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SEMI-INVARIANT SUBMANIFOLDS OF CODIMENSION 3 OF A COMPLEX PROJECTIVE SPACE IN TERMS OF THE JACOBI OPERATOR

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ABSTRACT. In this paper, we characterize some semi-invariant submanifolds of codimension 3 with almost contact metric structure (ϕ, ξ, g) in a complex projective space $\mathbb{C}P^{n+1}$ in terms of the structure tensor ϕ, the Ricci tensor S and the Jacobi operator R_ξ with respect to the structure vector ξ.

0. Introduction

A submanifold M is called a *CR submanifold* of a Kaehlerian manifold $\tilde{M}$ with complex structure J if there exists a differentiable distribution $\triangle : p \to \triangle_p \subset M_p$ on M such that $\triangle$ is J-invariant and the complementary orthogonal distribution $\triangle^\perp$ is totally real, where M_p denotes the tangent space at each point p in M ([1, 18]). In particular, M is said to be a *semi-invariant submanifold* provided that $\dim \triangle^\perp = 1$. The unit normal in $J\triangle^\perp$ is called the *distinguished normal* to the semi-invariant submanifold ([2, 16]). In this case, M admits an induced almost contact metric structure (ϕ, ξ, g). A typical example of a semi-invariant submanifold is real hypersurfaces. And new examples of nontrivial semi-invariant submanifolds in a complex projective space $\mathbb{C}P^n$ are constructed in [8] and [13]. Therefore we may expect to generalize some results which are valid in a real hypersurface to a semi-invariant submanifold.

For the real hypersurface case, when $\tilde{M}$ is a complex projective space, many results are known. One of them, Takagi ([14]) classified homogeneous real hypersurfaces of a complex projective space by means of

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six model spaces of type A_1, A_2, B, C, D and E, further he explicitly write down their principal curvatures and multiplicities in the table in [15]. Cecil and Ryan ([13]) extensively studied a real hypersurface which is realized a tube of constant radius r over a complex submanifold of $\mathbb{C}P^n$ on which ξ is principal curvature vector with principal curvature $\alpha = 2\cot 2r$ and the corresponding focal map φ_r has constant rank. From this point of view, Okumura ([12]) characterized real hypersurfaces of type A_1 and A_2 in a complex projective space $\mathbb{C}P^n$ by the property that the shape operator A and structure tensor ϕ commute.

Denoting by R the curvature tensor of the submanifold, we define the Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ with respect to the structure vector ξ. Then R_ξ is a self-adjoint endomorphism on the tangent space of a CR submanifold.

In the previous paper ([4]), Cho and one of the present authors gave another characterization of real hypersurfaces of type A_1 and A_2 in a complex projective space $\mathbb{C}P^n$ in terms of the shape operator A, the structure tensor ϕ and the Jacobi operator R_ξ. Namely, they proved the follwoing:

THEOREM CK([4]). *Let M be a connected real hypersurface of $\mathbb{C}P^n$. Suppose that M satisfies $R_\xi\phi = \phi R_\xi$ and at the same time $R_\xi A = AR_\xi$. Then ξ is a principal curvature vector field on M, that is, M is a Hopf real hypersurface. Further if $g(A\xi, \xi) \neq 0$, then M is locally congruent to one of the following spaces:*

(A_1) *a geodesic hypersphere (that is, a tube of radius r over a hyperplane $\mathbb{C}P^{n-1}$,*

where $0 < r < \frac{\pi}{2}$ and $r \neq \frac{\pi}{4}$),

(A_2) *a tube of radius r over a totally geodesic $\mathbb{C}P^k (1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$ and $r \neq \frac{\pi}{4}$.*

On the other hand, semi-invariant submanifolds of codimension 3 in a complex projective space $\mathbb{C}P^{n+1}$ have been studied in [6], [7], [8], [17] and so on by using properties of induced almost contact metric structure and those of the third fundamental form of the submanifold. In the preceding work, Song, Takagi and one of the present authors assert the following:

THEOREM KST([8]). *Let M be a real $(2n-1)$-dimensional semi-invariant submanifold of codimension 3 in a complex projective space $\mathbb{C}P^{n+1}$. If the structure vector ξ is an eigenvector for the shape operator in the direction of the distinguished normal and the third fundamental form n satisfies $dn = 2\theta\omega$ for a certain scalar $\theta(< \frac{c}{2})$, where*

$\omega(X,Y) = g(X,\phi Y)$ for any vectors X and Y on M, then M is a Hopf real hypersurface in a complex projective space $\mathbb{C}P^n$.

The main purpose of the present paper is to extend Theorem CK under certain conditoins on a semi-invariant submanifold of codimension 3 in a complex projective space.

All manifolds in this paper are assumed to be connected and of class C^∞ and the semi-invariant submanifolds are supposed to be orientable.

1. Preliminaries

At first we review fundamental facts on a semi-invariant submanifold of $\mathbb{C}P^{n+1}$. Let $\tilde{M}$ be a real $2(n+1)$-dimensional Kaehlerian manifold equipped with parallel almost complex structure J and a Riemannian metric tensor G and covered by a system of coordinate neighborhoods $\{W; y^A\}$.

Let M be a real $(2n-1)$-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; x^h\}$ and immersed isometrically in $\tilde{M}$ by immersion $i : M \to \tilde{M}$.

Throughout the present paper the following convention on the range of indices are used, unless otherwise stated:

$$A, B, \cdots = 1, 2, \cdots, 2n+2; \; i, j, \cdots = 1, 2, \cdots, 2n-1.$$

Henceforth the summation convention will be used with respect to those systems of indices. We represent the immersion i locally by $y^A = y^A(x^h)$ and $B_j = (B_j{}^A)$ are also $(2n-1)$-linearly independent local tangent vectors of M, where $B_j{}^A = \partial_j y^A$ and $\partial_j = \frac{\partial}{\partial x^j}$. Then mutually orthogonal unit normals, C, D and E may be chosen, and the induced Riemannian metric tensor g with components g_{ji} on M is given by $g_{ji} = G(B_j, B_i)$ since the immersion i is isometric.

As is well-known, a submanifold M of a Kaehlerian manifold $\tilde{M}$ is said to be a *CR submanifold* ([1, 18]) if it is endowed with a pair of mutually orthogonal complementary differentiable distribution $(\triangle, \triangle^\perp)$ such that for any $p \in M$ we have $J\triangle_p = M_p$, $J\triangle_p{}^\perp \subset M_p{}^\perp$, where $M_p{}^\perp$ denotes the normal space of M at p. In particular M is said to be a *semi-invariant submanifold* provided that $\dim \triangle^\perp = 1$ ([2, 16]). In this case the unit normal vector field in $J\triangle^\perp$ is called a *distinguished normal* to the semi-invariant submanifold and denoted this by C ([2, 16]). Then we have

$$(1.1)\quad JB_i = \phi_i{}^h B_h + \xi_i C, \quad JC = -\xi^h B_h, \quad JD = -E, \quad JE = D,$$

where we have put $\phi_{ji} = G(JB_j, B_i), \xi_i = G(JB_i, C), \xi^h$ being associated components of ξ_h (see [8]). A tensor field of type (1,1) with components $\phi_i{}^h$ will be denoted by ϕ. By the Hermitian property of J, it is seen that ϕ_{ji} is skew-symmetric, and that

$$(1.2)\quad \begin{cases} \phi_i{}^r \phi_r{}^h = -\delta_i{}^h + \xi_i \xi^h, & \xi^r \phi_r{}^h = 0, \quad \xi_r \phi_i{}^r = 0, \\ g_{rs} \phi_j{}^r \phi_i{}^s = g_{ji} - \xi_j \xi_i, & \xi_r \xi^r = 1, \end{cases}$$

namely, the aggregate (ϕ, ξ, g) defines *almost contact metric structure.*

Denoting by ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to the induced Riemannian metric tensor g, the equation of Gauss for M of $\tilde{M}$ is obtained:

$$(1.3)\quad \nabla_j B_i = A_{ji} C + K_{ji} D + L_{ji} E,$$

where A_{ji}, K_{ji} and L_{ji} are components of the second fundamental forms in the direction of normals C, D, E respectively. Equations of Weingarten are also given by

$$(1.4)\quad \begin{cases} \nabla_j C = -A_j{}^h B_h + l_j D + m_j E, \\ \nabla_j D = -K_j{}^h B_h - l_j C + n_j E, \\ \nabla_j E = -L_j{}^h B_h - m_j C - n_j E, \end{cases}$$

where $A = (A_j{}^h), A_{(2)} = (K_j{}^h)$ and $A_{(3)} = (L_j{}^h)$, which are related by $A_{ji} = A_j{}^r g_{ir}, K_{ji} = K_j{}^r g_{ir}$ and $L_{ji} = L_j{}^r g_{ir}$ respectively, and l_j, m_j and n_j being components of the third fundamental forms.

