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WEIGHTED BLOCH SPACES AND SOME OPERATORS INDUCED BY RADIAL DERIVATIVES

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Abstract. In the setting of the half-plane of the complex plane, we show that for $r \geq 0$, the dual space of the weighted bergman spaces $B^{1,r}$ is the Bloch space of the half-plane and we study some bounded linear operators induced by radial derivatives.

1. Introduction

Let $H = \{x+iy : y > 0\}$ be the half-plane and let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane . For $1 \leq p < \infty$ and $r \geq 0$, the (weighted) Bergman space $B^{p,r}(A^{p,r}, resp.)$ of the half plane (disk,*resp.*) is the space of analytic functions in $L^p(H, dA_r)(L^p(\mathbb{D}, dA_r), resp.)$, where dA denotes the usual two dimensional area measure and $dA_r(z) = (2r+1)K(z,z)^{-r}dA(z)((2r+1)K_{\mathbb{D}}(z,z)^{-r}dA(z), resp.)$. In this case, $K(\cdot, w)(K_{\mathbb{D}}(\cdot, w), resp.)$ is the reproducing kernel for $B^{2,r}(A^{2,0}, resp.)$. In fact, $K(z,w) = -\dfrac{1}{\pi(z-\overline{w})^2}$ and $K_{\mathbb{D}}(z,w) = \dfrac{1}{\pi(1-z\overline{w})^2}$ (see [3], [5]).

Moreover, for $r \geq 0, K(\cdot, w)^{1+r}(K_{\mathbb{D}}(\cdot, w)^{1+r}, resp.)$ is the reproducing kernel for $B^{2,r}(A^{2,r}, resp.)$ (see [5]).

An analytic function f on $H(\mathbb{D}, resp.)$ is said to be in the (weighted) Bloch space $\mathcal{B}^r(i)(\mathcal{B}(\mathbb{D}), resp.)$ if $\|f\|_H = \sup_{z=x+iy \in H} y^{1+2r}|f'(z)| < \infty$ and

$$f(i) = 0(\|f\|_{\mathbb{D}} = \sup_{z \in \mathbb{D}}(1-|z|^2)|f'(z)| < \infty, resp.).$$

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Then $\|\cdot\|_H$ and $\|\cdot\|_{\mathbb{D}}$ are seminorms. But $\mathcal{B}^r(i)(\mathcal{B}(\mathbb{D}), resp.)$ can be made into a Banach space by introducing the norm $\|f\|_{\mathcal{B}^r} = |f(i)| + \|f\|_H(\|f\|_{\mathcal{B}(\mathbb{D})} = |f(0)| + \|f\|_{\mathbb{D}}, resp.)$

Let $\partial^\infty H = \partial H \bigcup \{\infty\}$ and let

$$\mathcal{B}_0^r(i) = \{f \in \mathcal{B}^r(i) : \lim_{z\to\partial^\infty H} y^{1+2r}|f'(z)| = 0\},$$

where $z \to \partial^\infty H$ means $|z| \to \infty$ or $y \to 0$ where $z = x+iy$. Then $\mathcal{B}_0^r(i)$ is also a Banach space. Moreover, $\mathcal{B}^0(i)$ and $\mathcal{B}_0^0(i)$ are known to be the dual and predual, repecetively, of $B^{1,0}$.

