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ON THE RANGE CLOSURE OF AN ELEMENTARY OPERATOR

Jose Angel Canavati, Slavisa V. Djordjević, and Bhagwati Prasad Duggal

Abstract. Let $A, B \in B(H)$ be Hilbert space contractions, and let $\triangle_{AB}$ be the elementary operator $\triangle_{AB} : X \longrightarrow AXB - X$. A number of conditions which are equivalent to " $\triangle_{AB}$ has closed range" are proved.

1. Introduction

Given an infinite dimensional complex Hilbert space H, let $B(H)$ denote the algebra of operators (equivalently, bounded linear transformations) on H. For $A, B \in B(H)$, let $\triangle_{AB} \in B(B(H))$ be the *elementary operator* defined by

$$\triangle_{AB}(X) = AXB - X.$$

Let $\sigma(\triangle_{AB})$ and $\sigma_p(\triangle_{AB})$ denote, respectively, the spectrum and the point spectrum of $\triangle_{AB}$. It was proved in [5, Theorem 2] that if $A_1, A_2 \in B(H)$ are contractions for which $\lambda \in \sigma(A_i)$ and $|\lambda| = 1$ implies $\lambda \in \sigma_p(A_i)$ $(i = 1, 2)$, then the range $\triangle_{A_1A_2}(B(H))$ of $\triangle_{A_1A_2}$ is closed. This paper dispenses with the hypothesis on the points in the boundary of the spectrum of the contractions A_i to prove the following theorem. For a Banach space operator T, $T \in B(\mathcal{Y})$, let iso$\sigma(T)$ denote the *isolated points of the spectrum of* T, asc(T) denote the *ascent of* T, dsc(T) denote the *descent of* T, and let $H_0(T)$ denote the *quasinilpotent part*

$$H_0(T) = \{y \in \mathcal{Y} : \lim_{n\to\infty} ||T^n y||^{\frac{1}{n}} = 0\}$$

of T. The operator T is *semi-regular* if $T(\mathcal{Y})$ is closed and $T^{-1}(0) \subseteq \bigcap_{n=1}^{\infty} T^n(\mathcal{Y})$; T admits a *generalized Kato decomposition*, or GKD, if

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there exists a pair of T-invariant closed subspaces (N, M) such that $\mathcal{Y} = N \oplus M$, $T|_M$ is semiregular and $T|_N$ is quasinilpotent. If in this restriction $T|_N$ is nilpotent, then T is said to be *Kato type*. Let $\gamma(T)$,

$$\gamma(T) = \inf\{\frac{||Ty||}{\text{dist}(y, T^{-1}(0))} : y \in \mathcal{Y} \setminus T^{-1}(0)\}$$

(with the convention that $\gamma(T) = \infty$ if $T = 0$), denote the *reduced minimal modulus* of an operator T [9, p. 203]. Let $\alpha(T) = \dim T^{-1}(0)$ and $\beta(T) = \dim(\mathcal{Y} \setminus T\mathcal{Y})$ denote, respectively, the *null deficiency* and the *image deficiency* of T, and let $\text{ind}(T) = \alpha(T) - \beta(T)$ denote the *index* of T.

THEOREM 1.1. *If $A, B \in B(H)$ are contractions such that $0 \in \sigma(\triangle_{AB})$ then the following conditions are equivalent.*
(i) $B(H) = \triangle_{AB}^{-1}(0) \oplus \triangle_{AB}(B(H))$.
(ii) 0 is a pole of the resolvent of $\triangle_{AB}$.
(iii) $0 \in \text{iso}\sigma(\triangle_{AB})$ and $H_0(\triangle_{AB}) = \triangle_{AB}^{-1}(0)$.
(iv) $\triangle_{AB}(B(H))$ (equivalently, $\triangle_{AB}^*(B(H)^*)$) is closed.
(v) $\text{dsc}(\triangle_{AB}) < \infty$.
(vi) $\triangle_{AB}$ is Kato type.
(vii) $\gamma(\triangle_{AB}) > 0$.
(viii) $\{0\} \neq \triangle_{AB}^{-1}(0)$ is complemented by a $\triangle_{AB}$ invariant closed subspace of $\triangle_{AB}(B(H))$.
Furthermore: (a) If the contractions A (and B) are such that the points $\lambda \in \text{iso}\sigma(A)$ (resp., $\lambda \in \text{iso}\sigma(B)$) with $|\lambda| = 1$ are eigenvalues of A (resp., B), then 0 is a pole of the resolvent of $\triangle_{AB}$ if and only if the set $\{\alpha \in \sigma(A) : \alpha^{-1} \in \sigma(B)\}$ is finite.
(b) If $0 < \alpha(\triangle_{AB}) < \infty$, then 0 is a pole of the resolvent of $\triangle_{AB}$ if and only if $\text{ind}(\triangle_{AB}^*) = -\text{ind}(\triangle_{AB})$.

