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COUNTABLE RINGS WITH ACC ON ANNIHILATORS

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ABSTRACT. We consider countable rings with ascending chain condition on right annihilators. We determine the structure of a countable right p-injective Baer ring, a countable semiprime quasi-Baer ring and a countable quasi-Baer biregular ring.

1. Introduction

Rangaswamy[11, Theorem 1] proved that a countable regular Baer ring is a semisimple Artinian ring. Kim and Park[9] showed that in this theorem, the condition that "R is countable" can be replaced by the condition that "R has only countably many idempotents". Recently, Birkenmeier, Kim and Park[2] studied rings with countably many direct summands. Recall that a ring is orthogonally finite if it has no infinite sets of nonzero orthogonal idempotents. A Baer ring is a ring in which the left (and right) annihilator of every subset is generated by an idempotent (see [12, Lemma 3.8.1, p.78]). From the proof of [11, Theorem 1] we know that a Baer ring satisfies the ascending chain condition on right ideals if and only if it is orthogonally finite. Rangaswamy's proof consists of the following observation: (1) A countable Baer ring has only countably many right annihilators; (2) If a ring has only countably many right annihilators, it must be orthogonally finite; (3) An orthogonally finite von Neumann regular ring is a semisimple Artinian ring. In this paper, we refine the above observation, and then generalize Rangaswamy[11, Theorem 1]. First we prove some preliminary results. Next we consider p-injective rings. Clearly a von Neumann regular ring is a p-injective ring. We prove that a countable right p-injective Baer ring is a semisimple Artinian ring. Finally we consider semiprime quasi-Baer

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rings. We prove that a countable semiprime quasi-Baer ring is a finite direct sum of prime rings and that a countable quasi-Baer biregular ring is a finite direct sum of simple rings.

2. Preliminary results

For a subset X of a ring R, $r(X)$ (resp. $l(X)$) denote the right (resp. left) annihilator of X in R. First we consider the relationship between the orthogonally finiteness, the ascending chain condition (ACC) on right annihilators and the descending chain condition (DCC) on right annihilators.

THEOREM 2.1. *Let R be a ring. Then the following statements are equivalent:*

(1) *R satisfies ACC* (resp. DCC) *on right annihilators;*

(2) *R is orthogonally finite and R satisfies ACC* (resp. DCC) *on right annihilators containing no nonzero idempotents.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). Assume that R is orthogonally finite and R satisfies ACC on right annihilators containing no nonzero idempotents. Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of right annihilators. Since R is orthogonally finite, $I = \bigcup_i I_i$ contains no infinite set of orthogonal idempotents. Hence there exists an idempotent $e \in I$ such that $(1-e)I$ contains no nonzero idempotent. Then $e \in I_j$ for some j. By hypothesis on e, $(1-e)I_j \subseteq (1-e)I_{j+1} \subseteq \cdots$ is an ascending chain of right annihilators containing no nonzero idempotents. Then there exists $k \geq j$ such that $(1-e)I_k = (1-e)I_{k+1} = \cdots$. Since $e \in I_j$, $I_m = eR \oplus (1-e)I_m$ for all $m \geq j$. Therefore $I_k = eR \oplus (1-e)I_k = eR \oplus (1-e)I_{k+m} = I_{k+m}$ for all $m \geq 0$.

Next assume that R is orthogonally finite and R satisfies DCC on right annihilators containing no nonzero idempotents. Let $I_1 \supseteq I_2 \supseteq \cdots$ be an descending chain of right annihilators. Since R is orthogonally finite, $I = \bigcap_i I_i$ contains an idempotent e such that $(1-e)I$ contains no nonzero idempotent. We can easily see that there exists an positive integer j such that the right annihilator $(1-e)I_j$ contains no nonzero idempotent. Then there exists $k \geq j$ such that $(1-e)I_k = (1-e)I_{k+1} = \cdots$. Therefore $I_k = eR \oplus (1-e)I_k = eR \oplus (1-e)I_{k+m} = I_{k+m}$ for all $m \geq 0$. □

The following example shows that the conditions in (2) of Theorem 2.1

are not superfluous.

