Bull. Korean Math. Soc. **42** (2005), No. 3, pp. 591–608

SOME NEW RESULTS RELATED TO BESSEL AND GRÜSS INEQUALITIES IN 2-INNER PRODUCT SPACES AND APPLICATIONS

S. S. Dragomir, Y. J. Cho, and S. S. Kim

Abstract. Some new reverses of Bessel's inequality for orthonormal families in real or complex 2-inner product spaces are pointed out. Applications for some Grüss type inequalities and for determinantal integral inequalities are given as well.

1. Introduction

The concepts of 2-inner products and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in [1]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let X be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $(\cdot, \cdot|\cdot)$ is a $\mathbb{K}$-valued function defined on $X \times X \times X$ satisfying the following conditions:

(2I$_1$) $(x, x|z) \geq 0$ and $(x, x|z) = 0$ if and only if x and z are linearly dependent,
(2I$_2$) $(x, x|z) = (z, z|x)$,
(2I$_3$) $(y, x|z) = \overline{(x, y|z)}$,
(2I$_4$) $(\alpha x, y|z) = \alpha(x, y|z)$ for any scalar $\alpha \in \mathbb{K}$,
(2I$_5$) $(x + x', y|z) = (x, y|z) + (x', y|z)$.

Received June 1, 2004.
2000 Mathematics Subject Classification: 46C05, 46C99, 26D15, 26D10.
Key words and phrases: Bessel's inequality, Grüss' inequality, 2-inner product, Lebesgue integral, synchronous functions.

$(\cdot,\cdot|\cdot)$ is called a *2-inner product* on X and $(X,(\cdot,\cdot|\cdot))$ is called a *2-inner product space* (or *2-pre-Hilbert space*). Some basic properties of 2-inner product spaces can be immediately obtained as follows [2]:

(1) If $\mathbb{K}=\mathbb{R}$, then $(2\mathrm{I}_3)$ reduces to

$$(y,x|z)=(x,y|z).$$

(2) From $(2\mathrm{I}_3)$ and $(2\mathrm{I}_4)$, we have

$$(0,y|z)=0, \quad (x,0|z)=0$$

and also

$$(x,\alpha y|z)=\bar{\alpha}(x,y|z). \tag{1.1}$$

(3) Using $(2\mathrm{I}_2)$–$(2\mathrm{I}_5)$, we have

$$(z,z|x\pm y)=(x\pm y,x\pm y|z)=(x,x|z)+(y,y|z)\pm 2\operatorname{Re}(x,y|z)$$

and

$$\operatorname{Re}(x,y|z)=\frac{1}{4}\left[(z,z|x+y)-(z,z|x-y)\right]. \tag{1.2}$$

In the real case $\mathbb{K}=\mathbb{R}$, (1.2) reduces to

$$(x,y|z)=\frac{1}{4}\left[(z,z|x+y)-(z,z|x-y)\right] \tag{1.3}$$

and, using this formula, it is easy to see that, for any $\alpha\in\mathbb{R}$,

$$(x,y|\alpha z)=\alpha^2(x,y|z). \tag{1.4}$$

In the complex case, using (1.1) and (1.2), we have

$$\operatorname{Im}(x,y|z)=\operatorname{Re}[-i(x,y|z)]=\frac{1}{4}[(z,z|x+iy)-(z,z|x-iy)],$$

which, in combination with (1.2), yields

$$(x,y|z)=\frac{1}{4}\left[(z,z|x+y)-(z,z|x-y)\right]+\frac{i}{4}[(z,z|x+iy)-(z,z|x-iy)]. \tag{1.5}$$

Using the above formula and (1.1), we have, for any $\alpha\in\mathbb{C}$,

$$(x,y|\alpha z)=|\alpha|^2(x,y|z). \tag{1.6}$$

However, for $\alpha\in\mathbb{R}$, (1.6) reduces to (1.4). Also, from (1.6) it follows that

$$(x,y|0)=0.$$

(4) For any three given vectors $x,y,z\in X$, consider the vector $u=(y,y|z)x-(x,y|z)y$. By $(2\mathrm{I}_1)$, we know that $(u,u|z)\ge 0$ with the equality

if and only if u and z are linearly dependent. The inequality $(u, u|z) \geq 0$ can be rewritten as

$$(y, y|z)\left[(x, x|z)(y, y|z) - |(x, y|z)|^2\right] \geq 0. \tag{1.7}$$

For $x = z$, (1.7) becomes

$$-(y, y|z)|(z, y|z)|^2 \geq 0,$$

which implies that

$$(z, y|z) = (y, z|z) = 0 \tag{1.8}$$

provided y and z are linearly independent. Obviously, when y and z are linearly dependent, (1.8) holds too. Thus (1.8) is true for any two vectors $y, z \in X$. Now, if y and z are linearly independent, then $(y, y|z) > 0$ and, from (1.7), it follows that

$$|(x, y|z)|^2 \leq (x, x|z)(y, y|z). \tag{1.9}$$

Using (1.8), it is easy to check that (1.9) is trivially fulfilled when y and z are linearly dependent. Therefore, the inequality (1.9) holds for any three vectors $x, y, z \in X$ and is strict unless the vectors $u = (y, y|z)x - (x, y|z)y$ and z are linearly dependent. In fact, we have the equality in (1.9) if and only if the three vectors x, y and z are linearly dependent.