The equivalent conditions $(i)-(viii)$ of the theorem answer [12, Question 2] and improve [5, Theorem 2]. The main tools that we use in the proof of Theorem 1.1 are the Nirschl-Schneider theorem [3, Theorem 10.10] and elements of "*local spectral theory*" (for which we refer the reader to the excellent monograph [9]). Our notation and terminology is explained below, and we prove Theorem 1.1 in Section 2.

The *ascent of* $T \in B(\mathcal{Y})$, $\text{asc}(T)$, is the least non-negative integer n such that $T^{-n}(0) = T^{-(n+1)}(0)$ and the *descent of* T, $\text{dsc}(T)$, is the least non-negative integer n such that $T^n(\mathcal{Y}) = T^{n+1}(\mathcal{Y})$. Clearly, the operator T is injective if and only if $\text{asc}(T) = 0$ and T is surjective if and only if $\text{dsc}(T) = 0$. In the following, we shall denote the open unit disc, the closed unit disc and the boundary of the closed unit disc in the complex plane $\mathbf{C}$ by $\mathbf{D}$, $\overline{\mathbf{D}}$ and $\partial\mathbf{D}$, respectively. The *numerical range* $W(B(\mathcal{Y}),T)$ of $T \in B(\mathcal{Y})$ is the set

$$\{f(T) : f \in B(\mathcal{Y})^*,\ \|f\| = f(T) = 1\},$$

where $B(\mathcal{Y})^*$ denotes the (Banach space) dual of $B(\mathcal{Y})$. The numerical range of T is the closed convex hull $\overline{coV(T)}$ of the *spatial numerical range*

$$V(T) = \{F(Ty) : F \in \mathcal{Y}^*, y \in \mathcal{Y}, \|F\| = \|y\| = F(y) = 1\},$$

of T [3, Theorem 9.4]. If we denote the operator conjugate to $T \in B(\mathcal{Y})$ by T^*, then $\overline{coV(T)} = \overline{coV(T^*)}$ [3, Corollary 9.6(ii)]. Hence:

PROPOSITION 1.2. $W(B(\mathcal{Y}),T) = W(B(\mathcal{Y}^*),T^*)$.

We say that T is semi-Fredholm if $T(\mathcal{Y})$ is closed and either $\alpha(T) = \dim(T^{-1}(0))$ or $\beta(T) = \dim(\mathcal{Y}/T(\mathcal{Y}))$ is finite. If T is semi-Fredholm, then *the (Fredholm) index* of T, $\text{ind}(T)$, is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$; T is said to be *Fredholm* if $\text{ind}(T)$ is finite. The *analytic core* $K(T)$ of a Banach space operator $T \in B(\mathcal{Y})$ is defined by

$$\begin{aligned} K(T) \;=\; & \{y \in \mathcal{Y} : \text{there exists a sequence } \{y_n\} \subset \mathcal{Y} \text{ and } \delta > 0 \\ & \text{for which } y = y_0,\ T(y_{n+1}) = y_n \text{ and } \|y_n\| \le \delta^n \|y\| \\ & \text{for all } n = 1, 2, \ldots\}. \end{aligned}$$

The subspaces $H_0(T)$ and $K(T)$ are generally non-closed, $TK(T) = K(T) \subseteq \bigcap_{n=1}^{\infty} T^n(\mathcal{Y})$, and $T^{-n}(0) \subseteq H_0(T)$ for all $n = 1, 2, \ldots$ (cf. [10]). Observe that if $0 \in \text{iso}\sigma(T)$, then $H_0(T)$ and $K(T)$ are closed.