Section 2 is devoted to the relationship between $A^{1,r}$ and $B^{1,r}$, in fact, there is an isometry isomorphism from $B^{1,r}$ onto $A^{1,r}$. Moreover, we introduce weighted Bloch spaces which are Banach spaces. We note that the reproducing kernel for $B^{2,0}$ does not belong to $B^{1,0}$ and hence the projection P dose not map $L^{1,0}$ into $B^{1,0}$. We are going to define a projection from L^∞ into $\mathcal{B}^r(i)$. To do so, we need a modified reproducing kernel which is in $L^{1,0}$. In Section3, we let $\mathcal{R}$ denote the radial differentiation of $f \in C^1(H)$ defined by $\mathcal{R}f(z) = xD_1f(z) + yD_2f(z)$, where $z = x+iy \in H$ and $D_1(D_2, resp.)$ denotes the differentiation with respect to $x(y, resp.)$. For $t > 0$ and $f(z) = u(x,y) + iv(x,y)$, $\frac{d}{dt}f(tz) = xD_1f(tz) + yD_2f(tz) = \mathcal{R}f(tz)$ and hence $\mathcal{R}f = 0$ if and only if f is radially constant, that is, $f(z) = f(tz)$ for all $t > 0$ and all $z \in H$. We show that for any $f \in B^{p,r}$ and any $\varphi \in Aut(H), f\circ\varphi$ can be considered as the radial derivative of some element of $B^{p,r}$. We note that some Bloch functions do not have the vanishing property. In fact, we show that for any $f \in \mathcal{B}^r(i)$ and any $\varphi \in Aut(H), f \circ \varphi$ can be considered as a radial derivative of some element of $\mathcal{B}^r(i)$ if and only if f has the vanishing property along the ray. Moreover, we get some operators induced by radial derivatives which are bounded. Throughout this paper, we use the symbol $A \lesssim B$ for nonnegative constants A, B to indicate that A is dominated by B times some positive constant.

2. Weighted Bloch spaces

In this section, we construct several function spaces and we find the relationship between $\mathcal{B}(\mathbb{D})$ and $\mathcal{B}^0(i)$ and between $A^{1,r}$ and $B^{1,r}$ and hence we get the fact that for $r \geq 0$, the dual space of $B^{1,r}$ is $\mathcal{B}^0(i)$. To do so, we will show that $\mathcal{B}^r(i)$ is a Banach space. Take any $z = x+iy$ in H and any f in $\mathcal{B}^r(i)$. Suppose $f(z) = u(x,y) + iv(x,y)$. The mean value theorem for several variables implies that

$$\begin{aligned}
|f(z)| &\le |f(z)-f(i)|+|f(i)| \\
&\le |u(x,y)-u(0,1)|+|v(x,y)-v(0,1)|+|f(i)| \\
&= |\nabla u(x_0,y_0)||z-i|+|\nabla v(x_1,y_1)||z-i|+|f(i)| \\
&\quad \text{for some } (x_0,y_0) \text{ and } (x_1,y_1)\in\mathbb{R}^2 \\
&\le 2|f'(z_0)||z-i|+2|f'(z_1)||z-i|+|f(i)| \\
&= 2(\mathrm{Im} z_0)^{1+2r}|f'(z_0)|\frac{|z-i|}{(\mathrm{Im} z_0)^{1+2r}} \\
&\quad +2(\mathrm{Im} z_1)^{1+2r}|f'(z_1)|\frac{|z-i|}{(\mathrm{Im} z_1)^{1+2r}}+|f(i)| \\
&\le 2\|f\|_{\mathcal{B}^r}|zi|\Big(\frac{1}{(\mathrm{Im} z_0)^{1+2r}}+\frac{1}{(\mathrm{Im} z_1)^{1+2r}}\Big).
\end{aligned}$$

If K is a compact subset of H then f is uniformly bounded on K and hence we have the following :

PROPOSITION 2.1. *Each $\mathcal{B}^r(i)$ is a Banach space.*

Let's show that $B^{1,r}$ can be identified with $A^{1,r}$. To do so, we introduce some function spaces. Let B^r denote the set $B^{1,r}$ with the norm $\|f\|_B=\int_H|f(z)|K(z,z)^{-r}dA$ and A^r the set $A^{1,r}$ with the norm $\|f\|_A=\int_{\mathbb{D}}|g(z)|K_{\mathbb{D}}(z,z)^{-r}dA(z)$. Clearly, $B^{1,r}\cong B^r$ via $f\mapsto\frac{1}{2r+1}f$ and $A^{1,r}\cong A^r$ via $f\mapsto\frac{1}{2r+1}f$. Then we have the following:

PROPOSITION 2.2. *$B^r\cong A^r$ under the following function Ψ : for any $f\in B^r$, $\Psi(f)(w)=\dfrac{2^{2+4r}f(g(w))}{(1-w)^{4+4r}}$, where $g(w)=\dfrac{1+w}{1-w}i$.*