In any given 2-inner product space $(X, (\cdot, \cdot \mid \cdot))$, we can define a function $\|\cdot \mid \cdot\|$ on $X \times X$ by

$$\|x|z\| = \sqrt{(x, x|z)} \tag{1.10}$$

for all $x, z \in X$.

It is easy to see that this function satisfies the following conditions:

(2N$_1$) $\|x|z\| \geq 0$ and $\|x|z\| = 0$ if and only if x and z are linearly dependent,
(2N$_2$) $\|z|x\| = \|x|z\|$,
(2N$_3$) $\|\alpha x|z\| = |\alpha|\|x|z\|$ for any scalar $\alpha \in \mathbb{K}$,
(2N$_4$) $\|x + x'|z\| \leq \|x|z\| + \|x'|z\|$.

Any function $\|\cdot \mid \cdot\|$ defined on $X \times X$ and satisfying the conditions (2N$_1$)–(2N$_4$) is called a *2-norm* on X and $(X, \|\cdot|\cdot\|)$ is called a *linear 2-normed space* [5]. Whenever a 2-inner product space $(X, (\cdot, \cdot|\cdot))$ is given, we consider it as a linear 2-normed space $(X, \|\cdot \mid \cdot\|)$ with the 2-norm defined by (1.10).

Let $(X; (\cdot, \cdot|\cdot))$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. If $(f_i)_{1 \leq i \leq n}$ are linearly independent vectors in the 2-inner product space X, and, for a given $z \in X$, $(f_i, f_j|z) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$, where δ_{ij} is the Kronecker delta (we say that the family

$(f_i)_{1\le i\le n}$ is *z-orthonormal*), then the following inequality is the corresponding *Bessel's inequality* (see for example [2]) for the *z-orthonormal family* $(f_i)_{1\le i\le n}$ in the 2-inner product space $(X;(\cdot,\cdot|\cdot))$:

$$\sum_{i=1}^{n} |(x, f_i|z)|^2 \le \|x|z\|^2 \tag{1.11}$$

for any $x \in X$. For more details on this inequality, see the recent paper [2] and the references therein.

The following reverse of Bessel's inequality in 2-inner product spaces has been obtained in [4]:

THEOREM 1. *Let $\{e_i\}_{i\in I}$ be a family of z-orthonormal vectors in X and F a finite part of I, ϕ_i, Φ_i $(i \in F)$ real or complex numbers and $x, z \in X$ be so that either*

(i) $\mathrm{Re}\left(\sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F} \phi_i e_i | z\right) \ge 0$
or, equivalently,

(ii) $\left\| x - \sum_{i\in F} \frac{\phi_i+\Phi_i}{2} e_i | z \right\| \le \frac{1}{2}\left(\sum_{i\in F} |\Phi_i - \phi_i|^2\right)^{\frac{1}{2}}$
holds. Then we have the inequality:

$$\begin{aligned} 0 &\le \|x|z\|^2 - \sum_{i\in F} |(x, e_i|z)|^2 \\ &\le \frac{1}{4}\sum_{i\in F} |\Phi_i - \phi_i|^2 - \mathrm{Re}\left(\sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F} \phi_i e_i | z\right) \\ &\left(\le \frac{1}{4}\sum_{i\in F} |\Phi_i - \phi_i|^2\right). \end{aligned}$$

The constant $\frac{1}{4}$ is best possible.

The following different reverse of Bessel's inequality has been obtained in [3].

THEOREM 2. *Let $\{e_i\}_{i\in I}$ be a family of z-orthonormal vectors in X, F a finite part of I, ϕ_i, Φ_i $(i \in I)$ real or complex numbers. For $x \in X$, if either* (i) *or* (ii) *from Theorem 1 holds, then the following reverse of*

Bessel's inequality

$$
\begin{aligned}
0 &\le \|x|z\|^2 - \sum_{i\in F} |(x, e_i|z)|^2 \\
&\le \frac{1}{4}\sum_{i\in F} |\Phi_i - \phi_i|^2 - \sum_{i\in F}\left|\frac{\phi_i + \Phi_i}{2} - (x, e_i|z)\right|^2 \\
&\left(\le \frac{1}{4}\sum_{i\in F} |\Phi_i - \phi_i|^2\right)
\end{aligned}
$$

is valid. The constant $\frac{1}{4}$ *is best possible.*

The main aim of the present paper is to establish a different reverse inequality for (1.11) than those incorporated in the above two theorems. Some companion results and applications for determinantal integral inequalities are also given.

2. A new reverse of Bessel's inequality

The following reverse of Bessel's inequality holds.

THEOREM 3. *Let* $\{e_i\}_{i\in I}$ *be a family of* z*-orthonormal vectors in* X, F *a finite part of* I *and* ϕ_i, Φ_i $(i \in F)$ *real or complex numbers such that* $\sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right) > 0$. *If* $x \in X$ *is such that either*
(i) $\operatorname{Re}\left(\sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F} \phi_i e_i|z\right) \ge 0$
or, equivalently,
(ii) $\left\|x - \sum_{i\in F} \frac{\phi_i+\Phi_i}{2} e_i|z\right\| \le \frac{1}{2}\left(\sum_{i\in F} |\Phi_i - \phi_i|^2\right)^{\frac{1}{2}}$
holds, then one has the inequality

$$
\|x|z\|^2 \le \frac{1}{4}\cdot\frac{\sum_{i\in F} |\Phi_i + \phi_i|^2}{\sum_{i\in F} \operatorname{Re}\left(\Phi_i\overline{\phi_i}\right)} \sum_{i\in F} |(x, e_i|z)|^2 . \tag{2.1}
$$