2. Proof of Theorem 1.1

$(i) \Longleftrightarrow (ii)$. The equivalence $\mathcal{Y} = T^{-p}(0) \oplus T^p(\mathcal{Y}) \Longleftrightarrow 0$ is a pole of the resolvent operator holds for every Banach space operator $T \in B(\mathcal{Y})$ [8, Proposition 50.2]. To prove $(ii) \Longleftrightarrow (i)$, we prove that $\text{asc}(\triangle_{AB}) = \text{dsc}(\triangle_{AB}) \le 1$. To achieve this we show that 0 is in the boundary $\partial W(B(B(H)), \triangle_{AB})$ of the numerical range of $\triangle_{AB}$. An application of the Nirschl-Schneider theorem on the eigenvalues of a Banach space operator in the boundary of the numerical range of the operator [3, Theorem 10.10] will then prove that $\text{asc}(\triangle_{AB}) \le 1$.

Let L_A and R_B ($\in B(B(H))$) denote, respectively, the operators of "left multiplication by A" and "right multiplication by B". Then the operators A and B being Hilbert space contractions it follows that

$$W(B(B(H)), L_A R_B) \subseteq \overline{\mathbf{D}}$$

(see [2, Theorem 5.2], [3] and [4]). Since (cf. [2])

$$W(B(B(H)),\triangle_{AB}) = W(B(B(H)),L_AR_B-1) = W(B(B(H)),L_AR_B)-1,$$

it follows that

$$W(B(B(H)),\triangle_{AB}) \subseteq \{\lambda \in \mathbf{C} : |\lambda+1| \leq 1\}.$$

In particular,

$$0 \in \partial W(B(B(H)),\triangle_{AB}) \Longrightarrow \text{asc}(\triangle_{AB})(\text{and } \text{asc}(\triangle_{AB}^*)) \leq 1.$$

(Observe that if a Banach space operator T has finite ascent and descent, then the conjugate operator T^* satisfies $\text{asc}(T^*) = \text{dsc}(T)$ and $\text{dsc}(T^*) = \text{asc}(T)$. Hence $\text{asc}(\triangle_{AB}^*) = \text{dsc}(\triangle_{AB}^*) \leq 1$, which implies that if (i) holds then 0 is a simple pole of the resolvent of $\triangle_{AB}^*$. Hence 0 is an eigenvalue of both $\triangle_{AB}$ and $\triangle_{AB}^*$.)

$(ii) \Longleftrightarrow (iii)$. Evidently, $(ii) \Longrightarrow (iii)$. For the reverse implication, we observe that if $0 \in \text{iso}\sigma(\triangle_{AB})$, then $B(H) = H_0(\triangle_{AB}) \oplus K(\triangle_{AB})$. Thus, if $H_0(\triangle_{AB}) = \triangle_{AB}^{-1}(0)$, then

$$\begin{aligned} & B(H) = \triangle_{AB}^{-1}(0) \oplus K(\triangle_{AB}) \\ \Longrightarrow\ & \triangle_{AB}(B(H)) = 0 \oplus \triangle_{AB}(K(\triangle_{AB})) = K(\triangle_{AB}) \\ \Longrightarrow\ & B(H) = \triangle_{AB}^{-1}(0) \oplus \triangle_{AB}(B(H)). \end{aligned}$$

$(iv) \Longleftrightarrow (i)$. For an arbitrary subset M of the Banach space $\mathcal{Y}$, let

$$M^{\perp} = \{\phi \in \mathcal{Y}^* : \phi(y) = 0 \text{ for all } y \in M\},$$

denote the *annihilator* of M in $\mathcal{Y}^*$. Recall that if $T \in B(\mathcal{Y})$, then $T(\mathcal{Y})^{\perp} = T^{*-1}(0)$, and if $T(\mathcal{Y})$ is closed, then $T^{-1}(0)^{\perp} = T^*(\mathcal{Y}^*)$.

