Bull. Korean Math. Soc. **41** (2004), No. 4, pp. 599–608

A GENERALIZATION OF THE JACOBSON RADICAL

A. R. NAGHIPOUR AND A. H. YAMINI

ABSTRACT. Let R be an associative ring with identity and $J(R)$ be the Jacobson radical of R. In this paper we investigate the generalization of the Jacobson radical of R, $J^*(R)$ say. Also we study the rings that $J^*(R) = J(R)$.

0. Introduction

Throughout, R stands for an associative ring with identity, $J(R)$ for the Jacobson radical of R, $N(R)$ for the non-left-invertible elements of R and $U(R)$ for the left-invertible elements of R. Recall that R is said to have *stable range one* if for any a, $b \in R$ satisfying $Ra + Rb = R$, there exists $y \in R$ such that $a + yb$ is a unit. This definition is left-right symmetric by Vaserstein [**14**, Theorem 2]. R is (*strongly*) π-*regular* if for every element $a \in R$ there exists a positive integer $n = n(a)$, depending on a, such that ($a^n \in a^{n+1}R$) $a^n \in a^nRa^n$ (see for example [**1**], [**2**] and [**13**]), and R is a *semilocal* if $R/J(R)$ is a left artinian ring. We say that R is J-*semisimple* if its Jacobson radical is zero. R is said to be *decomposable* if it contains a central idempotent $\neq 0, 1$, and *indecomposable* otherwise. The collection of all $n \times n$ matrices over a ring R will be denoted by $M_n(R)$ and the rings of row finite and column finite matrices over R by $CFM_n(R)$ and $RFM_n(R)$, respectively, where n is allowed to be any finite or infinite cardinal number. A detailed exposition about infinite matrices and their Jacobson radicals is given in [**8**] and [**9**]. A ring R is called J^*-ring if for every $x \in R$, either $x \in J(R)$ or $x = a + b$ such that a is left-invertible and b is non-left-invertible.

Received April 9, 2003.
2000 Mathematics Subject Classification: 16D70, 16N20.
Key words and phrases: Jacobson radical, matrix ring, non-left-invertible, stable range one.
This research was supported in part by a grant from IPM.

The Jacobson radical that introduced by N. Jacobson [**5**] has been an important tool in several branches of mathematics. See for example [**3**], [**10**] and [**15**]. Now let $J^*(R) = \{x \in N(R) : x + N(R) \subseteq N(R)\}$. It is shown that $J^*(R)$ and $J(R)$ have similar properties and as application of this the following are equivalent:

(1) R is a J^*-ring,
(2) $J^*(R) = J(R)$,
(3) $J^*(R)$ is a left ideal,
(4) $\overline{R} = R/J(R)$ is a J^*-ring,
(5) trivial extension $S(R, M)$ of M by R is a J^*-ring,
(6) every upper (lower) triangular matrix ring $UTM_n(R)$ ($LTM_n(R)$) is a J^*-ring, where n is allowed to be any finite or countably infinite cardinal number.

Finally we show that $J^*(R) = J(R)$ for the following classes of rings: (1) semilocal rings, (2) stable range one rings, (3) rings which are generated by their units (see for example, [**4**] and [**11**]), (4) left artinian rings, (5) $CFM_n(R)$, where R is a J^*-ring, $J(CFM_n(R)) = CFM_n(J(R))$ and n is any finite or countably infinite cardinal number. Similarly, under the same conditions, $RFM_n(R)$ is also a J^*-ring, (7) every ring whose indecomposable J-semisimple factor rings are J^*-ring.

1. Basic properties of $J^*(R)$

In this section we study some basic properties of $J^*(R)$ which is defined by

$$J^*(R) = \{x \in N(R) : x + N(R) \subseteq N(R)\}.$$

The results of this section will be used in the next section. First we show that $J^*(R)$ is an associative ring.

THEOREM 1.1. *$J^*(R)$ is an associative ring.*

Proof. Let $x, y \in J^*(R)$. Then

$$x + y + N(R) \subseteq x + N(R) \subseteq N(R)$$

and

$$-x + N(R) = -(x + N(R)) \subseteq N(R),$$

so $J^*(R)$ is closed under addition. Now suppose that $xy \notin J^*(R)$. Therefore, there exists $z \in N(R)$ such that $xy + z$ is left-invertible, so there exists $a \in R$ such that $a(xy + z) = 1$. Thus $1 - axy$ is not left-invertible.

