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STABILITY OF A CUBIC FUNCTIONAL EQUATION ON GROUPS

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Abstract. In this note we will find out the general solution and investigate the generalized Hyers-Ulam-Rassias stability for the cubic functional equation $f(3x+y)+f(3x-y)=3f(x+y)+3f(x-y)+48f(x)$ on abelian groups.

1. Introduction

In 1940, S. M. Ulam [15] raised the following question concerning the stability of group homomorphisms: *Under what condition does there is an additive mapping near an approximately additive mapping between a group and a metric group* ?

In next year, D. H. Hyers [7] answers the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th. M. Rassias [13]. Since then, the stability problems of various functional equation have been extensively investigated by a number of authors (for example, [2, 3, 6, 8, 11, 14]). In particular, one of the important functional equations studied is the following functional equation:

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{1.1}$$

The quadratic function $f(x)=ax^2$ is a solution of this functional equation, and so one usually is said the above functional equation to be quadratic [1, 4, 10, 12].

Quadratic functional equation was used to characterize inner product spaces [1, 5, 9]. A square norm on an inner product space satisfies the important parallelogram law

$$\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2).$$

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It is well known that a function f between real vector spaces X and Y is quadratic if and only if there exists a unique symmetric bi-additive function $B : X \times X \to Y$ such that $f(x) = B(x, x)$ for all $x \in X$ (see [1, 12]), where the function B is given by

$$B(x,y) = \frac{1}{4}(f(x+y) - f(x-y)) \quad \text{for all} \quad x,\ y \in X. \tag{1.2}$$

The Hyers-Ulam stability problem for the quadratic functional equation (1.1) was first proved by F. Skof [14] for functions $f : X \to Y$, where X is a normed space and Y a Banach space. On the other hand, P. W. Cholewa [3] demonstrated that the theorem of Skof is still valid if X is replaced by an abelian group. In [4], S. Czerwik generalized the stability in the sense of Hyers and Ulam for the quadratic functional equation (1.1).

Now, let us introduce the following functional equation

$$f(3x+y) + f(3x-y) = 3f(x+y) + 3f(x-y) + 48f(x). \tag{1.3}$$

It is easy to see that the cubic function $f(x) = cx^3$ is a solution of the above functional equation. So, in this note, we promise that the equation (1.3) is called a cubic functional equation and every solution of the cubic functional equation (1.3) is said to be a cubic function. Here our purpose is to establish the general solution and to examine the generalized Hyers-Ulam-Rassias stability problem [6] for the equation (1.3) on abelian groups. Throughout this note, we will denote by $(G, +)$ an abelian group.

2. Solutions of Eq. (1.3)

In this section, let X be a real vector space. We precede the proof of our main theorem by two trivial lemmas.

LEMMA 2.1. *A function $f : G \to X$ satisfies the functional equation*

$$f(x+2y) + f(x-2y) = 2f(x) + 8f(y) \tag{2.1}$$

for all $x,\ y \in G$ if and only if f is quadratic.

Proof. (*Necessity*). Let $x = 0 = y$ in (2.1). Then we have $f(0) = 0$. Putting $x = 0$ in (2.1) gives

$$f(2y) + f(-2y) = 8f(y), \tag{2.2}$$

and setting $y = -y$ in (2.2), we obtain

$$f(-2y) + f(2y) = 8f(-y), \tag{2.3}$$

and so, by (2.2) and (2.3), we get $f(-y) = f(y)$, i.e., f is an even function.

Hence from (2.2) or (2.3) it follows that

$$f(2y) = 4f(y). \tag{2.4}$$

Substituting $x = 2x$ in (2.1), it follows from (2.4) that f is quadratic.

(*Sufficiency*). Putting $x = 0 = y$ in (1.1) yields $f(0) = 0$. Replacing y by x in (1.1), we get $f(2x) = 4f(x)$. Therefore, the substitution $y = 2y$ in (1.1) now gives the equation (2.1). □

LEMMA 2.2. *A function $f : G \to X$ satisfies the functional equation*

$$f(x + 2y) + f(x - 2y) = 2f(x), \quad f(0) = 0 \tag{2.5}$$

for all $x, y \in G$ if and only if f is additive.

Proof. (*Necessity*). Replacing x by $2y$ in (2.5), we get

$$f(4y) = 2f(2y). \tag{2.6}$$

Putting $x = 2x$ in (2.5) and taking account of (2.6), we obtain

$$f(2x + 2y) + f(2x - 2y) = f(4x). \tag{2.7}$$

From the substitutions $u := 2x + 2y$ and $v := 2x - 2y$ in (2.7), it follows that

$$f(u + v) = f(u) + f(v),$$

which means that f is additive.

