

## NEGATIVELY BOUNDED SOLUTIONS FOR A PARABOLIC PARTIAL DIFFERENTIAL EQUATION

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**ABSTRACT.** In this note, we introduce a new proof of the uniqueness and existence of a negatively bounded solution for a parabolic partial differential equation. The uniqueness in particular implies the finiteness of the Fourier spanning dimension of the global attractor and the existence allows a construction of an inertial manifold.

### 1. Introduction

The long-time behaviour of solutions of a parabolic partial differential equation is often-times investigated in the framework of an infinite dimensional dynamical system. The wide class of partial differential equations including reaction diffusion system, Kuramoto-Sivashinsky equation, and 2D Navier-Stokes equations has the dissipative structure and possesses the global attractor. Henceforth the general understanding of the dynamics of the underlying partial differential equations is reduced to the study of the geometric structure and dynamical property of the global attractor. See [3] and [4] for details.

It is known in many cases that the global attractor has finite Hausdorff and fractal dimension. The next important questions would be whether the Fourier spanning dimension of the global attractor is finite and whether we can find a finite dimensional smooth manifold containing it. These questions are rephrased as follows;

(1) The leading partial differential operator generates a complete orthogonal system of eigenfunctions and we can expand points on the global attractor as Fourier series. Such an eigenfunction expansion has

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a natural orthogonal decomposition and we want to find out whether the high frequency parts are uniquely determined by the low frequency modes.

(2) An inertial manifold is a positively invariant finite dimensional Lipschitz manifold which attracts every solutions at the exponential rates, thus contains the global attractor obviously. Once it is known that the Fourier spanning dimension is finite, we expect that the global attractor is a part of graph of a mapping from low modes to high modes.

Many good theories have been developed during the last decades but still much is unknown for important equations including 2D Navier-Stokes equations.

The main purpose of this note is to introduce two new observations regarding the above questions. We first introduce, in section 2, a new proof of the injectivity of the spectral projection from the global attractor to finite dimensional Fourier eigenspaces. We next introduce, in section 3, an existence proof of negatively bounded solutions and it follows an existence of an inertial manifold.

## 2. Injectivity of spectral projection

Let the ambient phase space  $H$  be a Hilbert space with an inner product  $\langle \cdot, \cdot \rangle$  and a norm  $\|\cdot\|$ . Let there be given a linear closed unbounded positive self-adjoint operator  $A$  in  $H$ , with domain  $D(A) \subset H$ . We assume that  $A^{-1}$  is compact in  $H$ . We denote  $\lambda_j$  the eigenvalues of  $A$  and  $\phi_j$  the corresponding eigenvectors and we assume that

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_j \rightarrow \infty \quad \text{as } j \rightarrow \infty$$

and the eigenvectors form an orthonormal basis of  $H$ .

The evolution equation that we will consider is of the form

$$(2.1) \quad \frac{du}{dt} = -Au + F(u),$$

where  $F(u)$  is a globally Lipschitz continuous in  $H$ , that is,

$$\|F(u) - F(v)\| \leq K\|u - v\|$$

for  $u, v \in H$  and for some positive constant  $K$ .

Let  $P$  be an orthogonal projection from  $H$  onto a finite dimensional space spanned by  $\{\phi_1, \phi_2, \dots, \phi_N\}$  and  $Q = I - P$ . Then  $H$  is orthogonally decomposed as  $H = PH \oplus QH$  and if we write  $u = p + q = Pu + Qu$ , then (2.1) can be rewritten as

$$(2.2) \quad \begin{aligned} p_t &= -Ap + PF(p + q), \\ q_t &= -Aq + QF(p + q). \end{aligned}$$

We now state the main result of this section.

**THEOREM 1.** *Under the additional assumption that*

$$(2.3) \quad \lambda_{N+1} - \lambda_N > 4K,$$

there exists at most one solution of (2.2) with  $p(0) = p_0$  and  $\|q(t)\|$  bounded for  $t \leq 0$ , for every  $p_0 \in PH$ .

*Proof.* Suppose, given  $p_0$ , there exist two negatively bounded solution  $u_1$  and  $u_2$  with  $u_1(t) = p_1(t) + q_1(t)$ ,  $u_2(t) = p_2(t) + q_2(t)$  and  $p_1(0) = p_2(0) = p_0$ .