As we saw above, $\text{asc}(\triangle_{AB})$ and $\text{asc}(\triangle_{AB}^*)$ are both less of equal of 1; hence $\triangle_{AB}^{-1}(0) \cap \triangle_{AB}(B(H)) = \{0\} = {\triangle_{AB}^*}^{-1}(0) \cap \triangle_{AB}^*(B(H)^*)$ [9, Lemma 4.10.1]. If $\triangle_{AB}(B(H))$ is closed, then $\text{asc}(\triangle_{AB}) \leq 1$ implies $\triangle_{AB}^{-1}(0) + \triangle_{AB}(B(H))$ is closed [9, Proposition 4.10.4]. We have:

$$\begin{aligned} \{\triangle_{AB}^{-1}(0) + \triangle_{AB}(B(H))\}^{\perp} &= \triangle_{AB}(B(H))^{\perp} \cap \triangle_{AB}^{-1}(0)^{\perp} \\ &= {\triangle_{AB}^*}^{-1}(0) \cap \triangle_{AB}^*(B(H)^*) \\ &= \{0\}. \end{aligned}$$

Hence, $\text{dsc}(\triangle_{AB}) \leq 1$, which implies that $\text{asc}(\triangle_{AB}) = \text{dsc}(\triangle_{AB}) \leq 1 \Longrightarrow (i)$. Evidently, $(i) \Longrightarrow (iv)$.

$(v) \Longleftrightarrow (i)$. Obvious, since $\text{asc}(\triangle_{AB}) \leq 1$.

$(vi) \Longleftrightarrow (i)$. The implication $(i) \Longrightarrow (vi)$ is obvious. For the implication $(vi) \Longrightarrow (i)$, we observe that both $\text{asc}(\triangle_{AB})$ and $\text{asc}(\triangle_{AB}^*)$ are finite. If $\triangle_{AB}$ is Kato type, then an application of [1, Theorem 2.9] shows that $\mathrm{dsc}(\triangle_{AB}) \leq 1$.

$(iv) \Longleftrightarrow (vii)$. Evident (see [9, p. 203]).

$(viii) \Longleftrightarrow (iii)$. Observe that if $(viii)$ holds, then $B(H) = \triangle_{AB}^{-1}(0) \oplus M$ for some closed subspace M of $\triangle_{AB}(B(H))$. Consequently,

$$\triangle_{AB}(B(H)) = 0 \oplus \triangle_{AB}(M) \subseteq M \subseteq \triangle_{AB}(B(H)).$$

Hence $B(H) = \triangle_{AB}^{-1}(0) \oplus \triangle_{AB}(B(H)) \Longrightarrow (iii)$. The reverse implication $(iii) \Longrightarrow (viii)$ being obvious (from the proof of $(iii) \Longrightarrow (ii)$), the equivalence follows.

To complete the proof of the theorem we now consider (a) and (b). To prove (a), we observe that if 0 is a pole of the resolvent of $\triangle_{AB}$, then $0 \in \mathrm{iso}\sigma(\triangle_{AB})$. Recall [7] that $\sigma(\triangle_{AB}) = \{\alpha\beta - 1 : \alpha \in \sigma(A), \beta \in \sigma(B)\}$. Hence, if $0 \in \mathrm{iso}\sigma(\triangle_{AB})$, then the set $\{\alpha \in \sigma(A) : \alpha^{-1} \in \sigma(B)\}$ is finite. Conversely, if $0 \in \mathrm{iso}\sigma(\triangle_{AB})$, and the isolated points of $\sigma(A)$ and $\sigma(B)$ in $\partial\mathbf{D}$ are eigenvalues of A and B (respectively), then the argument of the proof of Theorem 2 of [5] implies that $H_0(\triangle_{AB}) = \triangle_{AB}^{-1}(0)$ (i.e., (iii) is satisfied), which implies that 0 is a pole of the resolvent of $\triangle_{AB}$. To prove (b), we start by observing that if 0 is a pole of the resolvent of $\triangle_{AB}$, then $\mathrm{asc}(\triangle_{AB}) = \mathrm{dsc}(\triangle_{AB})(= \mathrm{asc}(\triangle_{AB}^*) = \mathrm{dsc}(\triangle_{AB}^*)) \leq 1$. Since $\mathrm{asc}(\triangle_{AB}) < \infty \Longrightarrow \beta(\triangle_{AB}) \geq \alpha(\triangle_{AB})$ and $\mathrm{dsc}(\triangle_{AB}) < \infty \Longrightarrow \beta(\triangle_{AB}) \leq \alpha(\triangle_{AB})$ [8, Proposition 38.5], our hypothesis $0 < \alpha(\triangle_{AB}) < \infty$ implies that $\triangle_{AB}$ is Fredholm. Hence $\mathrm{ind}(\triangle_{AB}^*) = -\mathrm{ind}(\triangle_{AB})$. Conversely, the implications $\mathrm{asc}(\triangle_{AB}) \leq 1 \Longrightarrow \mathrm{ind}(\triangle_{AB}) \leq 0$ and $\mathrm{asc}(\triangle_{AB}^*) \leq 1 \Longrightarrow \mathrm{ind}(\triangle_{AB}^*) = -\mathrm{ind}(\triangle_{AB}) \leq 0$ imply that $\alpha(\triangle_{AB}) = \beta(\triangle_{AB}) < \infty$. Hence $\triangle_{AB}$ is Fredholm of index 0. But then, see [1, Corollary 2.10], $\mathrm{asc}(\triangle_{AB}) = \mathrm{dsc}(\triangle_{AB}) \leq 1 \Longrightarrow 0$ is a pole of the resolvent of $\triangle_{AB}$. □