The constant $\frac{1}{4}$ *is best possible in the sense that it cannot be replaced by a smaller constant.*

Proof. Firstly, we observe that, for $y, a, A \in X$, the following are equivalent

$$
\operatorname{Re}(A - y, y - a|z) \ge 0 \tag{2.2}
$$

and

$$
\left\|y - \frac{a + A}{2}|z\right\| \le \frac{1}{2}\|A - a|z\| . \tag{2.3}
$$

Now, for $a = \sum_{i\in F} \phi_i e_i$ and $A = \sum_{i\in F} \Phi_i e_i$, we have

$$\begin{aligned} \|A - a|z\| = \left\|\sum_{i\in F} (\Phi_i - \phi_i)\, e_i|z\right\| &= \left[\left\|\sum_{i\in F} (\Phi_i - \phi_i)\, e_i|z\right\|^2\right]^{\frac{1}{2}} \\ &= \left(\sum_{i\in F} |\Phi_i - \phi_i|^2 \|e_i|z\|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{i\in F} |\Phi_i - \phi_i|^2\right)^{\frac{1}{2}}, \end{aligned}$$

which gives, for $y = x$, the desired equivalence. On the other hand, we have the identity

$$\begin{aligned} &\operatorname{Re}\left(\sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F} \phi_i e_i|z\right) \\ &= \sum_{i\in F} \operatorname{Re}\left[\Phi_i \overline{(x, e_i|z)} + \overline{\phi_i}\,(x, e_i|z)\right] - \|x|z\|^2 - \sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right), \end{aligned}$$

which gives, from (i), that

$$\|x|z\|^2 + \sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right) \le \sum_{i\in F} \operatorname{Re}\left[\Phi_i \overline{(x, e_i|z)} + \overline{\phi_i}\,(x, e_i|z)\right]. \tag{2.4}$$

Utilizing the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha} q^2 \ge 2pq, \quad \alpha > 0,\ p, q \ge 0,$$

we deduce

$$2\|x|z\| \le \frac{\|x|z\|^2}{\left[\sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right)\right]^{\frac{1}{2}}} + \left[\sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right)\right]^{\frac{1}{2}}. \tag{2.5}$$

Dividing (2.4) by $\left[\sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right)\right]^{\frac{1}{2}} > 0$ and using (2.5), we obtain

$$\begin{aligned} \|x|z\| &\le \frac{1}{2} \frac{\sum_{i\in F} \operatorname{Re}\left[\Phi_i \overline{(x, e_i|z)} + \overline{\phi_i}\,(x, e_i|z)\right]}{\left[\sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right)\right]^{\frac{1}{2}}} \\ &= \frac{1}{2} \frac{\sum_{i\in F} \operatorname{Re}\left[\left(\overline{\Phi_i} + \overline{\phi_i}\right)(x, e_i|z)\right]}{\left[\sum_{i\in F} \operatorname{Re}\left(\Phi_i \overline{\phi_i}\right)\right]^{\frac{1}{2}}} \end{aligned} \tag{2.6}$$

since it is obvious that

$$\operatorname{Re}\left[\Phi_i\overline{(x, e_i|z)}\right] = \operatorname{Re}\left[\overline{\Phi_i}\,(x, e_i|z)\right].$$

Note that (2.6) is also an interesting inequality in itself.

Utilizing the Cauchy-Bunyakovsky-Schwarz inequality for real numbers, we get

$$\begin{aligned}
&\sum_{i\in F}\operatorname{Re}\left[\left(\overline{\Phi_i}+\overline{\phi_i}\right)(x, e_i|z)\right] \\
&\leq \sum_{i\in F}\left|\left(\overline{\Phi_i}+\overline{\phi_i}\right)(x, e_i|z)\right| \\
&\leq \sum_{i\in F}\left(|\Phi_i+\phi_i|\right)|(x, e_i|z)| \\
&\leq \left[\sum_{i\in F}|\Phi_i+\phi_i|^2\right]^{\frac{1}{2}}\left[\sum_{i\in F}|(x, e_i|z)|^2\right]^{\frac{1}{2}}.
\end{aligned} \tag{2.7}$$

Making use of (2.6) and (2.7), we deduce the desired result (2.1).

To prove the sharpness of the constant $\frac{1}{4}$, let us assume that (2.1) holds with a constant $c > 0$, i.e.,

$$\|x|z\|^2 \leq c\cdot\frac{\sum_{i\in F}|\Phi_i+\phi_i|^2}{\sum_{i\in F}\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right)}\sum_{i\in F}|(x, e_i|z)|^2, \tag{2.8}$$

provided x, ϕ_i, Φ_i $(i\in F)$ satisfy (i).