Proof. Take any f in B^r. Let $g(z)=\dfrac{1+z}{1-z}i$, that is, g is a Riemann map from $\mathbb{D}$ onto H. Then $\Psi(f)$ is analytic on $\mathbb{D}$ and

$$\begin{aligned}
&\|\Psi(f)\|_A \\
=& \int_{\mathbb{D}}\Big|\frac{2^{2+4r}f(g(w))}{(1-w)^{4+4r}}\Big|K_{\mathbb{D}}(w,w)^{-r}dA(w) \\
=& \int_H\frac{2^{2+4r}|f(g(g^{-1}(z)))|}{|1-g^{-1}(z)|^{4+4r}}K_{\mathbb{D}}(g^{-1}(z),g^{-1}(z))^{-r}|(g^{-1})'(z)|^2dA(z) \\
=& \int_H\frac{|z+i|^{4+4r}}{4}|f(z)|\frac{(-1)^r\pi^r(2\mathrm{Im} z)^{2r}}{|z+i|^{4r}}\frac{4}{|z+i|^4}dA(z) \\
=& \int_H|f(z)|K(z,z)^{-r}dA(z)=\|f\|_B.\ \text{Hence } \Psi(f)\in A^r.
\end{aligned}$$

That is, Ψ is well-defined and Ψ is an isometry. Clearly, Ψ is linear. Let's show that Ψ is onto. Take any h in A^r . We define $f(z) = \frac{(1-g^{-1}(z))^{4+4r}}{2^{2+4r}} h(g^{-1}(z))$ for all $z \in H$. Then f is analytic on H and

$$\begin{aligned}
\|f\|_B &= \int_H |f(z)| K(z,z)^{-r} dA(z) \\
&= \int_H \frac{(1-g^{-1}(z))^{4+4r}}{2^{2+4r}} |h(g^{-1}(z))| K(z,z)^{-r} dA(z) \\
&= \int_{\mathbb{D}} \frac{|1-w|^{4+4r}}{2^{2+4r}} |h(w)| K(g(w),g(w))^{-r} |g'(w)|^2 dA(w) \\
&= \int_{\mathbb{D}} |h(w)| (\pi(1-|w|^2)^2)^r dA(w) \\
&= \int_{\mathbb{D}} |h(w)| K_{\mathbb{D}}(w,w)^{-r} dA(w) = \|h\|_A.
\end{aligned}$$

This implies that $f \in B^r$. Since

$$\Psi(f)(w) = \frac{2^{2+4r} f(g(w))}{(1-w)^{4+4r}} = \frac{2^{2+4r} \frac{(1-g^{-1}(g(w)))^{4+4r}}{2^{2+4r}}}{(1-w)^{4+4r}} h(g^{-1}(g(w))) = h(w),$$

Ψ is onto. Hence Ψ is an isometry isomorphism. □

We note that $K(\cdot,w) = -\frac{1}{\pi(z-\overline{w})^2}$ is the reproducing kernel for $B^{2,0}$. But $K(\cdot,w) \notin B^{1,0}$and P does not map $L^{1,0}$ into $B^{1,0}$. This implies that we need a modified reproducing kernel $M(\cdot,w) = K(\cdot,w) - K(\cdot,i)$. Then we can show that $M(\cdot,w) \in L^{1,0}$ and $f \mapsto \int_H f(z)\overline{M(z,w)} dA(z)$ is a bounded linear operator from L^∞ into $\mathcal{B}^0(i)$.

THEOREM 2.3. *We define* $Q : L^\infty \to \mathcal{B}^r(i)$ *by*

$$Q(b)(z) = \int_H b(w) \overline{M^{1+r}(w,z)} dA(w)$$

for all $b \in L^\infty$. *Then* Q *is a bounded linear operator.*

Proof. Since $K(w,z) = -\dfrac{1}{\pi(w-\overline{z})^2}$ and $M^{1+r}(w,z) = K(w,z)^{1+r} - K(w,i)^{1+r}$ for $j \in \{1,2\}$,

$$\begin{aligned} D_j\overline{M^{1+r}(w,z)} &= D_j\Big(-\frac{1}{\pi(\overline{w}-z)^2}\Big)^{1+r} \\ &= \Big(-\frac{1}{\pi}\Big)^{1+r} D_j\Big(-\frac{1}{(\overline{w}-z)^{2(1+r)}}\Big) \\ &= \Big(-\frac{1}{\pi}\Big)^{1+r}(-1)(2+2r)\frac{1}{(\overline{w}-z)^{3+2r}}. \end{aligned}$$