On the other hand, $aN(R) = N(R)$. Therefore, $ax + N(R) \subseteq N(R)$, hence

$$ax \in J^*(R).$$

So $1 - ax$ is left-invertible. Hence $(1 - ax)(1 + y) = 1 + y - ax - axy$ is left-invertible. Since $J^*(R)$ is a subgroup of R, $y - ax \in J^*(R)$. Thus $y - ax + N(R) \subseteq N(R)$ and hence $y - ax + 1 - axy \in N(R)$, which is a contradiction. □

THEOREM 1.2. *Let* $\overline{R} = R/J(R)$. *Then* $J^*(\overline{R}) = \overline{J^*(R)}$.

Proof. Let $\overline{x} \in J^*(\overline{R})$. Then $\overline{x} + N(\overline{R}) \subseteq N(\overline{R})$. Therefore,

$$\overline{x + N(R)} \subseteq \overline{N(R)}$$

and by [**6**, Proposition (4.8)] we have $x + N(R) \subseteq N(R)$. Thus $\overline{x} \in \overline{J^*(R)}$. Now let $\overline{x} \in \overline{J^*(R)}$. Then $x + N(R) \subseteq N(R)$, hence $\overline{x} + \overline{N(R)} \subseteq \overline{N(R)}$, whence $\overline{x} + N(\overline{R}) \subseteq N(\overline{R})$. Therefore,

$$\overline{x} \in J^*(\overline{R}).$$

□

THEOREM 1.3. *For any direct product* $\prod_{i\in I} R_i$ *of rings we have*

$$J^*(\prod_{i\in I} R_i) = \prod_{i\in I} J^*(R_i).$$

Proof. A routine argument shows that $\prod_{i\in I} J^*(R_i) \subseteq J^*(\prod_{i\in I} R_i)$. Now let $x \in J^*(\prod_{i\in I} R_i)$, $i \in I$ and $y \in N(R_i)$. Define $z \in J^*(\prod_{i\in I} R_i)$ such that $\pi_i(z) = y$ and $\pi_j(z) = 1 - \pi_j(x)$ for all $j \neq i$, where $\pi_k : \prod_{i\in I} R_i \to R_k$ is the canonical projection. We have $x + z \in J^*(\prod_{i\in I} R_i)$, so $\pi_i(x) + y \in N(R_i)$, whence $\pi_i(x) \in J^*(R_i)$. □

THEOREM 1.4. *Let* n *be any finite or countably infinite cardinal number. Then the following hold:*

(1) $J^*(CFM_n(R)) \subseteq CFM_n(J^*(R))$;

(2) $J^*(RFM_n(R)) \subseteq RFM_n(J^*(R))$.

Proof. Since the proofs of (1) and (2) are similar, we provide only the proof (1). Let $\mathrm{A} \in J^*(CFM_n(R))$. First we show that $\mathrm{a}_{k,k} \in J^*(R)$ for all k. If $\mathrm{a}_{k,k} \notin J^*(R)$ for some k, then there exists $y \in N(R)$ such that $a_{k,k} + y \notin N(R)$. Suppose $v(a_{k,k} + y) = 1$. Define

$$B = \begin{array}{r} \\ \\ \\ \\ k \\ \\ \\ \end{array} \begin{pmatrix} -a_{1,1}+1 & -a_{1,2} & \cdots & -a_{1,k-1} & \overset{k}{0} & -a_{1,k+1} & \cdots \\ -a_{2,1} & -a_{2,2}+1 & \cdots & -a_{2,k-1} & 0 & -a_{2,k+1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -a_{k-1,1} & -a_{k-1,2} & \cdots & -a_{k-1,k-1}+1 & 0 & -a_{k-1,k+1} & \cdots \\ -a_{k,1} & -a_{k,2} & \cdots & -a_{k,k-1} & y & -a_{k,k+1} & \cdots \\ -a_{k+1,1} & -a_{k+1,2} & \cdots & -a_{k+1,k-1} & 0 & -a_{k+1,k+1}+1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is easy to show that $B \in N(CFM_n(R))$ and

