Bull. Korean Math. Soc. **41** (2004), No. 2, pp. 269–274

DIMENSIONS OF A DERANGED CANTOR SET WITH SPECIFIC CONTRACTION RATIOS

In-Soo Baek

Abstract. We investigate a deranged Cantor set(a generalized Cantor set) using the similar method to find the dimensions of cookie-cutter repeller. That is, we will use a Gibbs measure which is a weak limit of a subsequence of discrete Borel measures to find the dimensions. The deranged Cantor set that will be considered is a generalized form of a perturbed Cantor set(a variation of the symmetric Cantor set) and a cookie-cutter repeller.

1. Introduction

We define a deranged Cantor set [2]. Let I_ϕ =[0,1]. We can obtain the left subinterval $I_{\tau,1}$ and the right subinterval $I_{\tau,2}$ of I_τ deleting middle open subinterval of I_τ inductively for each $\tau \in \{1,2\}^n$, where $n = 0, 1, 2, \cdots$. Consider $E_n = \cup_{\tau\in\{1,2\}^n} I_\tau$. Then (E_n) is a decreasing sequence of closed sets. For each n, we put $\mid I_{\tau,1} \mid / \mid I_\tau \mid= c_{\tau,1}$ and $\mid I_{\tau,2} \mid / \mid I_\tau \mid= c_{\tau,2}$ for all $\tau \in \{1,2\}^n$,where $\mid I \mid$ denotes the diameter of I. We call $F = \bigcap_{n=0}^\infty E_n$ a deranged Cantor set. We note that if $c_{\tau,1} = a_{n+1}$ and $c_{\tau,2} = b_{n+1}$ for all $\tau \in \{1,2\}^n$ for each n then $F = \bigcap_{n=0}^\infty E_n$ is called a perturbed Cantor set [1]. We recall the s-dimensional Hausdorff measure of F:

$$H^s(F) = \lim_{\delta\to 0} H^s_\delta(F),$$

where $H^s_\delta(F) = \inf\{\sum_{n=1}^\infty \mid U_n \mid^s : \{U_n\}_{n=1}^\infty$ is a δ-cover of $F\}$, and the Hausdorff dimension of F :

$$\dim_H(F) = \sup\{s > 0 : H^s(F) = \infty\}$$
$$(= \inf\{s > 0 : H^s(F) = 0\})(\text{see [3]}).$$

Received June 15, 2002.
2000 Mathematics Subject Classification: 28A78.
Key words and phrases: Cantor set, fractal dimensions, Gibbs measure.
This paper was supported by the research grants of Pusan University of Foreign Studies in 2002.

Also we recall the s-dimensional packing measure of F:

$$p^s(F) = \inf\{\sum_{n=1}^{\infty} P^s(F_n) : \bigcup_{n=1}^{\infty} F_n = F\},$$

where $P^s(F_n) = \lim_{\delta\to 0} P^s_\delta(F_n)$ and $P^s_\delta(E) = \sup\{\sum_{n=1}^{\infty} \mid U_n \mid^s : \{U_n\}$ is a δ-packing of E $\}$, and the packing dimension of F:

$$\dim_p(F) = \sup\{s > 0 : p^s(F) = \infty\}(= \inf\{s > 0 : p^s(F) = 0\} \text{ ([3])}.$$

We introduce functions $h^s(F) = \liminf_{n\to\infty} \sum_{\sigma\in\{1,2\}^n} \mid I_\sigma \mid^s$ and $q^s(F) = \limsup_{n\to\infty} \sum_{\sigma\in\{1,2\}^n} \mid I_\sigma \mid^s$ for $s \in (0,1)$ and a deranged Cantor set F. Clearly $h^s(F)$ and $q^s(F)$ are decreasing functions for s.