In the sequel, we denote the normal components of $\nabla_j C$ by $\nabla^\perp C$. The distinguished normal C is said to *parallel* in the normal bundle if we have $\nabla^\perp C = 0$, that is, l_j and m_j vanish identically.

Since J is parallel, by differentiating (1.1) covariantly along M and using (1.1), (1.3) and (1.4), and by comparing the tangential and normal parts, we find (see [17])

$$(1.5)\quad \nabla_j \phi_i{}^h = -A_{ji} \xi^h + A_j{}^h \xi_i,$$

$$(1.6)\quad \nabla_j \xi_i = -A_{jr} \phi_i{}^r,$$

$$K_{ji} = -L_{jr}\phi_i{}^r - m_j\xi_i, \tag{1.7}$$

$$L_{ji} = K_{jr}\phi_i{}^r + l_j\xi_i. \tag{1.8}$$

There is no loss of generality such that we may assume $TrA_{(3)} = 0$ (see [8]).

From (1.7) and (1.8), we have

$$K_{jr}\xi^r = -m_j, \quad L_{jr}\xi^r = l_j, \tag{1.9}$$

$$m_r\xi^r = -TrA_{(2)}, \quad l_r\xi^r = 0 \tag{1.10}$$

because of $TrA_{(3)} = 0$. Further we obtain

$$\phi_{jr}l^r = m_j + TrA_{(2)}\xi_j, \quad \phi_{jr}m^r = -l_j, \tag{1.11}$$

$$K_{jr}L_i{}^r + K_{ir}L_j{}^r + l_jm_i + l_im_j = 0. \tag{1.12}$$

REMARK 1. To write our formulas in a convention form, in the sequel we denote by $\alpha = A_{ji}\xi^j\xi^i, \beta = A_{ji}^2\xi^j\xi^i, TrA_{(2)} = k, \mu = n_t\xi^t$ and $\nu = \xi^t\nabla_t k$, and for a function f we denote by ∇f the gradient vector field of f.

Now we put $U_j = \xi^r\nabla_r\xi_j$. Then U is orthogonal to the structure vector ξ. Because of (1.6) and properties of the almost contact metric structure, it follows that

$$\phi_{jr}U^r = A_{jr}\xi^r - \alpha\xi_j, \tag{1.13}$$

$$U^r\nabla_j\xi_r = A_{jr}{}^2\xi^r - \alpha A_{jr}\xi^r. \tag{1.14}$$

From (1.13), we get $g(U,U) = \beta - \alpha^2$. Thus we easily see that $A\xi = \alpha\xi$ if and only if $\beta - \alpha^2 = 0$.

Differentiating (1.13) covariantly along M and making use of (1.5) and (1.6), we find

$$\xi_j(A_{kr}U^r + \nabla_k\alpha) + \phi_{jr}\nabla_kU^r = \xi^r\nabla_kA_{jr} - A_{jr}A_{ks}\phi^{rs} + \alpha A_{kr}\phi_j{}^r, \tag{1.15}$$

which shows that

$$(\nabla_k A_{rs})\xi^r\xi^s = 2A_{kr}U^r + \nabla_k\alpha. \tag{1.16}$$

By means of (1.5), (1.6) and (1.15), it is verified that

$$\xi^r\nabla_r U_i = -3U^r A_{rs}\phi_i{}^s + \alpha A_{ir}\xi^r - \beta\xi_i - \phi_i^r\nabla_r\alpha. \tag{1.17}$$

In the rest of this paper we shall suppose that $\tilde{M}$ is a Kaehlerian manifold of constant holomorphic sectional curvature c. Then equations of Gauss and Codazzi are given by

(1.18)

$$\begin{aligned} R_{kjih} = &\frac{c}{4}(g_{kh}g_{ji} - g_{jh}g_{ki} + \phi_{kh}\phi_{ji} - \phi_{jh}\phi_{ki} - 2\phi_{kj}\phi_{ih}) \\ &+ A_{kh}A_{ji} - A_{jh}A_{ki} + K_{kh}K_{ji} - K_{jh}K_{ki} + L_{kh}L_{ji} - L_{jh}L_{ki}, \end{aligned}$$

$$\begin{aligned} &\nabla_k A_{ji} - \nabla_j A_{ki} - l_k K_{ji} + l_j K_{ki} - m_k L_{ji} + m_j L_{ki} \\ &\quad = \frac{c}{4}(\xi_k\phi_{ji} - \xi_j\phi_{ki} - 2\xi_i\phi_{kj}), \end{aligned} \tag{1.19}$$

$$\nabla_k K_{ji} - \nabla_j K_{ki} = l_j A_{ki} - l_k A_{ji} + n_k L_{ji} - n_j L_{ki}, \tag{1.20}$$

$$\nabla_k L_{ji} - \nabla_j L_{ki} = m_j A_{ki} - m_k A_{ji} - n_k K_{ji} + n_j K_{ki}, \tag{1.21}$$

where R_{kjih} are covariant components of the Riemann-Christoffel curvature tensor R of M, and those of the Ricci by

$$\nabla_k l_j - \nabla_j l_k = A_{jr}K_k{}^r - A_{kr}K_j{}^r + m_j n_k - m_k n_j, \tag{1.22}$$

$$\nabla_k m_j - \nabla_j m_k = A_{jr}L_k{}^r - A_{kr}L_j{}^r + n_j l_k - n_k l_j, \tag{1.23}$$

$$\nabla_k n_j - \nabla_j n_k = K_{jr}L_k{}^r - K_{kr}L_j{}^r + l_j m_k - l_k m_j + \frac{c}{2}\phi_{kj}. \tag{1.24}$$

2. The Jacobi operator of semi-invariant submanifolds with $dn = \frac{c}{2}\omega$

In this section we suppose that M is a semi-invariant submanifold of codimension 3 in $\mathbb{C}P^{n+1}$ satisfying $dn = \frac{c}{2}\omega$, namely, $\nabla_j n_i - \nabla_i n_j = \frac{c}{2}\phi_{ji}$, where d denotes the exterior differential operator and the 2-form ω is defined by $\omega(X,Y) = g(X, \phi Y)$ for any vectors X and Y on M. In this case, it is known that

THEOREM K([6]). *Let M be a semi-invariant submanifold of codimension 3 satisfying $dn = \frac{c}{2}\omega$ in $\mathbb{C}P^{n+1}$. If $A\xi = \alpha\xi$, then we have $A_{(2)} = A_{(3)} = 0$.*

From (1.18) we have

$$\begin{aligned} -(R_\xi)_{ji} =& \frac{c}{4}(g_{ji} - \xi_j\xi_i) + \alpha A_{ji} - (A_{jr}\xi^r)(A_{is}\xi^s) \\ &+ kK_{ji} - m_jm_i - l_jl_i \end{aligned} \tag{2.1}$$

because of (1.9) and (1.10), where we denote by $(R_\xi)_{ji} = R_{jkih}\xi^k\xi^h$.
Suppose that $R_\xi\phi = \phi R_\xi$. Then from (2.1) we have

$$\alpha(A_{jr}\phi_i^{\ r} + A_{ir}\phi_j^{\ r}) + U_iA_{jr}\xi^r + U_jA_{ir}\xi^r + 2kL_{ji} = 2k(l_j\xi_i + l_i\xi_j), \tag{2.2}$$

where we have used (1.6), (1.8) and (1.11).
Now, we prove

LEMMA 2.1. *Let M be a semi-invariant submanifold of codimension 3 satisfying $dn = \frac{c}{2}\omega$ in $\mathbb{C}P^{n+1}$. Suppose that $R_\xi\phi = \phi R_\xi$, then we have $k = 0$.*