(*Sufficiency*). In the additive equation $f(x + y) = f(x) + f(y)$, by letting $y = x$ and then putting $x = x + 2y$, $y = x - 2y$, respectively, it is easy to see that f satisfies the equation (2.5). □

Our main result which presents the general solution of the equation (1.3) is

THEOREM 2.3. *A function $f : G \to X$ satisfies the functional equation (1.3) if and only if there exists a function $F : G \times G \to X$ such that $f(x) = F(x, x)$ for all $x \in G$, and for fixed $y \in G$, the function $A : G \to X$ defined by $A(x) = F(x, y)$ for all $x \in G$ is additive and for fixed $x \in G$, the function $Q : G \to X$ defined by $Q(y) = F(x, y)$ for all $y \in G$ is quadratic.*

Proof. (*Necessity*). Set $x = 0 = y$ in (1.3). Then we get $f(0) = 0$. Putting $x = 0$ in (1.3) gives $f(-y) = -f(y)$. By letting $y = 0$ in (1.3), we have

$$f(3x) = 27f(x). \tag{2.8}$$

Putting $y = x$ in (1.3), we get

$$f(4x) = 2f(2x) + 48f(x). \tag{2.9}$$

Replacing y by $3x$ in (1.3) and using (2.8), we obtain

$$10f(2x) = f(4x) + 16f(x), \tag{2.10}$$

which, by (2.9), yields

$$f(2x) = 8f(x). \tag{2.11}$$

Replacing x and y by $x + y$ and $x - y$ in (1.3), respectively, we have

$$f(4x + 2y) + f(2x + 4y) = 3f(2x) + 3f(2y) + 48f(x + y),$$

which, in view of (2.11), reduces to

$$f(2x + y) + f(x + 2y) = 3f(x) + 3f(y) + 6f(x + y). \tag{2.12}$$

Putting $x = x + 3y$ and $y = x - 3y$ in (2.12) and then using (2.8) and (2.11), we have

$$9f(x + y) + 9f(x - y) = f(x + 3y) + f(x - 3y) + 16f(x). \tag{2.13}$$

Let us interchange x with y in (2.13). Then we get the relation

$$9f(x + y) - 9f(x - y) = f(3x + y) - f(3x - y) + 16f(y). \tag{2.14}$$

Then, by adding (2.12) to (2.13), we lead to

$$18f(x+y) = f(x+3y)+f(x-3y)+f(3x+y)-f(3x-y)+16f(x)+16f(y). \tag{2.15}$$

On the other hand, if we interchange x with y in (1.3), we get

$$f(x + 3y) - f(x - 3y) = 3f(x + y) - 3f(x - y) + 48f(y). \tag{2.16}$$

Hence, according to (1.3) and (2.16), we obtain

$$6f(x+y) = f(3x+y)+f(3x-y)+f(x+3y)-f(x-3y)-48f(x)-48f(y). \tag{2.17}$$

Now, by adding (2.15) and (2.17), we arrive at

$$f(x + 3y) + f(3x + y) = 12f(x + y) + 16f(x) + 16f(y). \tag{2.18}$$

Using (1.3), we have

$$\begin{aligned} &16f(3x + z) + 16f(3x - z) + 16f(3y + z) + 16f(3y - z) \\ = \; &48f(x + z) + 48f(x - z) + 768f(x) + 48f(y + z) \\ &+48f(y - z) + 768f(y). \end{aligned} \tag{2.19}$$

Also, putting $x = 3x + z$ and $y = 3y + z$ in (2.18) and using (1.3), respectively, we deduce that

$$\begin{aligned} & 16f(3x+z)+16f(3y+z)+16f(3x-z)+16f(3y-z) \\ = & f(3x+9y+4z)+f(9x+3y+4z)-12f(3x+3y+2z) \\ & +f(3x+9y-4z)+f(9x+3y-4z)-12f(3x+3y-2z) \\ = & 3f(x+3y+4z)+3f(x+3y-4z)+48f(x+3y) \\ & +3f(3x+y+4z)+3f(3x+y-4z)+48f(3x+y) \\ & -36f(x+y+2z)-36f(x+y-2z)-576f(x+y), \end{aligned}$$

which yields, by virtue of (2.19), the relation

$$\begin{aligned} (2.20) \qquad & 3f(3x+y+4z)+3f(3x+y-4z)+48f(3x+y) \\ & +3f(x+3y+4z)+3f(x+3y-4z)+48f(x+3y) \\ = & 48f(x+z)+48f(x-z)+768f(x)+48f(y+z) \\ & +48f(y-z)+768f(y) \\ & +36f(x+y+2z)+36f(x+y-2z)+576f(x+y). \end{aligned}$$