$$A + B = \begin{array}{r} \\ \\ \\ \\ k \\ \\ \\ \end{array} \begin{pmatrix} 1 & 0 & \cdots & 0 & \overset{k}{a_{1,k}} & 0 & \cdots \\ 0 & 1 & \cdots & 0 & a_{2,k} & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{k-1,k} & 0 & \cdots \\ 0 & 0 & \cdots & 0 & a_{k,k}+y & 0 & \cdots \\ 0 & 0 & \cdots & 0 & a_{k+1,k} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since $C(A + B) = I$, where

$$C = \begin{array}{r} \\ \\ \\ \\ k \\ \\ \\ \end{array} \begin{pmatrix} 1 & 0 & \cdots & 0 & \overset{k}{-a_{1,k}v} & 0 & \cdots \\ 0 & 1 & \cdots & 0 & -a_{2,k}v & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{k-1,k}v & 0 & \cdots \\ 0 & 0 & \cdots & 0 & v & 0 & \cdots \\ 0 & 0 & \cdots & 0 & -a_{k+1,k}v & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and I is the identity matrix, $A + B$ is left-invertible which is a contradiction. Therefore, $a_{k,k} \in J^*(R)$. Now let $l < k$ and assume that $a_{k,l} \notin J^*(R)$. Therefore, there exist $v \in R$ and $y \in N(R)$ such that $v(a_{k,l} + y) = 1$. Define

$$B' = I - E_{ll} - E_{kk} + E_{lk} + yE_{kl} - A + \sum_{i=1}^{\infty} a_{i,l}E_{il},$$

where E_{ij}'s denote the matrix units. It is easy to show that $B' \in N(CFM_n(R))$ and

$$A+B' = \begin{array}{c@{}c} & \begin{array}{ccccccccccc} & & & & l & & & & k & & \end{array} \\ \begin{array}{c} \\ \\ \\ \\ l \\ \\ \\ \\ k \\ \\ \\ \end{array} & \begin{pmatrix} 1 & 0 & \cdots & 0 & a_{1,l} & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 1 & \cdots & 0 & a_{2,l} & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{l-1,l} & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & a_{l,l} & 0 & \cdots & 0 & 1 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & a_{l+1,l} & 1 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & \cdots & 0 & a_{k-1,l} & 0 & \cdots & 1 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & a_{k,l}+y & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & a_{k+1,l} & 0 & \cdots & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}.$$

Since $C'(A+B') = I$, where

$$C' = \begin{array}{c@{}c} & \begin{array}{cccccccccccc} & & & & l & & & & & k & & \end{array} \\ \begin{array}{c} \\ \\ \\ \\ l \\ \\ \\ \\ k \\ \\ \\ \end{array} & \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -a_{1,l}v & 0 & \cdots \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & -a_{2,l}v & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & -a_{l-1,l}v & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & v & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & -a_{l+1,l}v & 0 & \cdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & -a_{k-1,l}v & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -a_{l,l}v & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -a_{k+1,l}v & 1 & \cdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array},$$

$A+B'$ is left-invertible which is a contradiction. Therefore, $a_{k,l} \in J^*(R)$.
The proof of the case $k < l$ is similar. □

The following example shows that the inclusions in the Theorem 1.4 may be strict.

EXAMPLE 1.5 (see [**16**]). Let $(D, \mathfrak{m})$ be a commutative local domain that is not a field. Then $J(D) = \mathfrak{m} \neq 0$. It is easy to see that $J(D[x]) = 0$ and

$$N(D[x]) = N(D) \cup \{g \in D[x] \mid \deg(g) \geq 1\}.$$

Now we show that $J^*(D) \subseteq J^*(D[x])$. Assume that $a \in J^*(D)$ then $a \in N(D)$ and hence $a \in N(D[x])$. Let $f \in N(D[x])$. Then we have two cases:

(i) "$f \in N(D)$" in this case we have $a + f \in N(D)$ and hence $a + f \in N(D[x])$.