Proof. Since we have $dn = \frac{c}{2}\omega$, (1.24) implies

$$K_{jr}L_i^{\ r} - K_{ir}L_j^{\ r} + l_jm_i - l_im_j = 0,$$

which together with (1.12) gives $K_{jr}L_i^{\ r} + l_jm_i = 0$. □

Multiplying this by ϕ^{ji} and summing for j and i, and making use of (1.8), (1.9) and (1.11), we find

$$L_{ji} = 0, \quad l_j = 0. \tag{2.3}$$

Thus, (2.2) becomes

$$\alpha(A_{jr}\phi_i^r + A_{ir}\phi_j^r) + U_i A_{jr}\xi^r + U_j A_{ir}\xi^r = 0. \tag{2.4}$$

Further, (1.7) implies that $m_j\xi_i - m_i\xi_j = 0$ and consequently

$$m_j = -k\xi_j \tag{2.5}$$

because of the first equation of (1.10). Hence (1.7) turns out to be

$$K_{ji} = k\xi_j\xi_i. \tag{2.6}$$

Using (2.3), (2.5) and (2.6), the relationship (1.21) leads to

$$k(\xi_k A_{ji} - \xi_j A_{ki} + n_j\xi_k\xi_i - n_k\xi_j\xi_i) = 0,$$

which shows

$$k\{n_k + A_{kr}\xi^r - (\alpha + \mu)\xi_k\} = 0.$$

Now, let Ω_0 be a set of points such that $k \neq 0$ on M and suppose that Ω_0 be nonvoid. Then the last two equations imply

$$A_{ji} = \xi_j A_{ir}\xi^r + \xi_i A_{jr}\xi^r - \alpha\xi_j\xi_i$$

on Ω_0. Since U is orthogonal to ξ, it is seen that $AU = 0$ and $A_{jr}\phi_i{}^r = -\xi_j U_i$, which together with (2.4) yields

$$U_j(A_{ir}\xi^r - \alpha\xi_i) + U_i(A_{jr}\xi^r - \alpha\xi_j) = 0.$$

Hence we have $A\xi = \alpha\xi$ on Ω_0. Therefore by Theorem K we have $A_{(2)} = 0$ and consequently $k = 0$. Thus Ω_0 is empty. This completes the proof of Lemma 2.1.

3. Semi-invariant submanifolds satisfying $dn = 2\theta\omega$

In this section we shall suppose that M is a semi-invariant submanifold of codimension 3 in a complex projective space $\mathbb{C}P^{n+1}$ and that the third fundamental form n satisfies $dn = 2\theta\omega$ for a certain scalar θ on M, namely,

$$\nabla_j n_i - \nabla_i n_j = 2\theta\phi_{ji}. \tag{3.1}$$

Then we see that $dn = 2\theta\omega$ is independent of the choice of D and E and that from (1.24) we have

$$K_{jr}L_i^{\ r} - K_{ir}L_j^{\ r} + l_j m_i - l_i m_j = 2(\theta - \frac{c}{4})\phi_{ij},$$

which together with (1.12) yields

$$K_{jr}L_i^{\ r} + l_j m_i = (\theta - \frac{c}{4})\phi_{ij}. \tag{3.2}$$

Differentiating (3.1) covariantly and using the first Bianchi identity, it is verified that $\theta = const.$ if $n > 2$ (see [8]).

In the previous paper [8], Song, Takagi and one of the present authors proved the following:

LEMMA 3.1. [8] *Let M be a semi-invariant submanifold of codimension 3 in $\mathbb{C}P^{n+1}$ satisfying* (3.1). *If $\theta \neq \frac{c}{2}$, then we have $\nabla_j^{\perp}C = -k\xi_j E$ on M. Further if $A\xi = \alpha\xi$, then the distinguished normal is parallel in the normal bundle.*

In what follows, we assume that M satisfies (3.1) with $\theta \neq \frac{c}{2}$ and $n > 2$. Then by Lemma 3.1 and (1.4), we have

$$l_j = 0, \quad m_j = -k\xi_j. \tag{3.3}$$

Thus (1.7), (1.8) and (3.2) turn out respectively to

$$L_{jr}\phi_i^{\ r} = -K_{ji} + k\xi_j\xi_i, \tag{3.4}$$

$$K_{jr}\phi_i^{\ r} = L_{ji}, \tag{3.5}$$

$$K_{jr}L_i^{\ r} = (\theta - \frac{c}{4})\phi_{ij}. \tag{3.6}$$

From the last two equations, it is seen that

$$L_{ji}^{\ 2} = (\theta - \frac{c}{4})(g_{ji} - \xi_j\xi_i). \tag{3.7}$$

Furthermore, if we make use of (3.3), then the other structure equations (1.19)–(1.23) are reduced respectively to

$$\nabla_k A_{ji} - \nabla_j A_{ki} = k(\xi_j L_{ki} - \xi_k L_{ji}) + \frac{c}{4}(\xi_k\phi_{ji} - \xi_j\phi_{ki} - 2\xi_i\phi_{kj}), \tag{3.8}$$

$$\nabla_k K_{ji} - \nabla_j K_{ki} = n_k L_{ji} - n_j L_{ki}, \tag{3.9}$$

$$\nabla_k L_{ji} - \nabla_j L_{ki} = k(\xi_k A_{ji} - \xi_j A_{ki}) - n_k K_{ji} + n_j K_{ki}, \tag{3.10}$$

$$A_{jr} K_k{}^r - A_{kr} K_j{}^r = k(n_k \xi_j - n_j \xi_k), \tag{3.11}$$

$$A_{jr} L_k{}^r - A_{kr} L_j{}^r = \xi_k \nabla_j k - \xi_j \nabla_k k + k(A_{kr}\phi_j{}^r - A_{jr}\phi_k{}^r), \tag{3.12}$$

where we have used (1.6). Because of (1.9) and (3.3), it is clear that

$$K_{jr}\xi^r = k\xi_j, \quad L_{jr}\xi^r = 0. \tag{3.13}$$

Multiplying (3.11) and (3.12) with ξ^k and summing for the index k, we have respectively

$$\xi^s A_{sr} K_j{}^r = k A_{jr}\xi^r + k(n_j - \mu\xi_j), \tag{3.14}$$

$$K_{jr} U^r = \nu\xi_j - \nabla_j k + k U_j \tag{3.15}$$

by virtue of (1.6), (1.13), (3.4) and (3.13).

Transforming (3.14) by $\phi_k{}^j$ and taking account of (1.13), (3.5), (3.6) and (3.13), we find

$$K_{kr} U^r = k(\phi_{kr} n^r - U_k), \tag{3.16}$$

which together with (3.15) implies that

$$\nabla_j k = \nu\xi_j - k(\phi_{jr} n^r - 2U_j). \tag{3.17}$$

If we transform (3.12) by $\phi_i{}^k$ and make use of (3.4) and (3.17), then we obtain

$$\begin{aligned} A_{sr} L_j{}^r \phi_i{}^s + A_{jr} K_i{}^r = & k\{(n_i - \mu\xi_i)\xi_j + 2\xi_j(A_{ir}\xi^r - \alpha\xi_i) \\ & + 2\xi_i A_{jr}\xi^r - A_{ji} - A_{sr}\phi_j{}^r\phi_i{}^s\}, \end{aligned}$$

or, use (3.11)

$$A_{sr} L_j{}^r \phi_i{}^s = A_{sr} L_i{}^r \phi_j{}^s. \tag{3.18}$$

Since θ is constant if $n > 2$, by differentiation (3.7) covariantly gives

$$L_{jr}\nabla_k L_i{}^r + L_{ir}\nabla_k L_j{}^r = (\theta - \frac{c}{4})(\xi_j A_{kr}\phi_i{}^r + \xi_i A_{kr}\phi_j{}^r),$$

or using (3.6), (3.10) and the last equation, it is verified that (see. [7])

$$\begin{aligned} &2L_{jr}\nabla_k L_i{}^r \\ &= (\theta - \frac{c}{4})\{2n_k\phi_{ij} + \xi_j(A_{ir}\phi_k{}^r + A_{kr}\phi_i{}^r) + \xi_i(A_{kr}\phi_j{}^r - A_{jr}\phi_k{}^r) \\ &+ \xi_k(A_{ir}\phi_j{}^r - A_{jr}\phi_k{}^r\} + k\{\xi_j(A_{kr}L_i{}^r + A_{ir}L_k{}^r) \\ &- \xi_i(A_{kr}L_j{}^r + A_{jr}L_k{}^r) + \xi_k(A_{ir}L_j{}^r - A_{jr}L_i{}^r)\}, \end{aligned}$$

which together with (3.13) and (3.16) gives

$$\begin{aligned} &(\theta - \frac{c}{4})(A_{ir}\phi_k{}^r + A_{kr}\phi_i{}^r) + (k^2 + \theta - \frac{c}{4})(U_k\xi_i + U_i\xi_k) \\ &+ k\{A_{kr}L_i{}^r + A_{ir}L_k{}^r - k(\xi_i\phi_{kr}n^r + \xi_k\phi_{ir}n^r)\} = 0. \end{aligned} \tag{3.19}$$