On account of (2.18) and (1.3), the left hand side of (2.19) can be written in the form

$$\begin{aligned} (2.21) \qquad & 16f(3x+z)+16f(3y-z)+16f(3x-z)+16f(3y+z) \\ = & f(3x+9y-2z)+f(9x+3y+2z)-12f(3x+3y) \\ & +f(9x+3y-2z)+f(3x+9y+2z)-12f(3x+3y) \\ = & 3f(x+3y+2z)+3f(x+3y-2z)+48f(x+3y) \\ & +3f(3x+y+2z)+3f(3x+y-2z)+48f(3x+y) \\ & -648f(x+y). \end{aligned}$$

Replacing z by $2z$ in (2.21) and then applying (2.20), we obtain

$$\begin{aligned} (2.22)\ & 16f(3x+2z)+16f(3y-2z)+16f(3x-2z)+16f(3y+2z) \\ = & 3f(x+3y+4z)+3f(x+3y-4z)+48f(x+3y) \\ & +3f(3x+y+4z)+3f(3x+y-4z) \\ & +48f(3x+y)-648f(x+y) \\ = & 768f(x)+768f(y)+48f(x+z)+48f(x-z)+48f(y+z) \\ & +48f(y-z)+36f(x+y+2z)+36f(x+y-2z)-72f(x+y). \end{aligned}$$

Again, making use of (2.18) and (1.3), we get

$$
\begin{aligned}
(2.23)\quad & 16f(3x+2z)+16f(3x-2z)+16f(3y+2z)+16f(3y-2z)\\
= \;& f(12x+4z)+f(12x-4z)-12f(6x)\\
& +f(12y+4z)+f(12y-4z)-12f(6y)\\
= \;& 64f(3x+z)+64f(3x-z)-2592f(x)+64f(3y+z)\\
& +64f(3y-z)-2592f(y)\\
= \;& 64[3f(x+z)+3f(x-z)+48f(x)+3f(y+z)+3f(y-z)\\
& +48f(y)]-2592f(x)-2592f(y)\\
= \;& 192f(x+z)+192f(x-z)+480f(x)\\
& +192f(y+z)+192f(y-z)+480f(y).
\end{aligned}
$$

Finally, if we compare (2.22) with (2.23), then we conclude that

$$
\begin{aligned}
(2.24)\quad & f(x+y+2z)+f(x+y-2z)+8f(x)+8f(y)\\
= \;& 2f(x+y)+4f(x+z)+4f(x-z)+4f(y+z)+4f(y-z)
\end{aligned}
$$

for all $x,\ y \in G$.

Define the function $F : G \times G \to X$ by

$$F(x,y) = \frac{1}{36}[8f(x+y)+8f(x-y)-f(2x+y)-f(2x-y)]$$

for all $x,\ y \in G$. Then by an simple calculation, we see that

$$F(x,x) = \frac{1}{36}[8f(2x)-f(3x)-f(x)] = f(x)$$

for all $x \in G$. Now, we claim that for each fixed $x \in G$, the function $Q : G \to X$ defined by $Q(y) = F(x,y)$ for all $y \in G$ is quadratic.

Indeed, utilizing (2.24) and the oddness of f, we get

$$
\begin{aligned}
& 36[F(x,y+2z)+F(x,y-2z)-2F(x,y)-8F(x,z)]\\
= \;& 8f(x+y+2z)+8f(x-y-2z)-f(2x+y+2z)\\
& -f(2x-y-2z)+8f(x+y-2z)+8f(x-y+2z)\\
& -f(2x+y-2z)-f(2x-y+2z)-16f(x+y)-16f(x-y)\\
& +2f(2x+y)+2f(2x-y)-64f(x+z)-64f(x-z)\\
& +8f(2x+z)+8f(2x-z)
\end{aligned}
$$

$$
\begin{aligned}
= \; & 16f(x+y)+32f(y+z)+32f(y-z)+32f(x+z)+32f(x-z) \\
& -64f(x)-64f(y)+16f(x-y)-32f(y-z)-32f(y+z) \\
& +32f(x+z)+32f(x-z)-64f(x)+64f(y)-2f(2x+y) \\
& -4f(y+z)-4f(y-z)-4f(2x+z)-4f(2x-z)+64f(x) \\
& +8f(y)-2f(2x-y)+4f(y-z)+4f(y+z)-4f(2x+z) \\
& -4f(2x-z)+64f(x)-8f(y)-16f(x+y)-16f(x-y) \\
& +2f(2x+y)+2f(2x-y)-64f(x+z)-64f(x-z) \\
& +8f(2x+z)+8f(2x-z)=0.
\end{aligned}
$$

Therefore, it follows from Lemma 2.1 that Q is quadratic.