Suppose that $F=\{1\}, e_1 = e, \|e|z\| = 1, \Phi_1 = \Phi > 0, \phi_1 = \phi > 0$. If we choose $x = \Phi e$, then the condition (i) holds true and, by (2.8), for $F = \{1\}$, we get

$$\Phi^2 \leq c\cdot\frac{(\Phi+\phi)^2}{\Phi\phi}\Phi^2,$$

i.e., $\Phi\phi \leq c(\Phi+\phi)^2$ for any $\Phi, \phi > 0$. Now, if we choose $\Phi = 1+\varepsilon, \phi = 1-\varepsilon$ with $\varepsilon\in(0,1)$ in the last inequality and make $\varepsilon\to 0+$, then we get $c\geq\frac{1}{4}$ and so the proof is completed. □

REMARK 1. By the use of (2.6), the second inequality in (2.7) and the Hölder inequality, we may state the following reverses of Bessel's inequality as well:

$$\|x|z\| \le \frac{1}{2}\cdot\frac{1}{\left[\sum_{i\in F}\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right)\right]^{\frac{1}{2}}} \times \begin{cases} \max\limits_{i\in F}\{|\Phi_i+\phi_i|\}\sum\limits_{i\in F}|(x,e_i|z)| \\ \left[\sum\limits_{i\in F}|\Phi_i+\phi_i|^p\right]^{\frac{1}{p}}\left(\sum\limits_{i\in F}|(x,e_i|z)|^q\right)^{\frac{1}{q}}, \\ \qquad\qquad \text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \max\limits_{i\in F}|(x,e_i|z)|\sum\limits_{i\in F}|\Phi_i+\phi_i|. \end{cases} \tag{2.9}$$

The following corollary holds.

COROLLARY 1. *With the assumption of Theorem 3 and, if either* (i) *or* (ii) *holds, then*

$$0 \le \|x|z\|^2-\sum_{i\in F}|(x,e_i|z)|^2 \le \frac{1}{4}\cdot\frac{\sum_{i\in F}|\Phi_i-\phi_i|^2}{\sum_{i\in F}\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right)}\sum_{i\in F}|(x,e_i|z)|^2. \tag{2.10}$$

The constant $\frac{1}{4}$ is best possible.

Proof. The inequality (2.10) follows by (2.1) on subtracting the same quantity $\sum_{i\in F}|(x,e_i|z)|^2$ from both sides.

The best constant may be shown in a similar way to the one in the above Theorem 3 and we omit the details. □

REMARK 2. If $\{e_i\}_{i\in I}$ is an z-orthonormal family in the real 2-inner product space $(X;(\cdot,\cdot|\cdot))$ and $M_i, m_i\in\mathbb{R}$, $i\in F$ (F is a finite part of I) and $x\in X$ are such that $M_i, m_i\ge 0$ for $i\in F$ with $\sum_{i\in F}M_im_i>0$ and

$$\left(\sum_{i\in F}M_ie_i-x,x-\sum_{i\in F}m_ie_i|z\right)\ge 0,$$

then we have the inequality

$$\begin{aligned} 0\le\ &\|x|z\|^2-\sum_{i\in F}[(x,e_i|z)]^2 \\ &\le\frac{1}{4}\cdot\frac{\sum_{i\in F}(M_i-m_i)^2}{\sum_{i\in F}M_im_i}\cdot\sum_{i\in F}[(x,e_i|z)]^2. \end{aligned} \tag{2.11}$$

The constant $\frac{1}{4}$ is best possible.

The following reverse of the Schwarz's inequality in 2-inner product spaces holds.

COROLLARY 2. *Let* $x, y \in X$ *and* $\delta, \Delta \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ *with the property that* $\operatorname{Re}(\Delta\bar{\delta}) > 0$. *If either*

$$\operatorname{Re}(\Delta y - x, x - \delta y|z) \geq 0 \tag{2.12}$$

or, equivalently,

$$\left\| x - \frac{\delta + \Delta}{2} \cdot y|z \right\| \leq \frac{1}{2} |\Delta - \delta| \, \|y|z\| \tag{2.13}$$

holds, then we have the inequalities

$$\begin{aligned} \|x|z\| \, \|y|z\| &\leq \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{\Delta} + \bar{\delta}\right)(x, y|z)\right]}{\sqrt{\operatorname{Re}(\Delta\bar{\delta})}} \\ &\leq \frac{1}{2} \cdot \frac{|\Delta + \delta|}{\sqrt{\operatorname{Re}(\Delta\bar{\delta})}} \, |(x, y|z)|, \end{aligned} \tag{2.14}$$

$$\begin{aligned} 0 \leq \; & \|x|z\| \, \|y|z\| - |(x, y|z)| \\ \leq \; & \frac{1}{2} \cdot \frac{|\Delta + \delta| - 2\sqrt{\operatorname{Re}(\Delta\bar{\delta})}}{\sqrt{\operatorname{Re}(\Delta\bar{\delta})}} \, |(x, y|z)|, \end{aligned} \tag{2.15}$$

$$\|x|z\|^2 \, \|y|z\|^2 \leq \frac{1}{4} \cdot \frac{|\Delta + \delta|^2}{\operatorname{Re}(\Delta\bar{\delta})} \, |(x, y|z)|^2 \tag{2.16}$$

and

$$0 \leq \|x|z\|^2 \, \|y|z\|^2 - |(x, y|z)|^2 \leq \frac{1}{4} \cdot \frac{|\Delta - \delta|^2}{\operatorname{Re}(\Delta\bar{\delta})} \, |(x, y|z)|^2. \tag{2.17}$$

The constants $\frac{1}{2}$ *and* $\frac{1}{4}$ *are best possible.*

Proof. The inequality (2.14) follows from (2.6) on choosing $F = \{1\}$, $e_1 = e = \frac{y}{\|y|z\|}$, $\Phi_1 = \Phi = \Delta \|y|z\|$, $\phi_1 = \phi = \delta \|y|z\|$ (y, z are linearly independent). The inequality (2.15) is equivalent with (2.14). The inequality (2.16) follows from (2.1) for $F = \{1\}$ and the same choices as above. Finally, (2.17) is obviously equivalent with (2.16). □

3. Some Grüss type inequalities

The following result holds.