4. The Jacobi operator satisfying $R_\xi\phi = \phi R_\xi$

We continue now, our arguments under the same hypotheses $dn = 2\theta\omega$ for a scalar $\theta(\neq \frac{c}{2})$ as in section 3. Furthermore suppose, throughout this paper, that $R_\xi\phi = \phi R_\xi$, which means that the eigenspace of R_ξ is invariant by the structure operator ϕ. Then (2.2) is reduced to

$$\alpha(A_{jr}\phi_i{}^r + A_{ir}\phi_j{}^r) + U_i A_{jr}\xi^r + U_j A_{ir}\xi^r = -2kL_{ji} \tag{4.1}$$

since we have $l_j = 0$.

Applying (4.1) by L_k^i and using (1.13), (3.4) and (3.13), we find

$$\begin{aligned} &\alpha\{A_{jr}K_k{}^r - k\xi_k A_{jr}\xi^r + A_{sr}L_k{}^s\phi_j{}^r\} \\ &+ (L_{ks}U^s)(A_{jr}\xi^r) + U_j K_{kr}U^r = -2kL_{jk}^2, \end{aligned} \tag{4.2}$$

from which, taking the skew-symmetric part and making use of (3.11) and (3.18),

$$\begin{aligned} &k\alpha(\xi_j n_k - \xi_k n_j + \xi_j A_{kr}\xi^r - \xi_k A_{jr}\xi^r) \\ &+ (L_{ks}U^s)(A_{jr}\xi^r) - (L_{js}U^s)(A_{kr}\xi^r) + U_j K_{kr}U^r - U_k K_{jr}U^r = 0. \end{aligned}$$

Taking the inner product with ξ^k and using (3.13), we get

$$k\alpha(n_j - \mu\xi_j + A_{jr}\xi^r - \alpha\xi_j) + \alpha L_{jr}U^r = 0.$$

Thus the last two equations imply

$$(L_{ir}U^r)(\phi_{js}U^s) - (L_{jr}U^r)(\phi_{is}U^s) + U_j K_{ir}U^r - U_i K_{jr}U^r = 0 \tag{4.3}$$

because of (1.13).

In the sequel, we consider that a semi-invariant submanifold M with (3.1) for $\theta \neq \frac{c}{2}$ satisfying $R_\xi\phi = \phi R_\xi$, and suppose that

$$AU = \lambda U \tag{4.4}$$

holds for a function λ on M. Then by applying U^jU^i to (4.1), we have $kL_{ji}U^jU^i = 0$ because of (1.13) and (4.4).

We set $\Omega = \{p \in M : k(p) \neq 0\}$, and assume that Ω is nonempty. In the following, we discuss our arguments on the open subset Ω of M. Then by the discussion above we have $L_{ji}U^jU^i = 0$. Thus, multiplying (4.3) with U^i and summing for i, we find

$$K_{jr}U^r = xU_j, \tag{4.5}$$

where x is defined by $x(\beta - \alpha^2) = K_{ji}U^jU^i$ because of Lemma 3.1. Transforming (4.5) by $\phi_i{}^j$ and using (3.5), we obtain

$$L_{ir}U^r = x\phi_{ir}U^r. \tag{4.6}$$

Because of (3.6) and the last two equations, it follows that $x^2 = \theta - \frac{c}{4}$. By the way, it is seen that $\theta \neq \frac{c}{4}$ on Ω by Lemma 2.1. Thus x is nonzero constant on Ω if $n > 2$.

By (3.16) and (4.5), we have

$$\phi_{jr}n^r = (1 + \frac{x}{k})U_j. \tag{4.7}$$

Thus, by properties of the almost contact metric structure, it is seen that

$$n_j = \mu\xi_j - (1 + \frac{x}{k})\phi_{jr}U^r. \tag{4.8}$$

Using (3.19) and (4.7), it is clear that

$$\begin{aligned} &x^2(A_{jr}\phi_i{}^r + A_{ir}\phi_j{}^r) + x(x-k)(U_j\xi_i + U_i\xi_j) \\ &+ k(A_{jr}L_i{}^r + A_{ir}L_j{}^r) = 0. \end{aligned} \tag{4.9}$$

REMARK 2. $\alpha \neq 0$ on Ω.

In fact, if $\alpha = 0$ on Ω, then by (1.13), (4.1) and (4.6), we have $(\beta + 2kx)A_{jr}\xi^r = 0$ and hence $\beta + 2kx = 0$ by virtue of Lemma 3.1. Because of (1.13), (3.7), (4.5) and (4.6), we also have from (4.2)

$$x(A_{ks}\xi^s)(A_{jr}\xi^r) + xU_kU_j = -2kx^2(g_{jk} - \xi_j\xi_k),$$

from which, taking the trace $\beta x = -2(n-1)kx^2$. Therefore it follows that $(n-2)kx^2 = 0$, a contradiction. Thus $\alpha = 0$ is not imposible on Ω.

Applying (4.1) by U^i and using (1.13), (4.4) and (4.6), we find

$$A_{jr}^{\ 2}\xi^r = \varepsilon A_{jr}\xi^r - (2kx + \alpha\lambda)\xi_j, \tag{4.10}$$

where the function ε is defined by $\alpha\varepsilon = \beta + \alpha\lambda + 2kx$ with the aid of Remark 2.

Multiplying (4.9) to $A_s^{\ j}\xi^s$ and summing for j, and making use of (4.4), (4.5), (4.6) and (4.10), we obtain

$$(k - x)(\beta - \alpha^2) + 2k(\alpha\lambda + kx - x^2) = 0. \tag{4.11}$$

From this and the definition of ε, we have

$$(k - x)(\varepsilon - \alpha) + \lambda(k + x) = 0 \tag{4.12}$$

because of $\alpha \neq 0$.

Now, we put

$$A\xi = \alpha\xi + \rho W, \tag{4.13}$$

where ρ is a function on M, which does not vanish on Ω because of Lemma 3.1, and W is a unit vector field orthogonal to ξ. Then we have $\phi U = \rho W$ and $\rho^2 = \beta - \alpha^2$ because of (1.13). Thus W is also orthogonal to U. Further, we have from (4.10)

$$A_{jr}W^r = \rho\xi_j + (\varepsilon - \alpha)W_j \tag{4.14}$$

since we get $\rho \neq 0$ on Ω. Making use of (1.13) and (3.14), it is verified that

$$\rho K_{jr}W^r = kA_{jr}\xi^r + k(n_j - \mu\xi_j),$$

which together with (1.13) and (4.7) yields

$$K_{jr}W^r = -xW_j. \tag{4.15}$$

On the other hand, differentiating (4.5) covariantly, we find

$$(\nabla_k K_{jr})U^r + K_{jr}\nabla_k U^r = x\nabla_k U_j, \tag{4.16}$$

or, using (4.5), $(\nabla_k K_{ji})U^j U^i = 0$. Because of (3.9), (4.6) and this, it follows that $U^s U^r(\nabla_r K_{js}) = 0$.