EXAMPLE 2.2. Let F be a field and let $A = \prod_{i=1}^{\infty} A_i$, where $A_i = F[x]$ is the polynomial ring over F. Then A satisfies ACC (resp. DCC) on right annihilators containing no nonzero idempotents, but A is not orthogonally finite.

Next, let R be the subring of A generated by $\bigoplus_{i=1}^{\infty} S_i$ and 1_A, where $S_i = xF[x]$ is the ideal of A_i generated by x for all $i = 1, 2, \ldots$. Then R is a ring with only idempotents 0 and 1_A, but R does not satisfy ACC (resp. DCC) on right annihilators containing no nonzero idempotents.

As an immediate corollary of Theorem 2.1, we have the following.

COROLLARY 2.3. *For a ring R the following statements are equivalent:*

(1) *Every nonzero right annihilators in R contains a nonzero idempotent and R is orthogonally finite;*

(2) *R is an orthogonally finite Baer ring;*

(3) *Every nonzero right annihilator in R contains a nonzero idempotent and R satisfies ACC on right annihilators.*

In the proof of [11, Theorem 1], Rangaswamy used the fact that a countable Baer ring has only countably many right annihilators. More generally, if every right annihilator of a countable ring R is a right annihilator of some finite subset of R, then R has only countably many right annihilators. In fact a countable ring satisfying the descending chain condition on right annihilators has this property. The following proposition was proved by Faith [3, Corollary 2]. However we give a more direct proof of it.

PROPOSITION 2.4. *Then the following statements are equivalent:*

(1) *A ring R satisfies DCC on right annihilators;*

(2) *For each nonempty subset S of R, there exists a finite subset S' of S such that $r(S) = r(S')$.*

Proof. (1) $\Rightarrow$ (2). Let S be a nonempty subset of R. Let $a_1 \in S$. If $r(S) \subsetneq r(a_1)$, there exists $a_2 \in S$ such that $r(a_1) \supsetneq r(a_1, a_2)$. Continuing this process, we obtain a proper descending chain of right annihilators. Since R satisfies DCC on right annihilators, $r(S) = r(a_1, \cdots, a_n)$ for some $a_1, \cdots, a_n \in S$.

(2) $\Rightarrow$ (1). Let $r(S_1) \supset r(S_2) \supset \cdots$ be a descending chain of right annihilators. We can easily see that $\bigcap_i r(S_i) = r(\bigcup_i S_i)$. By hypothesis, there exists a finite subset S' of $\bigcup_i S_i$ such that $r(S') = r(\bigcup_i S_i)$. Since

S' is finite, there exist $i_1 < \cdots < i_k$ such that $S' \subseteq S_{i_1} \cup \cdots \cup S_{i_k}$. Then $r(S_{i_k}) = r(S_{i_{k+j}})$ for all $j = 1, 2, \cdots$. □

COROLLARY 2.5. *If a countable ring R satisfies ACC or DCC on right annihilators, then R has only countably many right annihilators.*

Proof. If R satisfies DCC on right annihilators, then the set of right annihilators is countable by Proposition 2.4. If R satisfies ACC on right annihilators, then R satisfies DCC on left annihilators.Then the set of left annihilators is countable. Since there is a one to one correspondence between the set of left annihilators and the set of right annihilators, the set of right annihilators is also countable. □

3. p-injective rings

Let R be a ring with identity. A right R-module M is said to be *p-injective* if given any principal right ideal I and any R-homomorphism $\sigma : I \to M$, there exists an R-homomorphism $\hat{\sigma} : R \to M$ that extends σ. A ring R is called a right p-injective ring if R_R is p-injective. This notion was first introduced by Ikeda and Nakayama[7]. It is easily seen that a von Neumann regular ring is nonsingular and right p-injective. For other examples of nonsingular p-injective rings, see [6]. The Jacobson radical of R is denoted by $J(R)$.

The following generalizes [9, Theorem 8].