THEOREM 4. *Let* $\{e_i\}_{i\in I}$ *be a family of* z*-orthonormal vectors in* X, F *a finite part of* I, $\phi_{i,j}, \Phi_{i,j} \in \mathbb{K}$ $(i \in F, j = 1, 2)$ *and* $x, y \in X$. *If either*

$$\operatorname{Re}\left(\sum_{i\in F}\Phi_{i,j}e_i - x, x - \sum_{i\in F}\phi_{i,j}e_i|z\right) \geq 0 \tag{3.1}$$

or, equivalently,

$$\left\|x - \sum_{i\in F}\frac{\Phi_{i,j}+\phi_{i,j}}{2}e_i|z\right\| \leq \frac{1}{2}\left(\sum_{i\in F}|\Phi_{i,j}-\phi_{i,j}|^2\right)^{\frac{1}{2}} \tag{3.2}$$

for $j = 1, 2$ *hold, then we have the inequality*

$$\begin{aligned} 0 &\leq \left|(x,y|z) - \sum_{i\in F}(x,e_i|z)(e_i,y|z)\right| \\ &\leq \frac{1}{4}\left(\frac{\sum_{i\in F}|\Phi_{i,1}-\phi_{i,1}|^2\sum_{i\in F}|\Phi_{i,2}-\phi_{i,2}|^2}{\sum_{i\in F}\operatorname{Re}\left(\Phi_{i,1}\overline{\phi_{i,1}}\right)\sum_{i\in F}\operatorname{Re}\left(\Phi_{i,2}\overline{\phi_{i,2}}\right)}\right)^{1/2} \\ &\quad\times\left(\sum_{i\in F}|(x,e_i|z)|^2\right)^{\frac{1}{2}}\left(\sum_{i\in F}|(y,e_i|z)|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{3.3}$$

The constant $\frac{1}{4}$ *is best possible.*

Proof. If we use Schwarz's inequality in 2-inner product space $(X, (\cdot,\cdot|\cdot))$, one has

$$\begin{aligned} &\left|\left(x - \sum_{i\in F}(x,e_i|z)e_i, y - \sum_{i\in F}(y,e_i|z)e_i|z\right)\right|^2 \\ &\leq \left\|x - \sum_{i\in F}(x,e_i|z)e_i|z\right\|^2\left\|y - \sum_{i\in F}(y,e_i|z)e_i|z\right\|^2 \end{aligned} \tag{3.4}$$

and, since a simple calculation shows that

$$\left(x - \sum_{i\in F}(x,e_i|z)e_i, y - \sum_{i\in F}(y,e_i|z)e_i|z\right) = (x,y|z) - \sum_{i\in F}(x,e_i|z)(e_i,y|z)$$

and

$$\left\| x - \sum_{i \in F} (x, e_i|z)\, e_i|z \right\|^2 = \|x|z\|^2 - \sum_{i \in F} |(x, e_i|z)|^2$$

for any $x, y \in X$, by (3.4) and by the reverse of Bessel's inequality in Corollary 1, we have

$$\begin{aligned}
& \left| (x, y|z) - \sum_{i \in F} (x, e_i|z)\,(e_i, y|z) \right|^2 \\
& \leq \left(\|x|z\|^2 - \sum_{i \in F} |(x, e_i|z)|^2 \right) \left(\|y|z\|^2 - \sum_{i \in F} |(y, e_i|z)|^2 \right) \\
(3.5) \quad & \leq \frac{1}{4} \frac{\sum_{i \in F} |\Phi_{i,1} - \phi_{i,1}|^2}{\sum_{i \in F} \operatorname{Re}\left(\Phi_{i,1}\overline{\phi_{i,1}}\right)} \sum_{i \in F} |(x, e_i|z)|^2 \\
& \quad \times \frac{\sum_{i \in F} |\Phi_{i,2} - \phi_{i,2}|^2}{\sum_{i \in F} \operatorname{Re}\left(\Phi_{i,2}\overline{\phi_{i,2}}\right)} \sum_{i \in F} |\langle y, e_i|z\rangle|^2 .
\end{aligned}$$

Taking the square root in (3.5), we deduce (3.3).

The fact that $\frac{1}{4}$ is the best possible constant follows by Corollary 1 and we omit the details. □

The following corollary for real 2-inner product spaces holds.

Corollary 3. *Let $\{e_i\}_{i \in I}$ be a family of z-orthonormal vectors in X, F a finite part of I, $M_{i,j}, m_{i,j} \geq 0$ $(i \in F, j = 1, 2)$ and $x, y \in X$ such that $\sum_{i \in F} M_{i,j} m_{i,j} > 0$ $(j = 1, 2)$ and*

$$\left(\sum_{i \in F} M_{i,j} e_i - x, x - \sum_{i \in F} m_{i,j} e_i | z \right) \geq 0. \tag{3.6}$$

Then we have the inequality

$$\begin{aligned}
0 & \leq \left| (x, y|z) - \sum_{i \in F} (x, e_i|z)\,(y, e_i|z) \right|^2 \\
(3.7) \quad & \leq \frac{1}{16} \cdot \frac{\sum_{i \in F} (M_{i,1} - m_{i,1})^2 \sum_{i \in F} (M_{i,2} - m_{i,2})^2}{\sum_{i \in F} M_{i,1} m_{i,1} \sum_{i \in F} M_{i,2} m_{i,2}} \\
& \quad \times \sum_{i \in F} |(x, e_i|z)|^2 \sum_{i \in F} |(y, e_i|z)|^2 .
\end{aligned}$$