Also, by using the similar argument, we can show that for each fixed $y \in G$,

$$36[F(x+2z,y)+F(x-2z,y)-2F(x,y)]=0$$

for all $x \in G$, and so we see that for each fixed $y \in G$, the function $A : G \to X$ defined by $A(x) = F(x,y)$ for all $x \in G$ is additive by Lemma 2.2.

(*Sufficiency*). Assume that there exists a function $F : G \times G \to X$ such that $f(x) = F(x,x)$ for all $x \in G$, and for fixed $y \in G$, $A : G \to X$ defined by $A(x) = F(x,y)$ for all $x \in G$ is additive and for fixed $x \in G$, $Q : G \to X$ defined by $Q(y) = F(x,y)$ for all $y \in G$ is quadratic. Then for fixed $w \in G$, the function $B_w : G \times G \to X$ defined by $B_w(x,y) = \frac{1}{4}[F(w,x+y) - F(w,x-y)]$ for all $x,\ y \in G$ is symmetric and biadditive because $Q = F(w,\cdot)$ is quadratic [1]. Therefore, we have

$$
\begin{aligned}
& f(3x+y)+f(3x-y)-3f(x+y)-3f(x-y)-48f(x) \\
= \; & F(3x+y,3x+y)+F(3x-y,3x-y) \\
& -3F(x+y,x+y)-3F(x-y,x-y)-48F(x,x) \\
= \; & F(3x,3x+y)+F(y,3x+y)+F(3x,3x-y)-F(y,3x-y) \\
& -3F(x,x+y)-3F(y,x+y)-3F(x,x-y) \\
& +3F(y,x-y)-48F(x,x) \\
= \; & 2F(3x,3x)+2F(3x,y)-6F(x,x)-6F(x,y)+4B_y(3x,y) \\
& -12B_y(x,y)-48F(x,x)=0.
\end{aligned}
$$

That is, f satisfies the equation (1.3). This completes the proof of the theorem. □

3. Stability of Eq. (1.3)

In this section, we will investigate the generalized Hyers-Ulam- Rassias stability problem [6] for the functional equation (1.3).

Let $\phi : G \times G \to [0, \infty)$ be a function such that

$$\sum_{i=0}^{\infty} \frac{\phi(3^i x, 0)}{27^i} < \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\phi(3^n x, 3^n y)}{27^n} = 0$$

for all $x, y \in G$.

THEOREM 3.1. *Let X be a Banach space. If a function $f : G \to X$ satisfies the inequality*

$$\|f(3x+y)+f(3x-y)-3f(x+y)-3f(x-y)-48f(x)\| \le \phi(x,y) \tag{3.1}$$

for all $x, y \in G$, then there exists a unique cubic function $C : G \to X$ which satisfies the equation (1.3) and the inequality

$$\|f(x) - C(x)\| \le \frac{1}{54}\sum_{i=0}^{\infty} \frac{\phi(3^i x, 0)}{27^i} \tag{3.2}$$

holds for all $x \in G$, where the function C is given by

$$C(x) = \lim_{n\to\infty} \frac{f(3^n x)}{27^n} \tag{3.3}$$

for all $x \in G$.

Proof. Putting $y = 0$ in (3.1) and dividing by 54, we have

$$\left\|\frac{f(3x)}{27} - f(x)\right\| \le \frac{1}{54}\phi(x, 0) \tag{3.4}$$

for all $x \in G$. Replacing x by $3x$ in (3.4) and dividing by 27 and summing the resulting inequality with (3.4), we get

$$\left\|\frac{f(3^2 x)}{27^2} - f(x)\right\| \le \frac{1}{54}[\phi(x, 0) + \frac{\phi(3x, 0)}{27}] \tag{3.5}$$

for all $x \in X$. Using the induction on n, we obtain that

$$\left\|\frac{f(3^n x)}{27^n} - f(x)\right\| \le \frac{1}{54}\sum_{i=0}^{n-1} \frac{\phi(3^i x, 0)}{27^i} \le \frac{1}{54}\sum_{i=0}^{\infty} \frac{\phi(3^i x, 0)}{27^i} \tag{3.6}$$

for all $x \in X$. In order to prove convergence of the sequence $\{\frac{f(3^n x)}{27^n}\}$, we divide the inequality (3.6) by 27^m and also replace x by $3^m x$ to find that for $n,\ m > 0$,

$$\begin{aligned} \left\| \frac{f(3^n 3^m x)}{27^{n+m}} - \frac{f(3^m x)}{27^m} \right\| &= \frac{1}{27^m} \left\| \frac{f(3^n 3^m x)}{27^n} - f(3^m x) \right\| \\ &\leq \frac{1}{54 \cdot 27^m} \sum_{i=0}^{n-1} \frac{\phi(3^{m+i}x, 0)}{27^i} \\ &\leq \frac{1}{54} \sum_{i=0}^{\infty} \frac{\phi(3^{m+i}x, 0)}{27^{m+i}}. \end{aligned}$$