THEOREM 3.1. *Let R be a right nonsingular right p-injective ring. Then the following are equivalent:*

(1) *R satisfies ACC on right annihilators;*

(2) *R has a finite right uniform dimension;*

(3) *R is a semisimple Artinian ring.*

Proof. (1) $\Rightarrow$ (3). By [7], a ring R is right p-injective if and only if every principal left ideal of R is a left annihilator. Since R satisfies ACC on right annihilators, R satisfies DCC on left annihilators. Hence R satisfies DCC on principal left ideals, and hence R is a right perfect ring. By [12, Corollary 8.5.4, p.190], R is semiprimary. Suppose that $J(R) \neq 0$. Let $r(a)$ be maximal in $\{r(x) \mid 0 \neq x \in J(R)\}$. Since R is left nonsingular, $r(a)$ is not essential. Hence we can choose a nonzero element $b \in R$ such that $r(a) \cap bR = 0$. Since R is right semi-artinian, we may assume that bR is a minimal right ideal of R. Since $ab \neq 0$, $bR \cong abR$. Since $r(ab) \supseteq r(b)$ and $r(b)$ is maximal right ideal of R,

$r(b) = r(ab)$. Hence $Rb = l(r(b)) = l(r(ab)) = Rab$. Hence $b = cab$ for some $c \in R$, and so $b \in r(a - aca)$. Hence $r(a) \subsetneq r(a - aca)$. Since $r(a)$ is maximal in $\{r(x) \mid 0 \neq x \in J(R)\}$, we conclude that $a - aca = 0$. Then $J(R)$ contains a nonzero idempotent ac, a contradiction. Therefore $J(R) = 0$, and hence R is a semisimple Artinian ring.

(3) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1). Let Q denote the maximal right ring of quotients of R. It is well-known that Q is a von Neumann regular ring. Since R_R is an essential R-submodule of Q_R and since R_R has finite uniform dimension, Q_Q is also finite uniform dimension. Since Q is regular, this implies Q is Artinian semisimple. Since Q satisfies ACC on right annihilators, the subring R also satisfies the same condition. □

Using Theorem 3.1, we can slightly generalize [11, Theorem 1] as follows.

COROLLARY 3.2. *A countable right p-injective Baer ring is a semisimple Artinian ring.*

Proof. It is well known that a Baer ring is right and left nonsingular. From his proof of [11, Theorem 1] we know that a countable Baer ring is orthogonally finite. Hence by Corollary 2.3, R satisfies ACC on right annihilators. Hence the assertion follows from Theorem 3.1. □

A ring R is said to be *of bounded index* (*of nilpotency*) if there exists a positive integer n such that $a^n = 0$ for each nilpotent element a of R. If n is the least such integer, we say that R has index n. For an example, it is well-known that any semiprime PI-ring is of bounded index ([6, Theorem 10.8.2]).

In [6, Proposition 1], we proved that if R is a semiprime p-injective ring of bounded index, then R is strongly π-regular. The following corollary shows that the strucure of a prime right p-injective ring of bounded index is more simple.

COROLLARY 3.3. *Let R be a prime right p-injective ring of bounded index. Then R is a simple Artinian ring.*

Proof. By [5, Proposition 4], R is right nonsingular. Then, by [5, Proposition 5], R satisfies ACC on right annihilators. Hence the assertion follows from Theorem 3.1. □

4. Semiprime rings

Let R be a ring. A family $\{S_i \subseteq R \mid i \in I\}$ of subsets of R is said to be independent with respect to right annihilators if, for any distinct subsets J, K of I, $r(\bigcup_{j\in J} S_j) \neq r(\bigcup_{k\in K} S_k)$. The following lemma is trivial.