Take any b in L^∞ and any closed contour C in H. Then

$$\begin{aligned} \int_C Q(b)(z)dA(z) &= \int_C\int_H b(w)\overline{M^{1+r}(w,z)}dA(w)dA(z) \\ &= \int_H b(w)\int_C \overline{M^{1+r}(w,z)}dA(z)dA(w) \\ &= \int_H b(w)\cdot 0dA(w) = 0. \end{aligned}$$

By Morera's Theorem, $Q(b)$ is analytic on H. Let $z = x+iy$ and $w = s+it$ be in H. Then for each $j \in \{1,2\}$,

$$\begin{aligned} & y^{1+2r}|D_jQ(b)(z)| \\ = \;& y^{1+2r}|D_j\int_H b(w)\overline{M^{1+r}(w,z)}dA(w)| \\ \leq \;& y^{1+2r}\frac{2+2r}{\pi^{1+r}}\int_H |b(w)|\frac{1}{|\overline{w}-z|^{3+2r}}dA(w) \\ \leq \;& \frac{2+2r}{\pi^{1+r}}\|b\|_\infty y^{1+2r}\int_H \frac{1}{|\overline{w}-z|^{3+2r}}dA(w) \\ = \;& \frac{2+2r}{\pi^{1+r}}\|b\|_\infty y^{1+2r}\int_0^\infty\int_{-\infty}^\infty \frac{1}{\{(s-x)^2+(t+y)^2\}^{\frac{3+2r}{2}}}dsdt \\ = \;& \frac{2+2r}{\pi^{1+r}}\|b\|_\infty y^{1+2r}\int_0^\infty\int_{-\infty}^\infty \frac{t+y}{(s-x)^2+(t+y)^2} \\ \times \;& \frac{1}{(t+y)\{(s-x)^2+(t+y)^2\}^{\frac{1+2r}{2}}}dsdt \\ \leq \;& \frac{2+2r}{\pi^r}\|b\|_\infty y^{1+2r}\int_0^\infty \frac{1}{(t+y)^{2+2r}}dt \\ = \;& \frac{2+2r}{(1+2r)\pi^r}\|b\|_\infty. \end{aligned}$$

This implies that $Q(b) \in \mathcal{B}^r(i)$ and Q is bounded. □

3. Some linear operators

We note that $Aut(H)$ is the Möbius group of bi-analytic mappings of H. For each $\varphi \in Aut(H)$, there are real numbers a and b such that $a > 0$ and $\varphi(z) = az + b$ for all $z \in H$ (see [2]). The Bergman kernel functions are intimately related to $Aut(H)$ of the half-plane . Moreover, for each $f \in \mathcal{B}^r(i)$ and any $\varphi \in Aut(H)$, $\|f\|_H \approx \|f \circ \varphi\|_H$, that is, $\|f\|_H \lesssim \|f \circ \varphi\|_H$ and $\|f\|_H \gtrsim \|f \circ \varphi\|_H$.

LEMMA 3.1. (1) *Suppose* $1 \le p < \infty$ *and* $r \ge 0$. *If* $f \in B^{p,r}$ *then* f *has the vanishing property along the ray, that is,* $\lim_{t \to \infty} f(tz) = 0$ *for all* $z \in H$.

(2) *Suppose* $1 \le p < \infty$ *and* $r \ge 0$. *If* $f \in B^{p,r}$ *and* $\varphi \in Aut(H)$ *then* $f \circ \varphi \in B^{p,r}$.

Proof. (1) See [4].