Since the right hand side of the inequality tends to 0 as $m \to \infty$, the sequence $\{\frac{f(3^n x)}{27^n}\}$ is a Cauchy sequence. Therefore, we may define $C(x) = \lim_{n\to\infty} 27^{-n} f(3^n x)$ for all $x \in G$. By letting $n \to \infty$ in (3.6), we arrive at (3.2). To show that C satisfies the equation (1.3), let us replace x and y by $3^n x$ and $3^n y$ in (3.1), respectively, and divide by 27^n. Then it follows that

$$\begin{aligned} 27^{-n} \| f(3^n(3x+y)) &+ f(3^n(3x-y)) - 3f(3^n(x+y)) \\ &- 3f(3^n(x-y)) - 48 f(3^n x)\| \\ &\leq 27^{-n} \phi(3^n x, 3^n y), \end{aligned}$$

and by taking the limit as $n \to \infty$, we see that C satisfies (1.3) for all $x,\ y \in G$.

To prove that the cubic function C is unique under the inequality (3.2), if we assume that there exists a cubic function $S : G \to X$ which satisfies (1.3) and (3.2), then we have $S(3^n x) = 27^n S(x)$ and $C(3^n x) = 27^n C(x)$ for all $x \in G$. Hence it follows from (3.2) that

$$\begin{aligned} \|S(x) - C(x)\| &= 27^{-n} \|S(3^n x) - C(3^n x)\| \\ &\leq 27^{-n} (\|S(3^n x) - f(3^n x)\| + \|f(3^n x) - C(3^n x)\|) \\ &\leq \frac{1}{27} \sum_{i=0}^{\infty} \frac{\phi(3^{n+i}x, 0)}{27^{n+i}} \end{aligned}$$

for all $x \in G$. By letting $n \to \infty$ in this inequality, it is immediate that C is unique. This completes the proof of the theorem. □

From Theorem 3.1, we obtain the following corollary concerning the stability of the equation (1.3) in the sense of Hyers, Ulam and Rassias [13].

COROLLARY 3.2. *Let X and Y be a real normed space and a Banach space, respectively, and let $\varepsilon \geq 0$ and $0 \leq p < 3$ be real numbers. If a function $f : X \to Y$ satisfies*

(3.7)
$$\|f(3x+y)+f(3x-y)-3f(x+y)-3f(x-y)-48f(x)\| \leq \varepsilon(\|x\|^p+\|y\|^p)$$

for all $x, y \in X$, then there exists a unique cubic function $C : X \to Y$ which satisfies the equation (1.3) and the inequality

$$\|f(x) - C(x)\| \leq \frac{\varepsilon}{2(27-3^p)}\|x\|^p$$

for all $x \in X$, where the function C is given by $C(x) = \lim_{n\to\infty} \frac{f(3^n x)}{27^n}$ for all $x \in X$.

Moreover, if for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $C(rx) = r^3 C(x)$ for all $r \in \mathbb{R}$.

The proof of the last assertion in the above corollary goes through in the same way as the one in [4]. Unfortunately, we don't know whether the cubic equation holds the Hyers-Ulam-Rassias stability if $p = 3$ is assumed in the inequality (3.7).

The following corollary is the Hyers-Ulam stability [7] of the equation (1.3) which also is an immediate consequence of Theorem 3.1.

COROLLARY 3.3. *Let X be a Banach space, and let $\varepsilon \geq 0$ be a real number. If a function $f : G \to X$ satisfies*

$$\|f(3x+y) + f(3x-y) - 3f(x+y) - 3f(x-y) - 48f(x)\| \leq \varepsilon$$

for all $x, y \in G$, then there exists a unique cubic function $C : G \to X$ defined by $C(x) = \lim_{n\to\infty} \frac{f(3^n x)}{27^n}$ which satisfies the equation (1.3) and the inequality

$$\|f(x) - C(x)\| \leq \frac{\varepsilon}{52}$$

for all $x \in G$. Moreover, if for each fixed $x \in G$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to X is continuous, then $C(rx) = r^3 C(x)$ for all $r \in \mathbb{R}$.

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