Define  $\rho = p_2 - p_1$ ,  $\sigma = q_2 - q_1$ , then  $\rho, \sigma$  solve

$$(2.4) \quad \begin{aligned} \rho_t &= -A\rho + PF(u_2) - PF(u_1), \\ \sigma_t &= -A\sigma + QF(u_2) - QF(u_1). \end{aligned}$$

We introduce  $\Lambda(t) = \|\sigma(t)\|^2 / \|\rho(t)\|^2$  and notice that  $\Lambda(0) = \infty$  and  $\Lambda(t)$  must be unbounded for  $t \leq 0$ .

$$\frac{1}{2} \frac{d\Lambda}{dt} = \frac{\langle \sigma_t, \sigma \rangle}{\|\rho\|^2} - \frac{\|\sigma\|^2}{\|\rho\|^4} < \rho_t, \rho \rangle.$$

A computation gives

$$(2.5) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\rho\|^2 &= \langle \rho_t, \rho \rangle \\ &= -\|A^{\frac{1}{2}}\rho\|^2 + \langle PF(u_2) - PF(u_1), \rho \rangle \\ &\geq -\lambda_N \|\rho\|^2 - K \|u_2 - u_1\| \|\rho\| \\ &= -\lambda_N \|\rho\|^2 - K \sqrt{\|\rho\|^2 + \|\sigma\|^2} \|\rho\| \end{aligned}$$

and

$$\begin{aligned}
 (2.6) \quad \frac{1}{2} \frac{d}{dt} \|\sigma\|^2 &= \langle \sigma_t, \sigma \rangle \\
 &= -\|A^{\frac{1}{2}} \sigma\|^2 + \langle PF(u_2) - PF(u_1), \sigma \rangle \\
 &\leq -\lambda_{N+1} \|\sigma\|^2 + K \|u_2 - u_1\| \|\sigma\| \\
 &= -\lambda_{N+1} \|\sigma\|^2 + K \sqrt{\|\rho\|^2 + \|\sigma\|^2} \|\sigma\|.
 \end{aligned}$$

Using (2.5) and (2.6), we have

$$\begin{aligned}
 \frac{1}{2} \frac{d\Lambda}{dt} &\leq -\lambda_{N+1} \Lambda + \lambda_N \Lambda + K \sqrt{1 + \Lambda} (\sqrt{\Lambda} + \Lambda) \\
 &\leq -(\lambda_{N+1} - \lambda_N - 2K) \Lambda + K \sqrt{\Lambda} + K \Lambda^{\frac{3}{2}}.
 \end{aligned}$$

Let  $0 < \alpha < \beta$  be roots of a quadratic equation

$$Kx^2 - (\lambda - \lambda_N - 2K)x + K = 0.$$

If  $\Lambda(t_0) \leq \beta$  for some  $t_0 < 0$ , then  $\Lambda(t) \leq \beta$  for  $t_0 \leq t < 0$ . Thus we may assume that  $\Lambda(t) = \|\sigma(t)\|^2 / \|\rho(t)\|^2 > \beta$  for  $t \leq 0$ .

From (2.6),

$$\frac{1}{2} \frac{d}{dt} \|\sigma\|^2 \leq -\lambda_{N+1} \|\sigma\|^2 + K \|\rho\| \|\sigma\| + K \|\sigma\|^2 \leq -\gamma \|\sigma\|.$$

Here  $\gamma = \lambda_{N+1} - K - K/\sqrt{\beta}$ .

Integrating from  $t$  to  $0$ , we obtain

$$0 < \|\sigma(0)\|^2 \leq e^{2\gamma t} \|\sigma(t)\|^2, \quad t \leq 0.$$

From the assumption  $\|\sigma(t)\|$  is bounded and taking  $t \rightarrow -\infty$  leads to a contradiction.

REMARK. (1) We first observe that  $\gamma \geq \lambda_{N+1} - 2K > 0$  and Theorem 1 still holds for the solutions with  $\|e^{(\gamma-1)t} q(t)\|$  bounded for  $t \leq 0$ .

(2) The method introduced here is new as far as we know and we hope it could be extended to the more general nonlinearity.