Multiplying $U^k W^j$ to (4.16) and summing for k and j, and taking account of (4.15), we obtain

$$W^r U^s \nabla_s U_r = 0. \tag{4.17}$$

By differentiating (4.4) covariantly, we find

$$(\nabla_k A_{jr})U^r + A_j{}^r\nabla_k U_r = (\nabla_k \lambda)U_j + \lambda\nabla_k U_j,$$

from which, taking the skew-symmetric part and using (1.13), (3.8), (4.6) and (4.13),

$$\begin{aligned}\rho(kx - \frac{c}{4})(\xi_j W_k - \xi_k W_j) + A_j{}^r\nabla_k U_r - A_k{}^r\nabla_j U_r \\ = U_j\nabla_k\lambda - U_k\nabla_j\lambda + \lambda(\nabla_k U_j - \nabla_j U_k).\end{aligned} \tag{4.18}$$

If we apply this by U^k and make use of (4.4), then we obtain

$$A_j{}^r U^s\nabla_s U_r - \lambda U^s\nabla_s U_j = d\lambda(U)U_j - \rho^2\nabla_j\lambda,$$

where we have put $d\lambda(U) = U^t\nabla_t\lambda$. Multiplying W^j to the last equation and summing for j, and taking account of (4.13) and (4.17), we get $\rho^2 d\lambda(W) + \rho\xi^r U^s\nabla_s U_r = 0$, which together with (4.10) implies that

$$d\lambda(W) = 0. \tag{4.19}$$

In the same way, we verify, using (1.13), (4.17) and (4.18), that

$$d\lambda(\xi) = 0. \tag{4.20}$$

From (3.15) and (4.5), we have

$$\nabla_j k = \nu\xi_j + (k - x)U_j, \tag{4.21}$$

which enables us to obtain

$$\nabla_k \nabla_j k = (\nabla_k \nu)\xi_j + \{\nu\xi_k + (k-x)U_k\}U_j - \nu A_{kr}\phi_j{}^r + (k-x)\nabla_k U_j.$$

Thus, it follows that

$$\begin{aligned}(4.22)\qquad &\xi_j \nabla_k \nu - \xi_k \nabla_j \nu + \nu(\xi_k U_j - \xi_j U_k + A_{jr}\phi_k{}^r - A_{kr}\phi_j{}^r)\\ &= (k-x)(\nabla_j U_k - \nabla_k U_j).\end{aligned}$$

On the other hand, differentiating (4.10) covariantly and making use of (1.6) and (4.21), we find

$$\begin{aligned}(4.23)\qquad &(\nabla_k A_{jr})A_s{}^r\xi^s + A_j{}^r(\nabla_k A_{rs})\xi^s - \varepsilon(\nabla_k A_{jr})\xi^r\\ &= (\nabla_k \varepsilon)A_{jr}\xi^r - \{2x(\nu\xi_k + (k-x)U_k) + \nabla_k(\alpha\lambda)\}\xi_j\\ &+ A_{jr}^2 A_{ks}\phi^{rs} - \varepsilon A_{jr}A_{ks}\phi^{rs} + (2xk + \alpha\lambda)A_{kr}\phi_j{}^r,\end{aligned}$$

which together with (1.16), (4.4) and (4.10) gives

$$(\nabla_k A_{sr})(A_t{}^r\xi^t)\xi^s = \frac{1}{2}\nabla_k \beta + \lambda\varepsilon U_k.$$

Thus, it follows that

$$(4.24)\qquad \xi^k(\nabla_k A_{jr})A_s{}^r\xi^s = \frac{1}{2}\nabla_j\beta + (\lambda\varepsilon - kx - \frac{c}{4})U_j,$$

where we have used (3.8), (3.13), (4.6) and (4.13).

By the way, we have from (1.16), (3.8) and (4.4)

$$\xi^k(\nabla_k A_{jr})\xi^r = 2\lambda U_j + \nabla_j \alpha.$$

Hence, multiplying (4.23) with ξ^k and summing for k, and taking account of (4.4), (4.13), (4.20) and (4.24), we obtain

$$\begin{aligned}(4.25)\qquad &A_j{}^r\nabla_r\alpha + \frac{1}{2}\nabla_j\beta - \varepsilon\nabla_j\alpha + (3\lambda^2 - \frac{c}{4} - 2\lambda\varepsilon + \alpha\lambda + kx)U_j\\ &= \rho d\varepsilon(\xi)W_j - \{2x\nu + \lambda d\alpha(\xi) - \alpha d\varepsilon(\xi)\}\xi_j,\end{aligned}$$

which connected with (4.14) implies that

$$(4.26)\qquad d\rho(W) = d\varepsilon(\xi) - d\alpha(\xi)$$

because $\rho \neq 0$ on Ω.

LEMMA 4.1. *$\nabla k = (k-x)U$ on Ω if $d\rho(W)=0$.*

Proof. From $g(U,U)=\rho^2$, we have $U^r\nabla_j U_r = \rho\nabla_j\rho$. So we have $W^rU^s\nabla_r U_s = 0$ because $d\rho(W)=0$ is assumed. Thus, applying (4.22) by U^jW^k, and making use of (4.4), (4.10) and (4.13), we get $\nu\rho(\lambda+\varepsilon-\alpha)=0$, which together with (4.12) yields $x(\varepsilon-\alpha)\nu=0$ and hence $(\varepsilon-\alpha)\nu=0$ by virtue of Lemma 2.1. Therefore we have $\lambda\nu=0$. □

Suppose that $\nu\neq 0$ on Ω. Then we have $\varepsilon=\alpha, \lambda=0$ and thus $AU=0$. By the definition of ε, we see that $\beta-\alpha^2+2kx=0$. Differentiating this covariantly and using (4.21), we find

$$\frac{1}{2}\nabla_j\beta-\alpha\nabla_j\alpha=-x\{\nu\xi_j+(k-x)U_j\},$$

which together with (4.25) gives

$$A_j{}^r\nabla_r\alpha=(\frac{c}{4}-x^2)U_j+\rho d\varepsilon(\xi)W_j+(\alpha d\varepsilon(\xi)-x\nu)\xi_j.$$

Applying the last equaiton by U^j, we see that $(x^2-\frac{c}{4})\rho=0$ and hence $x^2-\frac{c}{4}=0$. Therefore $\theta=\frac{c}{2}$, a contradiction. This completes the proof of Lemma 4.1.

From Lemma 4.1 and (4.22) we have

$$(k-x)(\nabla_jU_i-\nabla_iU_j)=0. \tag{4.27}$$

Differentiating (4.8) covariantly and using (1.5), (1.6), (4.4) and Lemma 4.1, we find

$$\begin{aligned}\nabla_k n_j =&\xi_j\nabla_k\mu-\mu A_{kr}\phi_j{}^r+\frac{x}{k^2}(k-x)U_k\phi_{jr}U^r\\ &-(1+\frac{x}{k})(\lambda U_k\xi_j+\phi_{jr}\nabla_kU^r),\end{aligned}$$

from which, taking the skew-symmetric part and using (3.1),

$$\begin{aligned}&2\theta\phi_{kj}+\frac{x}{k^2}(k-x)(U_j\phi_{kr}U^r-U_k\phi_{jr}U^r)\\ &=\xi_j\nabla_k\mu-\xi_k\nabla_j\mu-\mu(A_{kr}\phi_j{}^r-A_{jr}\phi_k{}^r)\\ &-(1+\frac{x}{k})\{\lambda(U_k\xi_j-U_j\xi_k)+\phi_{jr}\nabla_kU^r-\phi_{kr}\nabla_jU^r\}.\end{aligned}$$

On the other side, the skew-symmetric part of (1.15) gives

$$\phi_j{}^r\nabla_k U_r - \phi_k{}^r\nabla_j U_r + \lambda(U_k\xi_j - U_j\xi_k)$$
$$= -\frac{c}{2}\phi_{kj} - 2A_{jr}A_{ks}\phi^{rs} - \alpha(A_{jr}\phi_k{}^r - A_{kr}\phi_j{}^r) - (\nabla_k\alpha)\xi_j + (\nabla_j\alpha)\xi_k,$$

where we have used (3.8), (3.13) and (4.4). From the last two equations, it follows that

(4.28)
$$2\theta\phi_{kj} + \frac{x}{k^2}(k-x)(U_j\phi_{kr}U^r - U_k\phi_{jr}U^r) - \mu(A_{jr}\phi_k{}^r - A_{kr}\phi_j{}^r)$$
$$= \xi_j\nabla_k\mu - \xi_k\nabla_j\mu + (1+\frac{x}{k})$$
$$\{\frac{c}{2}\phi_{kj} + 2A_{jr}A_{ks}\phi^{rs} + \alpha(A_{jr}\phi_k{}^r - A_{kr}\phi_j{}^r) + \xi_j\nabla_k\alpha - \xi_k\nabla_j\alpha\}.$$