Using h^s and q^s, we define the lower Cantor dimension and the upper Cantor dimension of a deranged Cantor set F by $\dim_{\underline{C}}(F) = \sup\{s > 0 : h^s(F) = \infty\}$ and $\dim_{\overline{C}}(F) = \sup\{s > 0 : q^s(F) = \infty\}$. Then $\dim_{\underline{C}}(F) = \inf\{s > 0 : h^s(F) = 0\}$ and $\dim_{\overline{C}}(F) = \inf\{s > 0 : q^s(F) = 0\}$ since $h^s(F)$ and $q^s(F)$ are decreasing functions for s. We note $\dim_{\underline{C}}$ and $\dim_{\overline{C}}$ are just functions whose domains are the class of the deranged Cantor sets. We note that if c_τ are given, then a deranged Cantor set is determined. We also note that a perturbed Cantor set and a cookie-cutter repeller are special examples of deranged Cantor sets. We are now ready to study the ratio geometry of the deranged Cantor set.

2. Main results

In this section, F means a deranged Cantor set determined by $\{c_\tau\}$ with $\tau \in \{1,2\}^n$ where $n = 1, 2, \cdots$. Hereafter we only consider a deranged Cantor set whose contraction ratios $c_{\tau,1}$, $c_{\tau,2}$ and gap ratios $d_\tau(= 1 - (c_{\tau,1} + c_{\tau,2}))$ are uniformly bounded away from 0.

THEOREM 1. *Assume that for all $\tau,\sigma \in \{1,2\}^k$ where k is any integer,*

$$\frac{c_{\tau,l_1} c_{\tau,l_1,l_2} \cdots c_{\tau,l_1,l_2,\cdots,l_m}}{c_{\sigma,l_1} c_{\sigma,l_1,l_2} \cdots c_{\sigma,l_1,l_2,\cdots,l_m}} \geq B$$

for all m where $B > 0$ and $\sum_{\sigma\in\{1,2\}^n} \mid I_\sigma \mid^s > a$ for some $s > 0$ and for all n where some $a > 0$. Then $H^s(F) > 0$.

Proof. We may assume that $c_\tau, d_\tau(= 1 - (c_{\tau,1} + c_{\tau,2})) > \alpha > 0$ for some small α for all τ. By the assumption, there exists $a > 0$ such that $\sum_{\sigma\in\{1,2\}^n} \mid I_\sigma \mid^s > a$ for all n. Let $l(I_\sigma)$ be the set of left end points of I_σ for $\sigma \in \{1,2\}^n$ where $n = 1, 2, \cdots$.

For each n and any set A, we define a discrete measure

$$\mu_s^n(A) = \frac{\sum_{(\tau: l(I_\tau)\cap A\neq\phi, \tau\in\{1,2\}^n)} \mid I_\tau \mid^s}{\sum_{\tau\in\{1,2\}^n} \mid I_\tau \mid^s},$$

where $n = 1, 2, \cdots$.

Then μ_s^n is a Borel measure on [0,1] whose support is in $F = \bigcap_{n=1}^{\infty} E_n$. Clearly $\mu_s^n([0,1]) = 1$. By the weak convergence theorem of Borel measure,there is a weak limit μ_s (which is a Borel measure) of a subsequence of (μ_s^n) supported by F. Now, for $k \leq n$ and $\sigma \in \{1,2\}^k$,

$$\begin{aligned}\mu_s^n(I_\sigma) &= \frac{\sum_{(\tau: l(I_\tau)\cap I_\sigma\neq\phi, \tau\in\{1,2\}^n)} \mid I_\tau \mid^s}{\sum_{\tau\in\{1,2\}^n} \mid I_\tau \mid^s} \\ &\leq \frac{|I_\sigma|^s}{B^s \sum_{\sigma'\in\{1,2\}^k} \mid I_{\sigma'} \mid^s} \leq \frac{|I_\sigma|^s}{aB^s}.\end{aligned}$$

Considering a suitable open interval containing I_σ, we easily obtain

$$\frac{\mu_s(I_\sigma)}{|I_\sigma|^s} \leq \frac{1}{aB^s}.$$

Let $x \in F = \bigcap_{n=1}^{\infty} E_n$. Then there is a sequence $(I_{\sigma_n})_{n=1}^{\infty}$, where $\sigma_n \in \{1,2\}^n$ such that $\bigcap_{n=1}^{\infty} I_{\sigma_n} = \{x\}$. Given a small positive number r, there exists n such that $|I_{\sigma_{n+1}}| \leq r < |I_{\sigma_n}|$. Since $d_{j+1}|I_{\sigma_j}| \geq \alpha|I_{\sigma_n}| > \alpha r$ for $0 \leq j \leq n$, $B_{\alpha r}(x) \subset [\bigcup_{\tau(\neq\sigma_n)\in\{1,2\}^n} I_\tau]^c$, where $B_{\alpha r}(x)$ is the ball of radius αr with center x. Thus $\mu_s(B_{\alpha r}(x)) \leq \mu_s(I_{\sigma_n})$.