### 3. Inertial manifold

An inertial manifold which is positively invariant Lipschitz manifold which attracts all solutions with an exponential rate, introduced by Foias, Sell, and Temam in [2]. It has been constructed for a wide class of partial differential equations, see [3] and [4] for details. But it is not known whether there exists an inertial manifold for 2D Navier-Stokes equations. The reason for this is that the spectral gap condition

$$\Lambda_{N+1} - \Lambda_N \geq C\sqrt{\lambda_N}$$

does not hold for any  $N$  unless  $C$  is small. Henceforth a new existence theory must be developed.

In the previous section, we have seen that the negatively bounded solution (in the sense that  $e^{(\gamma-1)t}||q(t)||$  bounded for  $t \leq 0$ ) is unique (if exists). The main goal of this section is to prove an existence of such a solution and to construct an inertial manifold by using those solutions. It turns out that this existence proof is much concise and easy to follow relative to the previously developed methods. Much more importantly, the new method can be applied to an equation of the type

$$\begin{aligned} u_t &= -Au + Av + f(u, v), \\ v_t &= -Av + cu + g(v), \end{aligned}$$

where  $c$  is any constant and  $f, g$  are suitable functions. We recall that the above equation is very close to the transformed equations of 2D Navier-Stokes equations. See [1] for details.

We first recall an existence result for a linear equation.

**BASIC LEMMA.** *Let  $f(t)$  be given in  $L^\infty(-\infty, 0; D(A^{-\frac{1}{2}}))$ . Then there exists a unique function  $q(t)$  which is continuous and bounded from  $(-\infty, 0]$  into  $H$  and satisfies*

$$\frac{dq}{dt} = -Aq + f(t).$$

*Moreover if  $\sigma \in L^\infty(-\infty, 0; H)$ , then  $q(t)$  is continuous and bounded from  $(-\infty, 0]$  into  $D(A^{\frac{1}{2}})$ .*

*Proof.* The proof can be found in p.420 of [4].

We now introduce a weight function

$$\omega = \omega(t) = e^{\alpha t}, \quad \alpha = \frac{\lambda_N + \lambda_{N+1}}{2}$$

and a weighted Banach space

$$\mathbf{H} = C_\omega((-\infty, 0]; H) = \left\{ u : \sup_{-\infty < t \leq 0} \omega(t) \|u(t)\| < \infty \right\}.$$

Define  $\mathbf{p}(t) = \omega(t)p(t)$ ,  $\mathbf{q}(t) = \omega(t)q(t)$ . Then from (2.2), we get

$$(3.1) \quad \mathbf{p}_t = -A\mathbf{p} + \alpha\mathbf{p} + \omega(t)PF(p + q),$$

$$(3.2) \quad \mathbf{q}_t = -A\mathbf{q} + \alpha\mathbf{q} + \omega(t)QF(p + q).$$

Our strategy to find a negatively bounded solution is as follows;  
Step 1: Given any  $p_0 \in H$  and an initial guess  $q_0(t)$  with  $\sup_{-\infty < t \leq 0} \omega(t) \|q_0(t)\| \leq 1$ , solve

$$(3.3) \quad \mathbf{p}_t = -A\mathbf{p} + \alpha\mathbf{p} + \omega(t)PF(p + q_0(t))$$

with an initial data  $\mathbf{p}(0) = p(0) = p_0$ .

Step 2: Solve

$$(3.4) \quad \mathbf{q}_t = -A\mathbf{q} + \alpha\mathbf{q} + \omega(t)QF(p + q_0(t))$$

and define  $\mathbf{q}_1(t) = \mathbf{q}(t)$ .

Step 3: Prove that the mapping

$$\Phi : q_0(t) \rightarrow \frac{1}{\omega} \mathbf{q}_1(t)$$

is a strict contraction in  $\mathbf{H}$ .

In step 1, (3.3) is an ordinary differential equation and there exists a unique global solution  $\mathbf{p}(t) = \mathbf{p}(t; p_0, q_0(t))$ . In step 2,  $\omega(t)QF(p(t) + q_0(t)) \in L^\infty(-\infty, 0; H)$  and by applying Basic Lemma, one has a unique solution  $\mathbf{q}(t) \in C((-\infty, 0]; H)$ . It is also easy to see that  $\|\mathbf{q}(t)\| \leq 1$  for  $t \leq 0$ . Thus the mapping  $\Phi$  maps the unit ball in  $\mathbf{H}$  into itself.