(2) Since $\varphi \in Aut(H)$, $\varphi(z) = az + b$ for some $a > 0$ and $b \in \mathbb{R}$. Take any f in $B^{p,r}$. Then

$$\begin{aligned}
&\|f \circ \varphi\|_{p,r}^p \\
= &\int_H |f \circ \varphi|^p dA_r \\
= &(2r+1)\int_H |f(\varphi(z))|^p K(z,z)^{-r} dA(z) \\
= &(2r+1)\int_H |f(w)|^p K(\varphi^{-1}(w), \varphi^{-1}(w))^{-r} |(\varphi^{-1})'(w)|^2 dA(w) \\
= &(2r+1)\int_H |f(w)|^p \Big(\frac{a^2}{\pi(2\mathrm{Im} w)^2}\Big)^{-r} \Big(\frac{1}{a}\Big)^2 dA(w) \\
= &a^{-2-2r} \int_H (2r+1)|f(w)|^p K(w,w)^{-r} dA(w) \\
= &a^{-2-2r} \|f\|_{p,r}^p \text{ and hence } f \circ \varphi \in B^{p,r}.
\end{aligned}$$

□

The property of every element of $B^{p,r}$ in Lemma 3.1 implies that for $1 \le p < \infty$ and $r \ge 0$, each element of $B^{p,r}$ can be represented as a radial derivative of some element of $B^{p,r}$.

THEOREM 3.2. *Suppose $1 \leq p < \infty$ and $r \geq 0$. Then for each $f \in B^{p,r}$ and any $\varphi \in Aut(H)$, there is a unique $\widetilde{f \circ \varphi} \in B^{p,r}$ such that $\mathcal{R}\widetilde{f \circ \varphi} = f \circ \varphi$. Moreover, $f \longmapsto \widetilde{f \circ \varphi}$ is bounded on $B^{p,r}$.*

Proof. Suppose $\mathcal{R}\widetilde{f \circ \varphi} = \mathcal{R}\widetilde{g \circ \varphi}$. Since $\mathcal{R}(\widetilde{f \circ \varphi} - \widetilde{g \circ \varphi}) = 0, \widetilde{f \circ \varphi} - \widetilde{g \circ \varphi}$ is radially constant, that is, $(\widetilde{f \circ \varphi} - \widetilde{g \circ \varphi})(z) = (\widetilde{f \circ \varphi} - \widetilde{g \circ \varphi})(tz)$ for all $t > 0$ and $z \in H$. Since $\widetilde{f \circ \varphi} - \widetilde{g \circ \varphi}$ is in $B^{p,r}$, $(\widetilde{f \circ \varphi} - \widetilde{g \circ \varphi})(tz) \to 0$ as $t \to \infty$ and hence we get the uniqueness of $\widetilde{f \circ \varphi}$. Take any f in $B^{p,r}$ and any φ in $Aut(H)$. Then $\varphi(z) = az + b$ for some $a > 0$ and $b \in \mathbb{R}$. For each $z \in H$, we define $\widetilde{f}(z) = -\int_1^\infty \frac{f(tz)}{t} dt$. Since f has the vanishing property along the ray, $\widetilde{f}$ is well-defined and analytic on H. Minkowski's integral inequality implies that

$$\begin{aligned}
&\|\widetilde{f \circ \varphi}\|_{p,r} \\
= \;& \Big(\int_H |\widetilde{f \circ \varphi}(z)|^p (2r+1) K(z,z)^{-r} dA(z)\Big)^{\frac{1}{p}} \\
\leq \;& \Big(\int_H \int_1^\infty \frac{|f(\varphi(tz))|^p}{t^p} dt (2r+1) K(z,z)^{-r} dA(z)\Big)^{\frac{1}{p}} \\
\leq \;& \int_1^\infty \Big(\int_H \frac{|f(\varphi(tz))|^p}{t^p} (2r+1) K(z,z)^{-r} dA(z)\Big)^{\frac{1}{p}} dt \\
= \;& \int_1^\infty \frac{1}{t}\Big(\int_H |f(w)|^p \frac{|(\varphi^{-1})'(w)|^2}{t^2} (2r+1) \\
& \times K(\frac{\varphi^{-1}(w)}{t}, \frac{\varphi^{-1}(w)}{t})^{-r} dA(w)\Big)^{\frac{1}{p}} dt \\
= \;& \int_1^\infty \frac{1}{t}\Big(\int_H |f(w)|^p \frac{1}{t^2}\frac{1}{a^2} (2r+1) \Big(\frac{1}{\pi(2\mathrm{Im} w)^2}\Big)^r (a^2t^2)^{-r} dA(w)\Big)^{\frac{1}{p}} dt \\
= \;& \|f\|_{p,r} \frac{1}{a^{\frac{2+2r}{p}}} \int_1^\infty t^{-(1+\frac{2+2r}{p})} dt.
\end{aligned}$$