(ii) "$f \in N(D[x])$ with $\deg(f) \geq 1$" Let $f = \sum_{i=0}^{n} a_i x^i$ and let $a_n \neq 0$. Then $\deg(a+f) \geq 1$ and hence $a + f \in N(D[x])$.

Therefore $a \in J^*(D[x])$ and so $J^*(D) \subseteq J^*(D[x])$. Since $0 \neq J(D) \subseteq J^*(D) \subseteq J^*(D[x])$ we have that $J^*(D[x]) \neq 0$ (e.g., $D[x]$ is not a J^*-ring). So, $M_2(J^*(D[x])) \neq 0$. But, it is easy to show that $J^*(M_2(D[x])) = 0$.

The next theorem shows that $J(R)$ in Nakayama Lemma and Krull Intersection Theorem can be replaced by $J^*(R)$. See [**7**, Theorems 2.2, 8.10] and [**12**, Propositions VIII.1.3, VII.4.6].

THEOREM 1.6. *Let I be an ideal of R. Then $I \subseteq J^*(R)$ if and only if $I \subseteq J(R)$.*

Proof. Let $x \in I$ and $r \in R$. Then $rx \in I \subseteq J^*(R)$, and hence $1 - rx \notin N(R)$. Therefore, $x \in J(R)$. The converse is clear. □

2. J^*-rings

In this section, it is shown that the set of J^*-ring contains the classes of some important rings. Let M be a unital (R, R)-bimodule. Recall that the *split-null* (or *trivial extension*) $S(R, M)$ of M by R is the ring formed from the cartesian product $R \times M$ with componentwise addition and multiplication given by $(a, m)(b, k) = (ab, ak + mb)$.

THEOREM 2.1. *The following are equivalent:*

(1) *R is a J^*-ring,*

(2) *$J^*(R) = J(R)$,*

(3) *$J^*(R)$ is a left ideal,*

(4) *$\overline{R} = R/J(R)$ is a J^*-ring,*

(5) *$S(R, M)$ is a J^*-ring, where M is a unital (R, R)-bimodule,*

(6) *every upper (lower) triangular matrix ring $UTM_n(R)$ ($LTM_n(R)$) is a J^*-ring, where n is allowed to be any finite or countably infinite cardinal number.*

Proof. (1)$\Longrightarrow$(2): Let $x \in J^*(R) \setminus J(R)$. Then $x = a + b$ for some left-invertible element a and non-left-invertible element b. Hence $x - b = a$, which contradicts the definition of $J^*(R)$. Therefore, $J^*(R) = J(R)$.

(2)$\Longrightarrow$(3): Trivial.

(3)$\Longrightarrow$(2): It is enough to show that $J^*(R) \subseteq J(R)$. Suppose to the contrary that there exists an $x \in J^*(R) \setminus J(R)$. Then there exists $t \in R$ such that $1+tx$ is not left-invertible. Since $-tx \in J^*(R)$, we have $1 \in N(R)$, which is a contradiction. Therefore, $J^*(R) \subseteq J(R)$.

(2)$\Longrightarrow$(1): Let $x \notin J(R) = J^*(R)$. Then there exists $b \in N(R)$ such that $x + b \notin N(R)$. Therefore, $x = -b + a$, for some left-invertible element $a \in R$.

(4)$\Longleftrightarrow$(2): The assertion follows from Theorem 1.2.

(5)$\Longleftrightarrow$(2): The assertion follows immediately from $J(S(R,M)) = J(R) \times M$ and $J^*(S(R,M)) = J^*(R) \times M$.