LEMMA 4.2. *$du = 0$ on Ω if $d\rho(W) = 0$, where the 1-form u is defined by $u(X) = g(U, X)$ for any vector X on M.*

Proof. If $du \neq 0$ on Ω, then we have $k = x$ because of (4.27). Thus (4.12) implies $\lambda = 0$ and hence $AU = 0$. Further (4.28) turns out to be

$$(2\theta - c)\phi_{kj} = \xi_j(\nabla_k\mu + 2\nabla_k\alpha) - \xi_k(\nabla_j\mu + 2\nabla_j\alpha)$$
$$+ (\mu + 2\alpha)(A_{jr}\phi_k{}^r - A_{kr}\phi_j{}^r) + 4A_{jr}A_{ks}\phi^{rs},$$

which applying ξ^j,

$$\nabla_k\mu + 2\nabla_k\alpha = \{d\mu(\xi) + 2d\alpha(\xi)\}\xi_k + (\mu + 2\alpha)U_k. \tag{4.29}$$

Thus, it follows that

$$(2\theta - c)\phi_{kj} = (\mu + 2\alpha)(U_k\xi_j - U_j\xi_k + A_{jr}\phi_k{}^r - A_{kr}\phi_j{}^r) + 4A_{jr}A_{ks}\phi^{rs}.$$

If we apply this by ρW^k and make use of (4.14) and the fact that $AU = 0$, then we obtain

$$2\theta - c = (\mu + 2\alpha)(\varepsilon - \alpha) \tag{4.30}$$

because $\rho \neq 0$ on Ω, which together with (4.26) yields $(\varepsilon - \alpha)\{d\mu(\xi) + 2d\alpha(\xi)\} = 0$ because $d\rho(W) = 0$ is assumed and consequently $d\mu(\xi) + 2d\alpha(\xi) = 0$ by virtue of $\theta - \frac{c}{2} \neq 0$. Therefore (4.29) turns out to be

$$\nabla_k\mu + 2\nabla_k\alpha = (\mu + 2\alpha)U_k,$$

which implies

$$\nabla_j\nabla_k\mu + 2\nabla_j\nabla_k\alpha = (\mu + 2\alpha)(U_jU_k + \nabla_kU_j).$$

Thus, it follows that $(\mu+2\alpha)du = 0$, which connected with (4.30) implies that $du = 0$ because $\theta - \frac{c}{2} \neq 0$ is assumed. Hence Lemma 4.2 is proved.□

From Lemma 4.2, we have $\xi^j(\nabla_jU_i - \nabla_iU_j) = 0$, which together with (1.14), (1.17) and (4.4) gives

$$\phi_i{}^r\nabla_r\alpha = -3\lambda\phi_{ir}U^r - \beta\xi_i + A_{ir}{}^2\xi^r.$$

Therefore we see, using (1.13) and (4.10), that

$$\nabla_i\alpha = d\alpha(\xi)\xi_i + (\varepsilon - 3\lambda)U_i. \tag{4.31}$$

If we use (3.9), (4.6) and Lemma 4.2, then (4.16) implies

$$x(n_k\phi_{jr}U^r - n_j\phi_{kr}U^r) + K_{jr}\nabla_kU^r - K_{kr}\nabla_jU^r = 0.$$

Since U is orthogonal to the structure vector ξ, by applying ξ^k and making use of (1.13), (3.13) and Lemma 4.2, we obtain

$$x\mu(A_{jr}\xi^r - \alpha\xi_j) - K_j{}^r(U^k\nabla_r\xi_k) + kU^r\nabla_j\xi_r = 0.$$

Multiplying W^j to this and summing for j, and using (1.6), (4.14) and (4.15), we find $\rho\{x\mu + (k-x)(\varepsilon - \alpha)\} = 0$. So we have

$$x\mu + (k+x)(\varepsilon - \alpha) = 0, \quad \text{i.e.,}\ x(k-x)\mu - \lambda(k+x)^2 = 0 \tag{4.32}$$

because of (4.12).

Multiplying (4.28) with ξ^j and summing for j and making use of (4.4), (4.31) and (4.32), we find

$$\nabla_k\mu = d\mu(\xi)\xi_k + (1 + \frac{x}{k})(\lambda + \mu)U_k. \tag{4.33}$$

Substituting (4.31), (4.33) into (4.28) and using (4.12), we get

$$\begin{aligned}
&2\theta\phi_{kj} + \frac{x}{k^2}(k-x)(U_j\phi_{kr}U^r - U_k\phi_{jr}U^r)\\
&= (1 + \frac{x}{k})\{(\mu + \varepsilon - 2\lambda)(U_k\xi_j - U_j\xi_k) + \frac{c}{2}\phi_{kj} + 2A_{jr}A_{ks}\phi^{rs}\\
&+ (\mu + \varepsilon)(A_{jr}\phi_k{}^r - A_{kr}\phi_j{}^r)\}.
\end{aligned}$$

Now, we are going to prove that Ω is empty.
Multiplying this with ρW^j and summing for j, and using (4.4) and (4.14), we find

$$2\theta + \frac{x}{k^2}(k-x)(\beta - \alpha^2) = (1+\frac{x}{k})\{\frac{c}{2} - 2\lambda(\varepsilon - \alpha) + (\mu+\varepsilon)(\lambda+\varepsilon-\alpha)\}$$

because $\rho \neq 0$ on Ω, or using (4.11), (4.12) and (4.32),

$$(4.34) \qquad 2x(k-x)\alpha\lambda + \lambda^2(k+x)^2 = (\frac{c}{4} - x^2)(k-x)^2.$$

Differentiating this covariantly and making use of Lemma 4.1, we obtain

$$(4.35) \quad x(k-x)\nabla_j(\alpha\lambda) + \lambda(k+x)^2\nabla_j\lambda = x\lambda\{2\lambda(k+x) + \alpha(k-x)\}U_j,$$

which together with (4.20) yields $x(k-x)\lambda d\alpha(\xi) = 0$, or using (4.12)

$$(4.36) \qquad \lambda d\alpha(\xi) = 0.$$

By the hypotheses $d\rho(W) = 0$, we see, using (4.26), that $d\varepsilon(\xi) = d\alpha(\xi)$. Thus, applying (4.25) by ξ^j and making use of (4.13) and (4.36), we find

$$\frac{1}{2}d\beta(\xi) - hd\alpha(\xi) + \rho d\alpha(W) = 0$$

and hence $d\beta(\xi) = 2\varepsilon d\alpha(\xi)$, where we have used (4.31). Since we have $\alpha\varepsilon = \beta + 2xk + \alpha\lambda$, it is, using (4.20), (4.36) and Lemma 4.1, verified that $d\beta(\xi) = (\alpha+\varepsilon)d\alpha(\xi)$. From the last two equations, it follows that $(\varepsilon-\alpha)d\alpha(\xi) = 0$. Since $\varepsilon - \alpha \neq 0$ on Ω, we have $d\alpha(\xi) = 0$. Consequently (4.31) and (4.33) are reduced respectively to

$$(4.37) \qquad \nabla_j\alpha = (\varepsilon - 3\lambda)U_j,$$

$$(4.38) \qquad \nabla_j\mu = (1+\frac{x}{k})(\lambda+\mu)U_j.$$

Differentiating (4.32) covariantly yields

$$\lambda(k+x)^2U_j + x(k-x)\nabla_j\mu = (k+x)^2\nabla_j\lambda + 2(k+x)(k-x)\lambda U_j,$$

where we have used Lemma 4.1, which together with (4.32) and (4.38) gives

$$(k+x)\nabla_j\lambda = 6x\lambda U_j \tag{4.39}$$

because $k+x=0$ is not imposible on Ω. Thus (4.35) implies $(k-x)\nabla_j(\alpha\lambda) = \lambda\{\alpha(k-x)-4\lambda(k+x)\}U_j$ because of $\theta \neq \frac{c}{4}$ on Ω, which connected with (4.12) and (4.37) implies that

$$\alpha(k-x)\nabla_j\lambda + 6x\lambda^2 U_j = 0.$$

From this and (4.39) it follows that $x\lambda\{\alpha(k-x)+\lambda(k+x)\}=0$ and therefore $\lambda\{\alpha(k-x)+\lambda(k+x)\}=0$ because of Lemma 2.1.