Since $1 + \frac{2r+2}{p} > 1$, $\int_1^\infty t^{-(1+\frac{2+2r}{p})} dt$ is finite and hence $\widetilde{f \circ \varphi} \in B^{p,r}$ and $f \mapsto \widetilde{f \circ \varphi}$ is bounded on $B^{p,r}$. We note that for $j \in \{1,2\}$,

$$\begin{aligned}
D_j(\widetilde{f \circ \varphi})(z) &= D_j\Big(-\int_1^\infty \frac{f(\varphi(tz))}{t} dt\Big) \\
&= -\int_1^\infty \frac{1}{t} D_j f(\varphi(tz)) a t dt = -\int_1^\infty a D_j f(\varphi(tz)) dt
\end{aligned}$$

and $\frac{d(f(\varphi(tz))}{dt} = \frac{d(f(atz+b))}{dt} = D_1 f(\varphi(tz))ax + D_2 f(\varphi(tz))ay = a(xD_1 f(\varphi(tz)) + yD_2 f(\varphi(tz)))$, where $z = x+iy$ and hence $\mathcal{R}\widetilde{f\circ\varphi}(z) = xD_1\widetilde{f\circ\varphi}(z) + yD_2\widetilde{f\circ\varphi}(z) = -\int_1^\infty a(xD_1 f(\varphi(tz)) + yD_2 f(\varphi(tz)))dt = -\int_1^\infty \frac{d(f(\varphi(tz))}{dt}dt = f(\varphi(z))$ because $f\circ\varphi$ has the vanishing property along the ray. □

EXAMPLE 3.3. We note that

$$\mathcal{B}_0^r(i) = \{f \in \mathcal{B}^r(i) : \lim_{z\to\partial^\infty H} y^{1+2r}|f'(z)| = 0\}$$

and $M(z,w)^{1+r} = K(z,w)^{1+r} - K(z,i)^{1+r}$ is a modified reproducing kernel. Then $M(z,w)^{1+r} = \frac{(-1)^{1+r}}{\pi^{1+r}}\Big(\frac{1}{(z-\overline{w})^{2+2r}} - \frac{1}{(z+i)^{2+2r}}\Big)$. Put $z = x+iy$ and $w = s+it$. Since

$$\frac{d}{dz}M(z,w)^{1+r} = \frac{(-1)^{1+r}(-2-2r)}{\pi^{1+r}}\{\frac{1}{(z-\overline{w})^{3+2r}} - \frac{1}{(z+i)^{3+2r}}\},$$

$$y^{1+2r}|\frac{d}{dz}M(z,w)^{1+r}| \le \frac{2+2r}{\pi^{1+2r}}\Big(\frac{1}{|z-\overline{w}|^2} + \frac{1}{|z+i|^2}\Big) \le \frac{2+2r}{\pi^{1+2r}}(\frac{1}{t^2}+1).$$

Since $\lim_{z\to\partial^\infty H} y^{1+2r}|\frac{d}{dz}M(z,w)^{1+r}| = 0$, $M(z,w)^{1+r}$ belongs to $\mathcal{B}_0^r(i)$. Suppose that there is f in $\mathcal{B}_0^r(i)$ such that $M(z,w)^{1+r} = xD_1 f(z) + yD_2 f(z) = \mathcal{R}f(z)$. Consider $z = \frac{1}{n}i$. Then $\lim_{n\to\infty}\frac{1}{n}i = 0$ and $\lim_{n\to\infty} M(z,w)^{1+r} = \frac{(-1)^{1+r}}{\pi^{1+r}}(\frac{1}{\overline{w}^{2+2r}} - \frac{1}{i^{2+2r}})$. This contradicts to the fact that $\lim_{n\to\infty}(\frac{1}{n})^{1+2r}D_2 f(z) = 0$ and hence Theorem 3.2 does not hold on $\mathcal{B}_0^r(i)$. □

Every element of $\mathcal{B}_0^r(i)$ is a special element of $\mathcal{B}^r(i)$ which is related with some limit but Theorem 3.2 does not hold on $\mathcal{B}_0^r(i)$ and hence we consider a radial derivative of each element of $\mathcal{B}^r(i)$.