(6)$\Longleftrightarrow$(2): We only prove the upper triangular countably infinite case; the proof of other cases are similar. Let $S = UTM_n(R)$. It is easy to see that:

$$N(S) = \begin{pmatrix} N(R) & R & R & \cdots \\ 0 & N(R) & R & \cdots \\ 0 & 0 & N(R) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Also, one can cheek that:

$$J^*(S) = \begin{pmatrix} J^*(R) & R & R & \cdots \\ 0 & J^*(R) & R & \cdots \\ 0 & 0 & J^*(R) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

and

$$J(S) = \begin{pmatrix} J(R) & R & R & \cdots \\ 0 & J(R) & R & \cdots \\ 0 & 0 & J(R) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

The result follows. □

THEOREM 2.2. *We have the following:*

(1) *If $\{R_i\}_{i\in I}$ is a family of J^*-rings, then $\prod_{i\in I} R_i$ is a J^*-ring if and only if each R_i is a J^*-ring.*

(2) *If R is a ring in which every element is a sum of right-invertible elements, then R is a J^*-ring.*

(3) *$M_n(R)$ is a J^*-ring for any ring R, where $n > 1$ is finite.*

(4) *If R satisfies DCC on principal left ideals (right ideals), then R is a J^*-ring. In particular, left artinian rings are J^*-rings.*

(5) *If R has stable range one, then R is a J^*-ring. In particular, strongly π-regular and semilocal rings are J^*-rings.*

Proof. (1): The assertion follows from Theorem 1.3.

(2): By Theorem 2.1, it is enough to show that $J^*(R)$ is a left ideal. Let $x \in J^*(R)$ and $r \in R$. We have $r = u_1 + u_2 + \cdots + u_n$, where each u_i is right-invertible (and $n \geq 1$ is finite). It is easy to show that, $u_i N(R) = N(R)$ for each i. Therefore, $u_i x + N(R) \subseteq N(R)$, so $u_i x \in J^*(R)$ and hence $rx = u_1 x + \cdots + u_n x \in J^*(R)$.

(3): By [**11**, Lemma 5], for any ring R, every element of $M_n(R)$ ($n > 1$ is finite) can be written as a sum of an even number of units. So the assertion follows from (2).

(4): Since any descending chain of principal left ideals in $R/J(R)$ can be written in the form

$$R/J(R)\bar{x_1} \supseteq R/J(R)\bar{x_2}\bar{x_1} \supseteq R/J(R)\bar{x_3}\bar{x_2}\bar{x_1} \supseteq \cdots ,$$

the DCC on principal left ideals of R implies the same for $R/J(R)$. Since $R/J(R)$ is J-semisimple, $R/J(R)$ is a finite direct product of J^*-rings by [**6**, Theorem (3.5), (4.14)] and (3). Therefore, $R/J(R)$ is a J^*-ring by (1). So Theorem 2.1 shows that R is a J^*-ring. (5): Let $x \in J^*(R) \setminus J(R)$. Then there exists $a \in R$ such that $1 - ax$ is not left-invertible. Since $Rax + R(1 - ax) = R$, we have $Rx + R(1 - ax) = R$. Therefore, there exists $e \in R$ such that $x + e(1 - ax)$ is invertible, which is a contradiction. So R is a J^*-ring. Every strongly π-regular ring is stable range one by [**1**]. So, every strongly π-regular is a J^*-ring. Theorem 2.1 and (4) provide that a semilocal ring is a J^*-ring. □

Part (5) of the above theorem suggests a natural question: Is every π-regular ring a J^*-ring?

THEOREM 2.3. *Let n be any finite or countably infinite cardinal number, R be a J^*-ring and $J(CFM_n(R)) = CFM_n(J(R))$. Then $CFM_n(R)$ is a J^*-ring. In particular, if V_D is a right vector space of countably infinite dimension over the division ring D, then $End(V_D)$ is a J^*-ring.*

Proof. By Theorem 1.4, we have

$$J(CFM_n(R)) \subseteq J^*(CFM_n(R)) \subseteq CFM_n(J^*(R)) = CFM_n(J(R)).$$

Therefore $J(CFM_n(R)) = J^*(CFM_n(R))$. By Theorem 2.1, $CFM_n(R)$ is a J^*-ring. The final statement of the theorem follows from $End(V_D) \cong CFM_n(D)$ and $J(CFM_n(D)) = 0$. □

REMARK. The above theorem holds also for the ring $RFM_n(R)$.

In the Example 1.5, we have seen that if R is a J^*-ring, then $R[x]$ may not be a J^*-ring. On the other hand, the situation is much nicer for the ring $R[[x]]$ of formal power series over a Dedekind finite ring R ($ab = 1$ implies $ba = 1$). In fact, it is easy to show that if R is a Dedekind finite ring, then $R[[x]]$ is a J^*-ring if and only if R is a J^*-ring.