By arguing that  $\Phi$  is a strict contraction, one finds a negatively bounded solution looking for.

**THEOREM 2.** *Assuming  $\lambda_{N+1} - \lambda_N \geq 3K$ , we find an  $N$ -dimensional inertial manifold for (2.1).*

*Proof.* It is enough to show that  $\Phi$  is a strict contraction. In fact an inertial manifold is constructed by a finite dimensional graph of a mapping from  $p_0$  to  $q(0)$  as usual and it follows from the uniqueness of negatively bounded solution that the mapping is well-defined and the graph is invariant. The other properties can be proved by following the standard argument given in [3] and [4].

Let, given  $p_0$  and  $q_0(t)$ , we have solutions  $\mathbf{p}_0(t)$  and  $\mathbf{q}_1(t)$  of (3.3) and (3.4) respectively with  $\mathbf{p}_0(0) = p_0$ . We now replace  $q_0(t)$  by  $q_1(t)$  in (3.3) and (3.4) and let  $\mathbf{p}_1(t)$  and  $\mathbf{q}_2(t)$  be new solutions of replaced equations with  $p_1(0) = p_0$ . We define

$$\rho(t) = \mathbf{p}_1(t) - \mathbf{p}_0(t), \quad \sigma(t) = \mathbf{q}_2(t) - \mathbf{q}_1(t),$$

then  $\rho$  and  $\sigma$  solve

$$(3.5) \quad \rho_t = -A\rho + \alpha\rho + P\phi(t),$$

$$(3.6) \quad \sigma_t = -A\sigma + \alpha\sigma + Q\phi(t),$$

where

$$\phi(t) = \omega(t)(F(p_1(t) + q_1(t)) - F(p_0(t) + q_0(t))).$$

First of all,

$$\|\phi(t)\| \leq K\sqrt{\|\rho\|^2 + \|\mu\|^2},$$

for  $\mu = \sup_{-\infty < t \leq 0} \omega(t)\|q_1(t) - q_0(t)\|$ . We multiply  $\rho$  to (3.5) and integrate to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\rho\|^2 &= -\|A^{\frac{1}{2}}\rho\|^2 + \alpha\|\rho\|^2 + \langle \phi, \rho \rangle \\ &\geq -\|A^{\frac{1}{2}}\rho\|^2 + \alpha\|\rho\|^2 - \frac{K}{2}(\|\rho\|^2 + \|\mu\|^2) - \frac{K}{2}\|\rho\|^2 \end{aligned}$$

and thus

$$\frac{d}{dt} \|\rho\|^2 \geq \beta\|\rho\|^2 - K\|\mu\|^2,$$

where  $\beta = -2\lambda_N + 2\alpha - 2K$ . Integrating from  $t$  to 0 and recalling  $\rho(0) = 0$ , we have

$$(3.7) \quad \|\rho\|^2 \leq \frac{K}{\beta}\mu^2.$$

We now multiply  $\sigma$  to (3.6) and integrate to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\sigma\|^2 &= -\|A^{\frac{1}{2}}\sigma\|^2 + \alpha\|\sigma\|^2 + \langle \phi, \sigma \rangle \\ &\leq -\|A^{\frac{1}{2}}\sigma\|^2 + \alpha\|\sigma\|^2 + \frac{K}{2}(\|\rho\|^2 + \mu\|^2) + \frac{K}{2}\|\sigma\|^2. \end{aligned}$$

Using (3.7), we find

$$\frac{d}{dt} \|\sigma\|^2 \leq -(2\lambda_{N+1} - 2\alpha - K)\|\sigma\|^2 + K\left(\frac{K}{\beta} + 1\right)\mu^2$$

and

$$\begin{aligned} \|\sigma\|^2 &\leq K\left(\frac{K}{\beta} + 1\right) \frac{1}{2\lambda_{N+1} - 2\alpha - K} \mu^2 \\ &= \frac{K}{\lambda_{N+1} - \lambda_N - 2K} \mu^2 \\ &< \|\mu\|^2 \quad \text{from the assumption,} \end{aligned}$$

which completes the proof.

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