THEOREM 3.4. *Suppose that $r > 0, \varphi \in Aut(H)$, and $f \in \mathcal{B}^r(i)$. Then there is a unique $\widehat{f\circ\varphi} \in \mathcal{B}^r(i)$ such that $\mathcal{R}\widehat{f\circ\varphi} = f\circ\varphi$ if and only if $\lim_{t\to\infty} f(tz) = 0$ for all $z \in H$, that is, f has the vanishing property along the ray. Moreover, $f \longmapsto \widehat{f\circ\varphi}$ is bounded.*

Proof. We note that there are real numbers a and b such that $a > 0$ and $\varphi(z) = az + b$ for all $z \in H$. Take any f in $\mathcal{B}^r(i)$. Since

$$
\begin{aligned}
|\frac{d}{dt} f(\varphi(z))| = a|f'(az+b)|, \|f \circ \varphi\|_H &= \sup_{z=x+iy \in H} y^{1+2r} |\frac{d}{dz} f(\varphi(z))| \\
&= \frac{1}{a^{2r}} \sup_{z=x+iy\in H} (ay)^{1+2r} |f'(az+b)| = \frac{1}{a^{2r}} \|f\|_H
\end{aligned}
$$

and hence $f \circ \varphi \in \mathcal{B}^r(i)$. Let $\widehat{f \circ \varphi}(z) = -\int_1^\infty \frac{f(\varphi(tz)) - f(\varphi(ti))}{t} dt$ and let $z = x + iy$. Since

$$
D_j \widehat{f \circ \varphi}(z) = -\int_1^\infty \frac{D_j f(atz + b)}{t} at dt,
$$

$$
\begin{aligned}
|y^{1+2r} D_j \widehat{f \circ \varphi}(z)| &\le y^{1+2r} a \int_1^\infty |D_j f(atz+b)| dt \\
&= \int_1^\infty (aty)^{1+2r} \frac{1}{a^{2r}} \frac{1}{t^{1+2r}} |D_j f(atz+b)| dt \\
&\le \frac{\|f\|_H}{a^{2r}} \int_1^\infty \frac{1}{t^{1+2r}} dt.
\end{aligned}
$$

Since $r > 0$, $\widehat{f \circ \varphi} \in \mathcal{B}^r(i)$. Since $\lim_{t\to\infty} f(\varphi(tz)) = 0$, $\widehat{f \circ \varphi}$ is unique and

$$
\begin{aligned}
\mathcal{R}\widehat{f \circ \varphi} &= x D_1 \widehat{f \circ \varphi}(z) + y D_2 \widehat{f \circ \varphi}(z) \\
&= -\int_1^\infty (ax D_1 f(\varphi(tz)) + ay D_2 f(\varphi(tz))) dt \\
&= -\int_1^\infty \frac{df \circ \varphi}{dt}(tz) dt = f \circ \varphi(z).
\end{aligned}
$$

Conversely, we note that $\mathcal{R}\widehat{f \circ \varphi}(z) = -\int_1^\infty \frac{df \circ \varphi}{dt}(tz)dt = f \circ \varphi(z) - \lim_{s\to\infty} f \circ \varphi(sz) = f \circ \varphi(z)$ and $\mathcal{R}\widehat{f \circ \varphi}(z) = f \circ \varphi(z)$ for all $z \in H$ and hence $\lim_{s\to\infty} f \circ \varphi(sz) = 0$. By the above observation, $\|\widehat{f \circ \varphi}\|_H \lesssim \|f\|_H$. Thus $f \mapsto \widehat{f \circ \varphi}$ is bounded. □

COROLLARY 3.5. (See [4]) (1) *Suppose* $1 \le p < \infty$ *and* $r \ge 0$. *Then for each* $f \in B^{p,r}$, *there is a unique* $\widetilde{f} \in B^{p,r}$ *such that* $f = \mathcal{R}\widetilde{f}$. *Moreover,* $f \mapsto \widetilde{f}$ *is bounded on* $B^{p,r}$.