Then

$$\frac{\mu_s(B_{\alpha r}(x))}{(\alpha r)^s} \leq \frac{\mu_s(I_{\sigma_n})}{\alpha^s|I_{\sigma_{n+1}}|^s} \leq \frac{\mu_s(I_{\sigma_n})}{\alpha^{2s}|I_{\sigma_n}|^s} \leq \frac{1}{aB^s\alpha^{2s}}.$$

Then

$$\limsup_{r\to 0} \frac{\mu_s(B_r(x))}{r^s} \leq \frac{1}{aB^s\alpha^{2s}}.$$

Thus $H^s(F) > 0$ by the Hausdorff density theorem(Proposition 4.9 [3] or Proposition 2.2 [4]). □

THEOREM 2. *Assume that for all* $\tau,\sigma \in \{1,2\}^k$ *where k is any integer,*

$$\frac{c_{\tau,l_1} c_{\tau,l_1,l_2} \cdots c_{\tau,l_1,l_2,\cdots,l_m}}{c_{\sigma,l_1} c_{\sigma,l_1,l_2} \cdots c_{\sigma,l_1,l_2,\cdots,l_m}} \geq B$$

for all m *where* $B > 0$ *and* $\sum_{\sigma\in\{1,2\}^n} \mid I_\sigma \mid^s < b$ *for some* $s > 0$ *and for all* n *where some* $b < \infty$. *Then* $p^s(F) < \infty$.

Proof. We may assume that $c_\tau, d_\tau (= 1 - (c_{\tau,1} + c_{\tau,2})) > \alpha > 0$ for some small α for all τ. By the assumption, there exists $b < \infty$ such that

$\sum_{\sigma\in\{1,2\}^n} \mid I_\sigma \mid^s < b$ for all n. Let $l(I_\sigma)$ be the set of left end points of I_σ for $\sigma \in \{1,2\}^n$ where $n = 1, 2, \cdots$.

For each n and any set A, we define a discrete measure

$$\mu_s^n(A) = \frac{\sum_{(\tau:l(I_\tau)\cap A\neq\phi,\tau\in\{1,2\}^n)} \mid I_\tau \mid^s}{\sum_{\tau\in\{1,2\}^n} \mid I_\tau \mid^s},$$

where $n = 1, 2, \cdots$.

Then μ_s^n is a Borel measure on $[0,1]$ whose support is in $F = \bigcap_{n=1}^\infty E_n$. Clearly $\mu_s^n([0,1]) = 1$. By the weak convergence theorem of Borel measure, there is a weak limit μ_s (which is a Borel measure) of a subsequence of (μ_s^n) supported by F. Using a similar argument with the proof in Theorem 1, we obtain $\frac{\mu_s^n(I_\sigma)}{|I_\sigma|^s} \geq \frac{1}{bB^{-s}}$. Noting that I_σ is a compact set, we easily obtain $\frac{\mu_s(I_\sigma)}{|I_\sigma|^s} \geq \frac{1}{bB^{-s}}$.

Let $x \in F = \bigcap_{n=1}^\infty E_n$. Then there is a sequence $(I_{\sigma_n})_{n=1}^\infty$, where $\sigma_n \in \{1,2\}^n$ such that $\bigcap_{n=1}^\infty I_{\sigma_n} = \{x\}$.

Given a small positive number r, there exists n such that $|I_{\sigma_{n+1}}| \leq r < |I_{\sigma_n}|$.