THEOREM 2.4. *If every indecomposable J-semisimple factor ring of R is a J^*-ring, then R is a J^*-ring.*

Proof. Suppose R is not a J^*-ring. Then there exists $x \in R \setminus J(R)$ such that $x \notin U(R) + N(R)$. Let C be the set of ideals I of R such that $\bar{x} \notin U(R/I) + N(R/I)$. Then $J(R) \in C$ and C is inductive, so C has a maximal element, Q say. Replacing R by R/Q we may assume that for every proper factor ring $\bar{R}$ of R, $\bar{x} \in U(\bar{R}) + N(\bar{R})$. By our hypothesis there are three possibilities: R is decomposable, $J(R) \neq 0$, or R is a J^*-ring. If R is decomposable, say $R = S \times T$ with $x = (x_1, x_2)$, then $x_1 \in U(S) + N(S)$ and $x_2 \in U(T) + N(T)$, so $x \in U(R) + N(R)$, a contradiction. If $J(R) \neq 0$, then by [**6**, Proposition (4.8)] $x \in U(R) + N(R)$, a contradiction. Therefore, R is a J^*-ring. □

ACKNOWLEDGMENT. We are indebted to S. Yassemi and M. N. Ghosseiri for the present format of the paper, and for simplifying some of our proofs. The authors also thank the referee for many careful comments. The research of the first author was partially supported by a grant from IPM.

References

[1] P. Ara, *Strongly π-regular rings have stable range one*, Proc. Amer. Math. Soc. **124** (1996), 3293–3298.

[2] G. F. Birkenmeier, J. Y. Kim and J. K. Park, *Regularity conditions and the simplicity of prime factor rings*, J. Pure Appl. Algebra **115** (1997), no. 3, 213–230.

[3] A. P. Donsig, A. Katavolos and A. Manoussos, *The Jacobson radical for analytic crossed products*, J. Funct. Anal. **187** (2001), no. 1, 129–141.

[4] J. W. Fisher and R. L. Snider, *Rings generated by their units*, J. Algebra. **42** (1976), no. 2, 363–368.

[5] N. Jacobson, *The radical and semi-simplicity for arbitrary ring*, Amer. J. Math. **67** (1945), 300–320.

[6] T. Y. Lam, *A First Course in Noncommutative Rings*, Grad. Texts in Math. no. 131, Springer-Verlag, Berlin, Heildelberg, New York, 1991.

[7] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.

[8] E. M. Patterson, *On the radicals of certain rings of infinite matrices*, Proc. Roy. Soc. Edinburgh Sect. A **65** (1960), 263–271.

[9] ______, *On the radicals of rings of row-finite matrices*, Proc. Roy. Soc. Edinburgh Sect. A **66** (1961/62), 42–46.

[10] M. Prest and J. Schröer, *Serial functors, Jacobson radical and representation type*, J. Pure Appl. Algebra. **170** (2002), no. 2-3, 295–307.

[11] R. Raphael, *Rings which are generated by their units*, J. Algebra. **28** (1974), 199–205.

[12] B. Stenström, *Rings of quotients*, Springer-Verlag, 1975.

[13] A. A. Tuganbaev, *Semiregular, weakly regular, and π-regular ring*, Algebra, 16. J. Math. Sci. (New York). **109** (2002), no. 3, 1509–1588.

[14] L. N. Vaserstein, *Stable rank of rings and dimensionality of topological spaces*, Funct. Anal. Appl. **5** (1971), 102–110.

[15] A. R. Villena, *Automatic continuity in associative and nonassociative context*, Irish Math. Soc. Bull. **46** (2001), 43-76.

[16] S. Yassemi, *Maximal Elements of Support*, Acta Math. Univ. Comenian. **67** (1998), no. 2, 231–236.

A. R. Naghipour, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, and Institute for Studies in Theoretical Physics and Mathematics, Tehran, Iran
E-mail: arnaghip@ipm.ir

A. H. Yamini, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, Tehran, Iran