The constant $\frac{1}{16}$ *is best possible.*

In the case where the family $\{e_i\}_{i\in I}$ reduces to a single vector, we may deduce from Theorem 4 the following particular case:

COROLLARY 4. *Let* $e \in X, \|e\| = 1, \phi_j, \Phi_j \in \mathbb{K}$ *with* $\operatorname{Re}\left(\Phi_j \overline{\phi_j}\right) > 0$ $(j = 1, 2)$ *and* $x, y \in X$ *such that either*

$$\operatorname{Re}\left(\Phi_j e - x, x - \phi_j e | z\right) \geq 0 \tag{3.8}$$

or, equivalently,

$$\left\| x - \frac{\phi_j + \Phi_j}{2} e | z \right\| \leq \frac{1}{2} \left|\Phi_j - \phi_j\right| \tag{3.9}$$

holds, then

$$\begin{aligned} 0 \leq & \left|(x, y|z) - (x, e|z)\,(e, y|z)\right| \\ \leq & \frac{1}{4} \frac{|\Phi_1 - \phi_1|}{\sqrt{\operatorname{Re}\left(\Phi_1 \overline{\phi_1}\right)}} \frac{|\Phi_2 - \phi_2|}{\sqrt{\operatorname{Re}\left(\Phi_2 \overline{\phi_2}\right)}} \left|(x, e|z)\,(e, y|z)\right|. \end{aligned} \tag{3.10}$$

The constant $\frac{1}{4}$ *is best possible.*

REMARK 3. If X is real, $e \in X$, $\|e|z\| = 1$ and $a, b, A, B \in \mathbb{R}$ are such that $A > a > 0$, $B > b > 0$ and

$$\left\| x - \frac{a + A}{2} e | z \right\| \leq \frac{1}{2} (A - a), \quad \left\| y - \frac{b + B}{2} e | z \right\| \leq \frac{1}{2} (B - b), \tag{3.11}$$

then

$$\begin{aligned} & \left|(x, y|z) - (x, e|z)\,(e, y|z)\right| \\ \leq & \frac{1}{4} \cdot \frac{(A - a)\,(B - b)}{\sqrt{abAB}} \left|(x, e|z)\,(e, y|z)\right|. \end{aligned} \tag{3.12}$$

The constant $\frac{1}{4}$ is best possible.

If $(x, e|z)\,, (y, e|z) \neq 0$, then the following equivalent form of (3.12) also holds

$$\left| \frac{(x, y|z)}{(x, e|z)\,(e, y|z)} - 1 \right| \leq \frac{1}{4} \cdot \frac{(A - a)\,(B - b)}{\sqrt{abAB}}. \tag{3.13}$$

4. Some companion inequalities

The following companion of the Grüss inequality also holds.

THEOREM 5. *Let $\{e_i\}_{i\in I}$ be a family of z-orthonormal vectors in X, F a finite part of I, $\phi_i, \Phi_i \in \mathbb{K}$ $(i \in F)$, $x, y \in X$ and $\lambda \in (0,1)$ such that either*

$$\operatorname{Re}\left(\sum_{i\in F}\Phi_i e_i - (\lambda x + (1-\lambda)\,y)\,, \lambda x + (1-\lambda)\,y - \sum_{i\in F}\phi_i e_i|z\right) \geq 0 \tag{4.1}$$

or, equivalently,

$$\left\|\lambda x + (1-\lambda)\,y - \sum_{i\in F}\frac{\Phi_i + \phi_i}{2}\cdot e_i|z\right\| \leq \frac{1}{2}\left(\sum_{i\in F}|\Phi_i - \phi_i|^2\right)^{\frac{1}{2}} \tag{4.2}$$

holds. Then we have the inequality

$$\begin{aligned} &\operatorname{Re}\left[(x,y|z) - \sum_{i\in F}(x,e_i|z)\,(e_i,y|z)\right] \\ &\leq \frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}\frac{\sum_{i\in F}|\Phi_i - \phi_i|^2}{\sum_{i\in F}\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right)}\sum_{i\in F}|(\lambda x + (1-\lambda)\,y, e_i|z)|^2 . \end{aligned} \tag{4.3}$$

The constant $\frac{1}{16}$ is the best possible constant in (4.3) in the sense that it cannot be replaced by a smaller constant.

Proof. Using the known inequality

$$\operatorname{Re}(z,u|v) \leq \frac{1}{4}\|z + u|v\|^2 ,$$

we may state, for any $a, b \in X$ and $\lambda \in (0,1)$, that

$$\operatorname{Re}(a,b|z) \leq \frac{1}{4\lambda(1-\lambda)}\|\lambda a + (1-\lambda)\,b|z\|^2 . \tag{4.4}$$