Now suppose $\lambda \neq 0$ on Ω. Then we have $\lambda(k+x)+\alpha(k-x)=0$ and hence $\varepsilon = 2\alpha$ because of (4.32) on this set. Furthermore we have $(k+x)\nabla_j\lambda + (k-x)\nabla_j\alpha = 0$ on the set, which together with (4.37) and (4.39) implies that $6x\lambda + (k-x)(3\alpha-2\lambda)=0$, a contradiction. Therefore we have $\lambda = 0$ on Ω. So (4.34) implies $k=x$ because $\theta - \frac{c}{2} \neq 0$. Thus (4.32), (4.37) and (4.38) are reduced respectively to $\mu = 2(\alpha-\varepsilon)$, $\nabla\alpha = \varepsilon U$ and $\nabla\mu = 2\mu U$. From these facts, it is verified that $\nabla\varepsilon = (3\varepsilon - 2\alpha)U$.

On the other hand, in the proof of Lemma 4.2, we see that $2\theta - c = (\mu+2\alpha)(\varepsilon-\alpha)$ and hence $\theta - \frac{c}{2} = (2\alpha-\varepsilon)(\varepsilon-\alpha)$. By differentiation gives $(\varepsilon-\alpha)(2\alpha-\varepsilon)U = 0$, which produces a contradiction because $\theta - \frac{c}{2} \neq 0$ is assumed. Thus Ω is an empty set.

Summing up, we have

LEMMA 4.3. *Let M be a real $(2n-1)$-dimensional semi-invariant submanifold of codimension 3 in $\mathbb{C}P^{n+1}$ satisfying $dn = 2\theta\omega$ for a certain function $\theta \neq \frac{c}{2}$. If M satisfies $R_\xi\phi = \phi R_\xi$ and at the same time $d\rho(W) = 0$ and $AU = \lambda U$ for a function λ on M. Then the distinguished normal is parallel in the normal bundle.*

Let M be a connected real $(2n-1)$-dimensional $(n>2)$ semi-invariant submanifold of codimension 3 satisfying $dn = 2\theta\omega$ for a certain scalar $\theta < \frac{c}{2}$ in $\mathbb{C}P^{n+1}$. Suppose that $R_\xi\phi = \phi R_\xi$ and at the same time $d\rho(W)=0$ and $AU = \lambda U$. Then by Lemma 4.3 we have $k=0$ on M. Thus, (3.3) tells us that the distinguished normal C is parallel in the normal bundle. Hence, by Lemma 4.1 of [8], we have $A_{(2)} = A_{(3)} = 0$. Therefore, by the reduction theorem in [5], [10], M is a real hypersurface in a complex projective space $\mathbb{C}P^n$. Since we have $\nabla^\perp C = 0$, equations (1.19) and (2.2) are reduced respectively to

$$\nabla_k A_{ji} - \nabla_j A_{ki} = \frac{c}{4}(\xi_k\phi_{ji} - \xi_j\phi_{ki} - 2\xi_i\phi_{kj}),$$

$$\alpha(A_{jr}\phi_i{}^r + A_{ir}\phi_j{}^r) + U_iA_{jr}\xi^r + U_jA_{ir}\xi^r = 0.$$

Making use of (1.5), (1.6) and above two equations, it is proved in [4] that $g(U, U) = 0$, that is, M is a Hopf real hypersurface. Hence we have $A\phi = \phi A$ provided that $\alpha \neq 0$. Thus, by Theorem CK and Theorem KST, we have

LEMMA 4.4. *Let M be a real $(2n-1)$-dimensional semi-invariant $(n > 2)$ submanifold of codimension 3 in $\mathbb{C}P^{n+1}$ such that the third fundamental form n satisfies $dn = 2\theta\omega$ for a certain scalar $\theta(< \frac{c}{2})$, where $\omega(X,Y) = g(X, \phi Y)$ for any vectors X and Y on M. Suppose that M satisfies $R_\xi\phi = \phi R_\xi$ and at the same time $d\rho(W) = 0$ and $AU = \lambda U$ for a function λ on M, where $U = \nabla_\xi\xi$. Further if $g(A\xi, \xi) \neq 0$, then M is locally congruent to one of the following spaces in $\mathbb{C}P^n$:*

(A_1) *a geodesic hypersphere (that is, a tube of radius r over a hyperplane $\mathbb{C}P^{n-1}$,*
where $0 < r < \frac{\pi}{2}$ and $r \neq \frac{\pi}{4}$),

(A_2) *a tube of radius r over a totally geodesic $\mathbb{C}P^k(1 \leq k \leq n-2)$ where*
$0 < r < \frac{\pi}{2}$, *and* $r \neq \frac{\pi}{4}$

REMARK 3. It is proved in [7] that a semi-invariant submanifold of codimension 3 with $dn = 2\theta\omega$ for $\theta \neq \frac{c}{2}$ satisfying $R_\xi A\phi = \phi AR_\xi$ in a complex projective space $\mathbb{C}P^{n+1}$ implies $A\phi = \phi A$.

5. Theorems

First of all, we prove

THEOREM 5.1. *Let M be a connected real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 in a complex projective space $\mathbb{C}P^{n+1}$ such that the third fundamental form n satisfies $dn = 2\theta\omega$ for a certain scalar $\theta(< \frac{c}{2})$, where $\omega(X,Y) = g(X, \phi Y)$ for any vectors X and Y on M. If M satisfies $R_\xi\phi = \phi R_\xi$ and at the same time $R_\xi A = AR_\xi$, A being the shape opertor in the direction of the distinguished normal, then M is locally congruent to one of the following spaces in $\mathbb{C}P^n$ provided that $g(A\xi, \xi) \neq 0$:*

(A_1) *a geodesic hypersphere (that is, a tube of radius r over a hyperplane $\mathbb{C}P^{n-1}$,*
where $0 < r < \frac{\pi}{2}$ and $r \neq \frac{\pi}{4}$),

(A_2) *a tube of radius r over a totally geodesic $\mathbb{C}P^k (1 \leq k \leq n-2)$ where*

$0 < r < \frac{\pi}{2}$, *and* $r \neq \frac{\pi}{4}$.

Proof. By the hypotheses $dn = 2\theta\omega$ for $\theta < \frac{c}{2}$ and Lemma 3.1, we have (3.1)–(3.19). Thus $R_\xi A = AR_\xi$ gives

$$(A_{ir}\xi^r)(A_{js}{}^2\xi^s) - (A_{jr}\xi^r)(A_{is}{}^2\xi^s) + (k^2 + \frac{c}{4})(\xi_i A_{jr}\xi^r - \xi_j A_{ir}\xi^r) = k^2(n_i\xi_j - n_j\xi_i),$$

where we have used (2.1), (3.3) and (3.11). If we apply this by ξ^j, then we obtain

$$(5.1)\quad -\alpha A_{ir}{}^2\xi^r + (\beta - k^2 - \frac{c}{4})A_{ir}\xi^r = k^2 n_i - \{k^2\mu + \alpha(k^2 + \frac{c}{4})\}\xi_i.$$

Combining the last two equations, it follows that

$$(A_{js}{}^2\xi^s)(A_{ir}\xi^r - \alpha\xi_i) - (A_{is}{}^2\xi^s)(A_{jr}\xi^r - \alpha\xi_j) = \beta(\xi_j A_{ir}\xi^r - \xi_i A_{jr}\xi^r).$$

Multiplying this with $A_t{}^j\xi^t$ and summing for j, we find

$$(5.2)\qquad \rho^2 A_{ir}{}^2\xi^r = (\gamma - \beta\alpha)A_{ir}\xi^r + (\beta^2 - \alpha\gamma)\xi_i,$$

where we have put $\gamma = A_{ji}{}^3\xi^j\xi^i$.

