1-2c+\eta^2\xi^2} + \sqrt{1+2d+a^2b^2}; \tag{4.16}$$

If there exists a constant $\rho > 0$ satisfying

$$k + \rho m < 1, \tag{4.17}$$

and one of the following conditions

(4.18)

$$\beta\mu > m,\ |\alpha-(1-k)m| > k(2-k)(\beta^2\mu^2-m^2),$$
$$\left|\rho - \frac{\alpha-(1-k)m}{\beta^2\mu^2-m^2}\right| < \frac{\sqrt{(\alpha-(1-k)m)^2-k(2-k)(\beta^2\mu^2-m^2)}}{\beta^2\mu^2-m^2};$$

$$\beta\mu = m,\ \alpha > (1-k)m,\ \rho > \frac{k(2-k)}{2[\alpha-(1-k)m]}; \tag{4.19}$$

$$\begin{aligned}\beta\mu < m, &\left|\rho - \frac{(1-k)m-\alpha}{m^2-\beta^2\mu^2}\right| \\ &> \frac{\sqrt{(m^2-\beta^2\mu^2)k(2-k)+((1-k)m-\alpha)^2}}{m^2-\beta^2\mu^2},\end{aligned} \tag{4.20}$$

then for each given $f \in H$, there exist $w, u \in H$, $x \in Au$, $y \in Bu$ and $z \in Cu$ with $w = gu + \rho f - \rho N(x,y,z)$ satisfying the general resolvent equation (2.9) and the sequences $\{w_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, and $\{z_n\}_{n=0}^{\infty}$ generated by Algorithm 3.1 converge, respectively, to w, u, x, y and z strongly in H.

Proof. As in the proof of Theorem 4.1, we have

$$\|u_n - u_{n-1}\| \leq \left(1-2^{-1}k\right)^{-1}\|w_n - w_{n-1}\|, \tag{4.21}$$

and

(4.22)

$$\begin{aligned}
&\|w_{n+1} - w_n\| \\
&\le (1-\lambda)\|w_n - w_{n-1}\| + \lambda\|u_n - u_{n-1} - (gu_n - gu_{n-1})\| \\
&\quad + \lambda\|u_n - u_{n-1} - \rho(N(x_n, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_{n-1}))\| \\
&\le (1-\lambda)\|w_n - w_{n-1}\| + \lambda 2^{-1}k\|u_n - u_{n-1}\| \\
&\quad + \lambda\|(u_n - u_{n-1} - \rho(N(x_n, y_n, z_n) - N(x_{n-1}, y_n, z_n))\| \\
&\quad + \lambda\rho\|N(x_{n-1}, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_n) + u_n - u_{n-1}\| \\
&\quad + \lambda\rho\|N(x_{n-1}, y_{n-1}, z_n) - N(x_{n-1}, y_{n-1}, z_{n-1}) - (u_n - u_{n-1})\| \\
&\le (1-\lambda)\|w_n - w_{n-1}\| \\
&\quad + \lambda\left(2^{-1}k + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\mu^2(1+n^{-1})^2}\right)\|u_n - u_{n-1}\| \\
&\quad + \lambda\rho\|N(x_{n-1}, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_n) + u_n - u_{n-1}\| \\
&\quad + \lambda\rho\|N(x_{n-1}, y_{n-1}, z_n) - N(x_{n-1}, y_{n-1}, z_{n-1}) - (u_n - u_{n-1})\|.
\end{aligned}$$

Since B is H-Lipschitz continuous and relaxed Lipschitz with respect to the second argument of N, and C is H-Lipschitz continuous and relaxed monotone with respect to the third argument of N, by (3.5) we conclude that

(4.23)

$$\begin{aligned}
&\|N(x_{n-1}, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_n) + u_n - u_{n-1}\|^2 \\
&= \|u_n - u_{n-1}\|^2 + 2\langle N(x_{n-1}, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_n), u_n - u_{n-1}\rangle \\
&\quad + \|N(x_{n-1}, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_n)\|^2 \\
&\le (1 - 2c + \eta^2\xi^2(1+n^{-1})^2)\|u_n - u_{n-1}\|^2,
\end{aligned}$$

and

$$\begin{aligned}
&\|N(x_{n-1}, y_{n-1}, z_n) - N(x_{n-1}, y_{n-1}, z_{n-1}) - (u_n - u_{n-1})\|^2 \\
&= \|u_n - u_{n-1}\|^2 - 2\langle N(x_{n-1}, y_{n-1}, z_n) \\
&\quad - N(x_{n-1}, y_{n-1}, z_{n-1}), u_n - u_{n-1}\rangle \\
&\quad + \|N(x_{n-1}, y_{n-1}, z_n) - N(x_{n-1}, y_{n-1}, z_{n-1})\|^2 \\
&\le (1 + 2d + a^2b^2(1+n^{-1})^2)\|u_n - u_{n-1}\|^2.
\end{aligned} \tag{4.24}$$

By virtue of (4.21)–(4.24), we know that

$$\begin{aligned}
&\|w_{n+1} - w_n\| \\
&\le (1-\lambda)\|w_n - w_{n-1}\| + \lambda\Big[2^{-1}k + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\mu^2(1+n^{-1})^2}
\end{aligned}$$

$$
\begin{aligned}
&+\rho\sqrt{1-2c+\eta^2\xi^2(1+n^{-1})^2} \\
&+\rho\sqrt{1+2d+a^2b^2(1+n^{-1})^2}\Big] \|u_n-u_{n-1}\| \\
\le{}& \theta_n \|w_n-w_{n-1}\|,
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_n = 1-\lambda+\lambda\left(1-2^{-1}k\right)^{-1}\big(2^{-1}k+\sqrt{1-2\rho\alpha+\rho^2\beta^2\mu^2(1+n^{-1})^2} \\
+\rho\sqrt{1-2c+\eta^2\xi^2(1+n^{-1})^2}+\rho\sqrt{1+2d+a^2b^2(1+n^{-1})^2}\big).
\end{aligned}
$$

Set

$$
\theta = 1-\lambda+\lambda\left(1-2^{-1}k\right)^{-1}\left(2^{-1}k+\sqrt{1-2\rho\alpha+\rho^2\beta^2\mu^2}+\rho m\right).
$$

Then $\theta_n \downarrow \theta$ as $n \to \infty$. The remaining portion of the proof can be derived as in Theorem 4.1. This completes the proof. □

REMARK 4.2. Theorem 3.1 in [9] and Theorem 3.1 in [10] are special cases of Theorem 4.2.

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