Since

$$(x,y|z) - \sum_{i\in F}(x,e_i|z)\,(e_i,y|z) = \left(x - \sum_{i\in F}(x,e_i|z)\,e_i, y - \sum_{i\in F}(y,e_i|z)\,e_i|z\right)$$

for any $x, y \in X$, by (4.4), we get

$$\begin{aligned} &\operatorname{Re}\left[(x,y|z) - \sum_{i\in F}(x,e_i|z)\,(e_i,y|z)\right] \\ &= \operatorname{Re}\left[\left(x - \sum_{i\in F}(x,e_i|z)\,e_i, y - \sum_{i\in F}(y,e_i|z)\,e_i|z\right)\right] \end{aligned} \tag{4.5}$$

$$\leq \frac{1}{4\lambda(1-\lambda)}\left\|\lambda\left(x-\sum_{i\in F}(x,e_i|z)\,e_i\right)\right.$$
$$\left.+(1-\lambda)\left(y-\sum_{i\in F}(y,e_i|z)\,e_i\right)|z\right\|^2$$
$$=\frac{1}{4\lambda(1-\lambda)}\left\|\lambda x+(1-\lambda)\,y-\sum_{i\in F}(\lambda x+(1-\lambda)\,y,e_i|z)\,e_i|z\right\|^2$$
$$=\frac{1}{4\lambda(1-\lambda)}\left[\|\lambda x+(1-\lambda)\,y|z\|^2-\sum_{i\in F}|(\lambda x+(1-\lambda)\,y,e_i|z)|^2\right].$$

If we apply the reverse of Bessel's inequality from Corollary 1 for $\lambda x+(1-\lambda)\,y$, we may state that

$$\|\lambda x+(1-\lambda)\,y|z\|^2-\sum_{i\in F}|(\lambda x+(1-\lambda)\,y,e_i|z)|^2$$
$$\leq \frac{1}{4}\frac{\sum_{i\in F}|\Phi_i-\phi_i|^2}{\sum_{i\in F}\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right)}\sum_{i\in F}|(\lambda x+(1-\lambda)\,y,e_i|z)|^2. \tag{4.6}$$

Now, by making use of (4.5) and (4.6), we deduce (4.3).

The fact that $\frac{1}{16}$ is the best possible constant in (4.3) follows by the fact that, if in (4.1) we choose $x=y$, then it becomes (i) of Theorem 3, implying for $\lambda=\frac{1}{2}$ the inequality (2.10), for which we have shown that $\frac{1}{4}$ is the best constant. □

REMARK 4. If, in Theorem 5, we choose $\lambda=\frac{1}{2}$, then we get

$$\operatorname{Re}\left[(x,y|z)-\sum_{i\in F}(x,e_i|z)\,(e_i,y|z)\right]$$
$$\leq \frac{1}{4}\frac{\sum_{i\in F}|\Phi_i-\phi_i|^2}{\sum_{i\in F}\operatorname{Re}\left(\Phi_i\overline{\phi_i}\right)}\sum_{i\in F}\left|\left(\frac{x+y}{2},e_i|z\right)\right|^2 \tag{4.7}$$

provided

$$\operatorname{Re}\left(\sum_{i\in F}\Phi_ie_i-\frac{x+y}{2},\frac{x+y}{2}-\sum_{i\in F}\phi_ie_i|z\right)\geq 0$$

or, equivalently,

$$\left\|\frac{x+y}{2}-\sum_{i\in F}\frac{\Phi_i+\phi_i}{2}\cdot e_i|z\right\|\leq\frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}}. \tag{4.8}$$

5. Applications for determinantal integral inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, Σ a σ-algebra of subsets of Ω and μ a countably additive and positive measure on Σ with values in $\mathbb{R} \cup \{\infty\}$.

Denote by $L^2_\rho(\Omega)$ the Hilbert space of all real-valued functions f defined on Ω that are 2-ρ-integrable on Ω, i.e., $\int_\Omega \rho(s) |f(s)|^2 d\mu(s) < \infty$, where $\rho : \Omega \to [0, \infty)$ is a measurable function on Ω.

We can introduce the following 2-inner product on $L^2_\rho(\Omega)$ by formula

$$(f,g|h)_\rho := \frac{1}{2}\int_\Omega\int_\Omega \rho(s)\rho(t) \begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix} \begin{vmatrix} g(s) & g(t) \\ h(s) & h(t) \end{vmatrix} d\mu(s)d\mu(t), \tag{5.1}$$

where

$$\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix}$$

denotes the determinant of the matrix

$$\begin{bmatrix} f(s) & f(t) \\ h(s) & h(t) \end{bmatrix},$$

which generates the 2-norm on $L^2_\rho(\Omega)$ expressed by

$$\|f|h\|_\rho := \left(\frac{1}{2}\int_\Omega\int_\Omega \rho(s)\rho(t) \begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix}^2 d\mu(s)d\mu(t)\right)^{1/2}. \tag{5.2}$$

A simple calculation with integrals reveals that

$$(f,g|h)_\rho = \begin{vmatrix} \int_\Omega \rho f g \, d\mu & \int_\Omega \rho f h \, d\mu \\ \int_\Omega \rho g h \, d\mu & \int_\Omega \rho h^2 \, d\mu \end{vmatrix} \tag{5.3}$$

and

$$\|f|h\|_\rho = \begin{vmatrix} \int_\Omega \rho f^2 \, d\mu & \int_\Omega \rho f h \, d\mu \\ \int_\Omega \rho f h \, d\mu & \int_\Omega \rho h^2 \, d\mu \end{vmatrix}^{1/2}, \tag{5.4}$$

where, for simplicity, instead of $\int_\Omega \rho(s)f(s)g(s)d\mu(s)$, we have written $\int_\Omega \rho f g d\mu$.

We recall that the pair of functions $(q,p) \in L^2_\rho(\Omega) \times L^2_\rho(\Omega)$ is called *synchronous* if

$$(q(x) - q(y))(p(x) - p(y)) \geq 0$$

for *a.e.* $x, y \in \Omega$.