Let Ω be a set of point in M such that $k \neq 0$. Then (5.2) is reduced to

$$(5.3)\qquad A_{ir}{}^2\xi^r = \varepsilon A_{ir}\xi^r + (\beta - \varepsilon\alpha)\xi_i$$

because of Lemma 3.1, where we have defined the function ε by $\rho^2\varepsilon = \gamma - \beta\alpha$ on Ω. Hence (5.1) implies

$$(\beta - \varepsilon\alpha - k^2 - \frac{c}{4})(A_{jr}\xi^r - \alpha\xi_j) = k^2(n_j - \mu\xi_j),$$

which together with (3.16) yields

$$(5.4)\qquad K_{jr}U^r = xU_j$$

on Ω, where $kx = \frac{c}{4} + \varepsilon\alpha - \beta$, which together with (4.12) gives

$$(5.5)\qquad \{\rho^2 + kx - \frac{c}{4}\}(k - x) + \alpha\lambda(k + x) = 0.$$

Accordingly (5.3) becomes on Ω

$$(5.6) \qquad A_{jr}^{2}\xi^r = \varepsilon A_{jr}\xi^r - (kx - \frac{c}{4})\xi_j.$$

Transforming (5.4) by $\phi_i^{\ j}$ and using (3.5), we have

$$L_{ir}U^r = x\phi_{ir}U^r,$$

which together with (3.6), (5.4) and the last equation yields $x^2 = \theta - \frac{c}{4}$. So x is constant on Ω if $n > 2$.

Since the hypotheses $R_\xi\phi = \phi R_\xi$ gives (4.1), by applying $A_t^{\ i}\xi^t$ and summing for i, and using (1.13), (3.4), (5.4) and (5.6), we find

$$(5.7) \qquad \alpha A_{jr}U^r + (kx + \frac{c}{4})U_j = 0$$

on Ω. If $\alpha = 0$ on Ω, then we have $kx + \frac{c}{4} = 0$. So k is constant on Ω. Thus, we see, using (3.16), (3.17) and (5.4), that $k = x$ on Ω and hence $k^2 + \frac{c}{4} = 0$, a contradiction. Therefore $\alpha = 0$ is not imposible on Ω. By (5,7), it is clear that $AU = \lambda U$, where $\alpha\lambda = -\frac{c}{4} - kx$. Further, it is, using (4.12), seen that $k - x$ does not vanish on Ω. Thus (5.5) becomes

$$\rho^2 = \frac{2k}{k - x}(x^2 + \frac{c}{4}),$$

which connected with (4.21) implies that $d\rho(W) = 0$. Hence Ω is empty because of Lemma 4.3. According to Lemma 4.4, we have our Theorem 5.1. □

THEOREM 5.2. *Let M be a connected real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 with constant mean curvature in a complex projective space $\mathbb{C}P^{n+1}$ such that $dn = 2\theta\omega$ for a certain scalar $\theta(< \frac{c}{2})$. Suppose M satisfies $R_\xi\phi = \phi R_\xi$ and at the same time $S\xi = g(S\xi, \xi)\xi$. Further if $g(A\xi, \xi) \neq 0$, the M is locally congruent to the same type as those stated in Theorem 5.1, where S denotes the Ricci tensor of M.*

REMARK 4. For a real hypersurface satisfying $S\phi \pm \phi S = 2\tau\phi$ for a function τ in a complex projective space, many results are known under some additional hypotheses ([9]). However, we notice here that these conditions imply always $S\xi = g(S\xi, \xi)\xi$.

Proof. Since $dn = 2\theta\omega$ for $\theta < \frac{c}{2}$, by Lemma 3.1 we see that (3.3)–(3.13) are valid.

From (1.18) the Ricci tensor S of M is given by

$$S_{ji} = \frac{c}{4}\{(2n+1)g_{ji} - 3\xi_j\xi_i\} + hA_{ji} - A_{ji}^{\ 2} + kK_{ji} - K_{ji}^{\ 2} - L_{ji}^{\ 2}$$

because of the second equation of (1.10), where we have put $h = TrA$. Applying this by ξ^i and making use of (3.13), we find

$$S_{jr}\xi^r = \frac{c}{2}(n-1)\xi_j + hA_{jr}\xi^r - A_{jr}^{\ 2}\xi^r.$$

Since $S\xi$ is proportional to ξ, it follows that

$$A_{jr}^{\ 2}\xi^r = hA_{jr}\xi^r + (\beta - h\alpha)\xi_j. \tag{5.8}$$

From now on we discuss our arguments on the set $\Omega = \{p : k(p) \neq 0\}$.

If we multiply $A_t^{\ j}\xi^t$ to (4.1) and sum for j, and take account of (1.13), (3.4) and the last equation, then we get

$$\alpha A_{ir}U^r + (\beta - h\alpha)U_i = -2kK_{jr}U^r. \tag{5.9}$$

From this and (4.3), it follows that

(5.10)

$$k\{(L_{ir}U^r)(\phi_{js}U^s) - (L_{jr}U^r)(\phi_{is}U^s)\} - \frac{1}{2}\alpha(U_jA_{ir}U^r - U_iA_{jr}U^r) = 0.$$

Applying (4.1) by U^i and using (1.13),we find

$$-\alpha A_{jr}^{\ 2}\xi^r + \beta A_{jr}\xi^r + \alpha U^iA_{ir}\phi_j^{\ r} = -2kL_{jr}U^r,$$

which together with (1.13) gives $kL_{ji}U^jU^i = 0$. Thus, by (5.10) we have $\rho^2A_{jr}U^r - (A_{rs}U^rU^s)U_j = 0$ by virture of $\alpha \neq 0$. If $\rho = 0$, then by Lemma 3.1 and (4.1), we have $A\phi = \phi A$. Therefore M is of type A_1 or A_2. So we have $AU = \lambda U$, where the function λ is defined by $\rho^2\lambda^2 = A_{sr}U^rU^s$. Thus, (5.9) implies that

$$\beta - h\alpha + \alpha\lambda + 2kx = 0. \tag{5.11}$$

If we take account of (4.9), (5.8), (5.9) and the fact that $AU = \lambda U$, then we find

$$(k - x)(h - \alpha) + \lambda(k + x) = 0, \tag{5.12}$$

which enables us to obtain

$$(h-\alpha+\lambda)\nabla_j k+(k+x)\nabla_j\lambda+(k-x)(\nabla_j h-\nabla_j\alpha)=0.$$

From this, (4.19) and (4.21), we get

$$(k-x)\{dh(W)-d\alpha(W)\}=0. \tag{5.13}$$

Since the mean curvature of M is constant, it follows that $h\nabla_j h+k\nabla_j k=0$, which together with (4.21) gives

$$h\nabla_j h+k\{\nu\xi_j+(k-x)U_j\}=0, \tag{5.14}$$

which implies $dh(W)=0$ and hence

$$(k-x)d\alpha(W)=0 \tag{5.15}$$

because of (5.14).

On the other hand, using the quite same method as that used to (4.25) from (4.10), we can drive from (5.8) the following:

$$\begin{aligned}&A_j{}^r\nabla_r\alpha+\frac{1}{2}\nabla_j\beta-h\nabla_j\alpha+(3\lambda^2-\frac{c}{4}-2\lambda h+\alpha\lambda)U_j\\&=\rho\, dh(\xi)W_j-\{2x\nu+\lambda d\alpha(\xi)\}\xi_j.\end{aligned} \tag{5.16}$$

Now, suppose that $d\alpha(W)\neq 0$ on Ω. Then we have from (5.15) $k=x$. Hence (5.14) means that $h=const.$ So we have $\lambda=0$ on this set because of (5.12). Therefore (5.11) is reduced to $\beta-h\alpha+2x^2=0$, which implies $\nabla_j\beta=h\nabla_j\alpha$. Thus, (5.16) turns out to be

$$A_j{}^r\nabla_r\alpha-\frac{1}{2}h\nabla_j\alpha=\frac{c}{4}U_j.$$

Applying this by ξ^j and W^j, and making use of (4.13) and (4.14), we have respectively

$$(2\alpha-h)d\alpha(\xi)+2\rho\, d\alpha(W)=0, 2\rho\, d\alpha(\xi)+(h-2\alpha)d\alpha(W)=0,$$

which produces a contradiction. Thus $d\alpha(W)=0$ on Ω. So we easily see, using (4.19), (4.21) and (5.15), that $d\beta(W)=0$ on Ω. Accordingly we have $d\rho(W)=0$. By Lemma 4.3, it follows that the distinguished normal is parallel in the normal bundle. Therefore (4.1) turns out to be

$$\alpha(A_{jr}\phi_i{}^r+A_{ir}\phi_j{}^r)+U_iA_{jr}\xi^r+U_jA_{ir}\xi^r=0.$$

Thus, in the proof of Lemma 4.4 we arrive at the conclusion. □

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