(2) *For $r > 0$and $f \in \mathcal{B}^r(i)$, there is a unique $\widehat{f} \in \mathcal{B}^r(i)$ such that $\mathcal{R}\widetilde{f} = f$ and only if $\lim_{t\to\infty} f(tz) = 0$ for all $z \in H$, that is, f has vanishing property along the ray. Moreover, $f \longmapsto \widehat{f}$ is bounded.*

Proof. It is immediate from the fact that $\varphi(z) = z$ is in $Aut(H)$. □

Theorem 3.2 says that each element of $B^{p,r}$ can be considered as the radial derivative of some element of $B^{p,r}$. We do not quarantee this fact on $\mathcal{B}^r(i)$.

PROPOSITION 3.6. *There is a function $f \in \mathcal{B}^r(i)$ such that $f \neq \mathcal{R}g$ for all $g \in \mathcal{B}^r(i)$.*

Proof. Fix $w \in H$ and we consider a modified reproducing kernel $M(z,w)^{1+r} = K(z,w)^{1+r} - K(z,i)^{1+r}$. Example 3.3 implies that $M(\cdot,w)^{1+r} \in \mathcal{B}^r_0(i)$. Suppose that there is $g \in \mathcal{B}^r(i)$ such that $\mathcal{R}g = M(\cdot,w)^{1+r}$. Then $(\frac{-1}{\pi})^{1+r}\Big(\frac{1}{(z-\overline{w})^{2+2r}} - \frac{1}{(z+i)^{2+2r}}\Big) = M(z,w)^{1+r}$ $= xD_1g(z) + yD_2g(z)$, where $z = x+iy$. Put $z = \frac{1}{n}$. Since

$$\lim_{n\to\infty} \frac{1}{n} D_1 g(\frac{1}{n}) = 0,$$

$$\lim_{n\to\infty} (\frac{-1}{\pi})^{1+r}\Big(\frac{1}{(\frac{1}{n}-\overline{w})^{2+2r}} - \frac{1}{(\frac{1}{n}+i)^{2+2r}}\Big)$$

$$= (\frac{-1}{\pi})^{1+r}\Big(\frac{1}{(\overline{w})^{2+2r}} - \frac{1}{(-1)^{1+r}}\Big) = 0$$

which is a contradiction. This completes the proof. □

THEOREM 3.7. *Suppose that $r > 0, f \in L^\infty$, and we define $Q : L^\infty \to \mathcal{B}^r(i)$ by $Q(b)(z) = \int_H b(w)\overline{M(w,z)^{1+r}}dA(w)$ for all $b \in L^\infty$. Then there is a unique element $\overline{f} \in \mathcal{B}^r(i)$ such that $\mathcal{R}\overline{f} = Qf$ if and only if $\lim_{t\to\infty} Qf(tz) = 0$ for all $z \in H$, that is, Qf has the vanishing property along the ray. Moreover, $f \mapsto \overline{f}$ is bounded.*

Proof. Take any f in L^∞. By Theorem 2.4, $Q(f) \in \mathcal{B}^r(i)$. Theorem 3.4 implies that there is a unique $\overline{f} \in \mathcal{B}^r(i)$ such that $\mathcal{R}\overline{f} = Qf$ and hence $\|\overline{f}\|_{\mathcal{B}^r} \lesssim \|Qf\|_{\mathcal{B}^r}$. Since Q is bounded, $\|\overline{f}\|_{\mathcal{B}^r} \lesssim \|f\|_\infty$. Thus $f \mapsto \overline{f}$ is bounded. □

References

[1] B. R. Choe, H. Koo, and H. Yi, *Harmonic Bergman functions as radial derivatives of Bergman functions*, Proc. Amer. Math. Soc. **131** (2003), No. 2, 401–408.
[2] S. G. Gindkin, *Analysis in homogeneous domains*, Uspehi Mat. Nauk **19** (1964), no. 4, 3–92.
[3] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman spaces*, Springer-Verlag, New York, 2000.
[4] S. H. Kang and J. Y. Kim, *The radial derivatives on weighted Bergman spaces*, Commun. Korean Math. Soc. **18** (2003), no. 2, 243–249.
[5] J. Y. Kim, *Weighted analytic Bergman spaces of the half plane and their Toeplitz operators*, Ph. D. Thesis, Sookmyung Women's University, 2001.

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