Then

$$\frac{\mu_s(B_r(x))}{r^s} \geq \frac{\mu_s(I_{\sigma_{n+1}})}{|I_{\sigma_n}|^s}.$$

Since $|I_{\sigma_{n+1}}|/|I_{\sigma_n}| > \alpha > 0$ for all n,

$$\frac{\mu_s(B_r(x))}{r^s} \geq \frac{\mu_s(I_{\sigma_{n+1}})}{(\frac{1}{\alpha})^s|I_{\sigma_{n+1}}|^s} \geq \frac{\alpha^s}{bB^{-s}}.$$

Then

$$\liminf_{r\to 0} \frac{\mu(B_r(x))}{r^s} \geq \liminf_{n\to\infty} \frac{\alpha^s}{bB^{-s}}.$$

Thus $p^s(F) < \infty$ by the packing density theorem (Proposition 2.2 [4]). □

THEOREM 3. *If f is a positively oriented cookie-cutter map in the sense that $f' > 1$, then the repeller of f satisfies the assumption of Theorems 1 and 2 (in this paper, we assume that the cookie-cutter map f is of differentiability of class C^2).*

Proof. Given integers m and k, consider $I_{l_1,\cdots,l_m}$ and f^k. If $\tau, \sigma \in \{1,2\}^k$, then $(f^k)^{-1}(I_{l_1,\cdots,l_m}) \cap I_\tau = I_{\tau,l_1,\cdots,l_m}$ and $(f^k)^{-1}(I_{l_1,\cdots,l_m}) \cap I_\sigma = I_{\sigma,l_1,\cdots,l_m}$.

Using the mean value theorem, we easily obtain $|I_\tau| = \frac{1}{(f^k)'(x)}$ and $|I_\sigma| = \frac{1}{(f^k)'(y)}$ for some $x \in I_\tau$ and $y \in I_\sigma$. Similarly we obtain

$|I_{\tau,l_1,\cdots,l_m}| = \frac{|I_{l_1,\cdots,l_m}|}{(f^k)'(x')}$ and $|I_{\sigma,l_1,\cdots,l_m}| = \frac{|I_{l_1,\cdots,l_m}|}{(f^k)'(y')}$ for some $x' \in I_{\tau,l_1,\cdots,l_m}$ and $y' \in I_{\sigma,l_1,\cdots,l_m}$. Noting that there is a positive number B such that $B^{-1} \leq \frac{(f^k)'(z')}{(f^k)'(z)} \leq B$ for all $z, z' \in I_\tau$ where $\tau \in \{1,2\}^k$(cf. [4]), we easily see that

$$B^{-2} \leq \frac{c_{\tau,l_1} c_{\tau,l_1,l_2} \cdots c_{\tau,l_1,l_2,\cdots,l_m}}{c_{\sigma,l_1} c_{\sigma,l_1,l_2} \cdots c_{\sigma,l_1,l_2,\cdots,l_m}} = \frac{|I_{\tau,l_1,\cdots,l_m}||I_\sigma|}{|I_{\sigma,l_1,\cdots,l_m}||I_\tau|} \leq B^2.$$

Further, we easily see that there are $a > 0$ and $b < \infty$ such that $a < \sum_{\sigma\in\{1,2\}^n} \mid I_\sigma \mid^s < b$ for all n(cf. (5.21) in [4]) where s is the unique real number satisfying the equation that the topological pressure of $-s\log f'$ is zero (cf. (5.5) and (5.19) in [4]). □

COROLLARY 4. *Assume that for all* $\tau,\sigma \in \{1,2\}^k$ *where* k *is any integer,*

$$\frac{c_{\tau,l_1} c_{\tau,l_1,l_2} \cdots c_{\tau,l_1,l_2,\cdots,l_m}}{c_{\sigma,l_1} c_{\sigma,l_1,l_2} \cdots c_{\sigma,l_1,l_2,\cdots,l_m}} \geq B$$

for all m *where* $B > 0$.

Then $\dim_{\underline{C}}(F) = \dim_H(F)$.