We note that, if $\Omega = [a, b]$, then a sufficient condition for synchronicity is that the functions are both monotonic increasing or decreasing. This condition is not necessary.

Now, suppose that $h \in L^2_\rho(\Omega)$ is such that $h(x) \neq 0$ for $\mu - a.e.$ $x \in \Omega$. Then, by the definition of 2-inner product $(f, g|h)_\rho$, we have

$$(f,g|h)_\rho = \frac{1}{2}\int_\Omega\int_\Omega \rho(s)\rho(t)h^2(s)h^2(t) \times \left(\frac{f(s)}{h(s)} - \frac{f(t)}{h(t)}\right)\left(\frac{g(s)}{h(s)} - \frac{g(t)}{h(t)}\right) d\mu(s)d\mu(t) \tag{5.5}$$

and thus a *sufficient condition* for the inequality

$$(f,g|h)_\rho \geq 0 \tag{5.6}$$

to hold, that is, the functions $(f/h, g/h)$ are synchronous. It is obvious that this condition is not necessary.

Using the representations (5.3), (5.4) and the inequalities for 2-inner products and 2-norms established in the previous sections, we have some interesting determinantal integral inequalities.

PROPOSITION 1. *Let $h \in L^2_\rho(\Omega)$ be such that $h(x) \neq 0$ for $\mu - a.e.$ $x \in \Omega$ and $(f_i)_{i\in I}$ a family of functions in $L^2_\rho(\Omega)$ with the property that*

$$\begin{vmatrix} \int_\Omega \rho f_i f_j d\mu & \int_\Omega \rho f_i h d\mu \\ \int_\Omega \rho f_j h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix} = \delta_{i,j}$$

for any $i, j \in I$, where $\delta_{i,j}$ is the Kronecker delta.

If we assume that there exists the real numbers M_i, m_i $(i \in F)$ with $\sum_{i\in F} M_i m_i > 0$, where F is a given finite part of I, such that the functions

$$\sum_{i\in F} M_i \cdot \frac{f_i}{h} - \frac{f}{h}, \frac{f}{h} - \sum_{i\in F} m_i \cdot \frac{f_i}{h}$$

are synchronous on Ω, then we have the inequalities

$$\begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix} \leq \frac{1}{4}\frac{\sum_{i\in F}(M_i + m_i)^2}{\sum_{i\in F} M_i m_i} \sum_{i\in F} \begin{vmatrix} \int_\Omega \rho f_i f d\mu & \int_\Omega \rho f_i h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}^2$$

and

$$0 \leq \begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix} - \sum_{i\in F} \begin{vmatrix} \int_\Omega \rho f_i f d\mu & \int_\Omega \rho f_i h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}^2$$

$$\leq \frac{1}{4} \frac{\sum_{i\in F} (M_i - m_i)^2}{\sum_{i\in F} M_i m_i} \sum_{i\in F} \begin{vmatrix} \int_\Omega \rho f_i f d\mu & \int_\Omega \rho f_i h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}^2 .$$

The constant $\frac{1}{4}$ *is best possible in both inequalities.*

The proof follows by Theorem 3 and Corollary 1 applied for the 2-inner product $(\cdot, \cdot|\cdot)_\rho$ and we omit the details.

Similar determinantal integral inequalities may be stated if one uses the other results for 2-inner products obtained above, but we do not present them here.

ACKNOWLEDGEMENT 1. S. S. Dragomir and Y. J. Cho greatly acknowledge the financial support from the Brain Pool Program (2002) of the Korean Federation of Science and Technology Societies. The research was performed under the "Memorandum of Understanding" between Victoria University and Gyeongsang National University.

ACKNOWLEDGEMENT 2. The authors would like to thank the anonymous referee for his remarks that have been incorporated in the final version of this paper.

References

[1] Y. J. Cho, Paul C. S. Lin, S. S. Kim, and A. Misiak, *Theory of 2-inner product spaces*, Nova Science Publishers, Inc., New York, 2001.

[2] Y. J. Cho, M. Matić, and J. E. Pečarić, *On Gram's determinant in 2-inner product spaces*, J. Korean Math. Soc. **38** (2001), no. 6, 1125–1156.

[3] S. S. Dragomir, Y. J. Cho, S. S. Kim, and Y.-H. Kim, *On Bessel's and Gruss' inequalities for orthonormal families in 2-inner product spaces and applications*, Preprint on line: http://www.mathpreprints.com/math/Preprint/Sever/200309 29/1/

[4] S. S. Dragomir, Y. J. Cho, S. S. Kim, and J. Roumeliotis, *A reverse of Bessel's inequality in 2-inner product spaces and some Grüss type related results with applications*, preprint on line: http://www.mathpreprints.com/math/Preprint/Sever/ 20030923.1/2.

[5] R. W. Freese and Y. J. Cho, *Geometry of linear 2-normed spaces*, Nova Science Publishers, Inc., New York, 2001.

S. S. Dragomir, School of Computer Science and Mathematics, Victoria University of Technology, P.O. Box 14428, MCMC 8001, Victoria, Australia
E-mail: sever.dragomir@vu.edu.au

Y. J. Cho, Department of Mathematics Education, The Research Institute of Natural Sciences, College of Education, Gyeongsang National University, Chinju 660-701, Korea
E-mail: yjcho@gsnu.ac.kr

S. S. Kim, Department of Mathematics, Dongeui University, Pusan 614-714, Korea
E-mail: sskim@dongeui.ac.kr