LEMMA 4.1. *If R has only countably many right annihilators* (resp. right annihilators of ideals), *then R has no infinite family of subsets* (resp. ideals) *of R which is independent with respect to right annihilators.*

A ring R is called *quasi-Baer if the right annihilator of every ideal of R is generated by an idempotent of R.*

PROPOSITION 4.2. *Let R be a countable semiprime ring. Then R is a quasi-Baer ring if and only if R is a finite direct sum of prime rings.*

Proof. Obviously a finite direct sum of prime rings is quasi-Baer. Conversely suppose that R is quasi-Baer. Then, since R is countable, R has only countably many right annihilators of ideals. Then by Lemma 4.1, the R-R-bimodule R has finite uniform dimension, say n. Hence R contains a direct sum $I_1 \oplus \cdots \oplus I_n$ where each I_i is a nonzero ideal of R. If we set $R_k = r(\sum_{i\neq k} I_i)$ for each $k = 1, \cdots, n$, then each R_i is a prime ring and $R = R_1 \oplus \cdots \oplus R_n$. □

A ring R is *biregular* if the principal ideal (a) generated by every a has the form (e) where e is a central idempotent (see [8, p.210]).

COROLLARY 4.3. *A countable biregular quasi-Baer ring is a finite direct sum of simple rings.*

Proof. Since a biregular ring is semiprime, by Proposition 4.2 R is a finite direct sum of a prime rings. Obviously a prime biregular ring is a simple ring. □

PROPOSITION 4.4. *Let R be a countable semiprime ring of bounded index. Then the following are equivalent:*

(1) *R has only countably many right annihilators;*
(2) *There exists a positive integer n such that every chain of right annihilators in R has at most n proper inclusions;*
(3) *R satisfies ACC on right annihilators.*

Proof. (1) $\Rightarrow$ (2). By Lemma 4.1, there are uniform ideals $I_1, \cdots, I_m$ of R such that $I_1 \oplus \cdots \oplus I_n$ is an essential (right) ideal of R. We shall show that $r(I_i)$ is a prime ideal of R for each $i = 1, 2, \cdots, m$. Let I, J be ideals of R such that $IJ \subseteq r(I_i)$ and suppose that $I_i I \neq 0$. Then $(I_1 \oplus \cdots \oplus I_{i-1} \oplus I_i I \oplus I_{i+1} \oplus \cdots \oplus I_m) I_i J = 0$. Since $(I_1 \oplus \cdots \oplus I_{i-1} \oplus I_i I \oplus I_{i+1} \oplus \cdots \oplus I_m)$ is an essential right ideal of R and since R is nonsingular by [5, Proposition 4], we have $I_i J = 0$. Hence each $r(I_i)$ is a prime ideal of R. We also have that $\bigcap_{i=1}^n r(I_i) = r(I_1 \oplus \cdots \oplus I_m) = 0$. Hence R is embedded in $\bigoplus_{i=1}^n R/r(I_i)$. Let k be the nilpotency index of R. Then the index of each $R/r(I_i)$ is equal to or less than k by [1, Lemma 3], Hence, by [5, Proposition 5], every chain of right annihilators in $R/r(I_i)$ has at most k proper inclusions. Since R is embedded in $\bigoplus_i R/r(I_i)$, every chain of right annihilators in R has at most mk proper inclusions.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (1). This follows from Corollary 2.5. □

Lanski [10, Theorem 2] proved that if R is a semiprime PI-ring with infinite center C then $|R| \leq 2^{|C|}$, where $|R|$ (resp. $|C|$) denotes the cardinality of R (resp. C).

The following shows that if C is countable and if C has only countably many right annihilators then $|R| = |C|$.

COROLLARY 4.5. *Let R be a semiprime PI-ring with center C. Then the following are equivalent:*

(1) *R is countable and it has only countably many right annihilators;*
(2) *C is countable and it has only countably many right annihilators;*
(3) *R is a countable semiprime Goldie ring.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3). Since R is semiprime, C has no nonzero nilpotent elements. By Proposition 4,4, C satisfies ACC on annihilators in C. Then by Formanek [4, Theorem 9] R is semiprime Goldie. Now we can easily see that R is also countable.

(3) $\Rightarrow$ (1). By [6, Theorem 10.8.2] R is of bounded index. Hence this follows from Proposition 4.4. □

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