Proof. Suppose that $0 < s < \dim_{\underline{C}}(F)$. Then $h^s(F) = \infty$. Then there exists $a > 0$ such that $\sum_{\sigma\in\{1,2\}^n} \mid I_\sigma \mid^s > a$ for all n. By Theorem 1, $H^s(F) > 0$. On the other hand, $H^s(F) \leq h^s(F)$. □

COROLLARY 5. *Assume that for all* $\tau,\sigma \in \{1,2\}^k$ *where* k *is any integer,*

$$\frac{c_{\tau,l_1} c_{\tau,l_1,l_2} \cdots c_{\tau,l_1,l_2,\cdots,l_m}}{c_{\sigma,l_1} c_{\sigma,l_1,l_2} \cdots c_{\sigma,l_1,l_2,\cdots,l_m}} \geq B$$

for all m *where* $B > 0$.

Then $\dim_{\overline{C}}(F) = \dim_p(F)$.

Proof. Suppose that $\dim_{\overline{C}}(F) < s$. Then $q^s(F) = 0$. Then there exists $b < \infty$ such that $\sum_{\sigma\in\{1,2\}^n} \mid I_\sigma \mid^s < b$ for all n. By Theorem 2, $p^s(F) < \infty$. On the other hand, for $0 < t < \dim_{\overline{C}}(F)$, we have $p^t(F) = \infty$ using the Baire category theorem (cf. [1], Theorem 5). □

COROLLARY 6. *If* f *is a positively oriented cookie-cutter map in the sense that* $f' > 1$, *then* $0 < H^s(F) \leq p^s(F) < \infty$ *for the repeller* F *of* f *and the solution* s *of the equation* $P(-s\log f') = 0$ *where* $P(\phi)$ *is the topological pressure of* ϕ.

Proof. It follows from Theorems 1, 2 and 3. □

REMARK 7. In Corollary 6, we only considered the repeller of a positively oriented cookie-cutter map f, but we also have the same result for any cookie-cutter map. To avoid technical difficulties, we substitute X_τ and γ_τ for I_τ and c_τ. Let $X = [0,1]$, $X_1 = I_1$ and $X_2 = I_2$. We define $X_{i_1,\cdots,i_k} = F_{i_1} \circ \cdots \circ F_{i_k}(X)$, where $f(x) = \begin{cases} F_1^{-1}(x) \text{ if } x \in X_1 \\ \\ F_2^{-1}(x) \text{ if } x \in X_2, \end{cases}$

and $\gamma_{\tau,i} = \frac{|X_{\tau,i}|}{|X_\tau|}$, where $\tau \in \{1,2\}^n$ and $i = 1$ or 2. Then the repeller of any cookie-cutter map f satisfies the condition that for all τ,$\sigma \in \{1,2\}^k$ where k is any integer,

$$\frac{\gamma_{\tau,l_1}\gamma_{\tau,l_1,l_2}\cdots\gamma_{\tau,l_1,l_2,\cdots,l_m}}{\gamma_{\sigma,l_1}\gamma_{\sigma,l_1,l_2}\cdots\gamma_{\sigma,l_1,l_2,\cdots,l_m}} \geq B$$

for all m where $B > 0$ with $a < \sum_{\sigma\in\{1,2\}^n} \mid X_\sigma \mid^s < b$ for some $0 < a < b < \infty$ for all n where s is the solution of the equation $P(-s\log|f'|) = 0$. Hence it follows from the same arguments with Corollary 6.

REMARK 8. In the perturbed Cantor set, we note that

$$\frac{c_{\tau,l_1}c_{\tau,l_1,l_2}\cdots c_{\tau,l_1,l_2,\cdots,l_m}}{c_{\sigma,l_1}c_{\sigma,l_1,l_2}\cdots c_{\sigma,l_1,l_2,\cdots,l_m}} = 1.$$

References

[1] I. S. Baek, *Dimensions of the perturbed Cantor set*, Real Anal. Exchange **19** (1993/94), no. 1, 269–273.

[2] ———, *Dimensions of weakly convergent deranged Cantor set*, Real Anal. Exchange **23** (1997/98), no. 2, 689–696.

[3] K. J. Falconer, *Fractal geometry*, John Wiley and Sons, 1990.

[4] ———, *Techniques in fractal geometry*, John Wiley and Sons, 1997.

DEPARTMENT OF MATHEMATICS, PUSAN UNIVERSITY OF FOREIGN STUDIES, PUSAN 608-738, KOREA

E-mail: isbaek@pufs.ac.kr