REMARK. If M and N are subspaces of a Banach space $\mathcal{Y}$, then we say that *M is orthogonal to N* (in the sense of Garret Birkhoff - see [6, Page 93]), denoted $M \perp N$, if $||m|| \leq ||m+n||$ for all $m \in M$ and $n \in N$. This asymmetric definition of orthogonality agrees with the usual definition of orthogonality in the case in which $\mathcal{Y}$ is a Hilbert space. Observe, from the proof of Theorem 1.1, that if $0 \in \sigma_p(\triangle_{AB})$, then $0 \in \partial W(B(B(H)), \triangle_{AB})$, which implies by a result of Sinclair [11, Proposition 1] that $\triangle_{AB}^{-1}(0) \perp \triangle_{AB}(B(H))$. Thus, if $\mu \in \sigma_p(\triangle_{AB}) \cap \partial W(B(B(H)), \triangle_{AB})$, then $(\triangle_{AB} - \mu)^{-1}(0) \perp (\triangle_{AB} - \mu)(B(H))$. For distinct eigenvalues μ_1 and $\mu_2 \in \partial W(B(B(H)), \triangle_{AB})$, A and $B \in B(H)$ are some operators, we have the following.

PROPOSITION 2.1. *If* $\mu_i \in \sigma_p(\triangle_{AB}) \cap \partial W(B(B(H)), \triangle_{AB})$, $i = 1, 2$ *and* $\mu_1 \neq \mu_2$, *then* $(\triangle_{AB} - \mu_1)^{-1}(0)$ *and* $(\triangle_{AB} - \mu_2)^{-1}(0)$ *are mutually orthogonal.*

Proof. The hypothesis $\mu_1 \in \sigma_p(\triangle_{AB}) \cap \partial W(B(B(H)), \triangle_{AB})$ implies $(\triangle_{AB} - \mu_1)^{-1}(0) \perp (\triangle_{AB} - \mu_1)(B(H))$. Let $\triangle_{AB}T = \mu_2 T$; $T \in B(H)$. Since

$$(\triangle_{AB} - \mu_1)(T) = (\triangle_{AB} - \mu_2)(T) + (\mu_2 - \mu_1)T,$$

$$T \in (\triangle_{AB} - \mu_1)(B(H)) \Longrightarrow (\triangle_{AB} - \mu_1)^{-1}(0) \perp (\triangle_{AB} - \mu_2)^{-1}(0).$$

Reversing the roles of μ_1 and μ_2, we have $(\triangle_{AB} - \mu_2)^{-1}(0) \perp (\triangle_{AB} - \mu_1)^{-1}(0)$, and the proof is complete. □

References

[1] P. Aiena and O. Monsalve, *The single valued extension property and the generalized Kato decomposition property*, Acta Sci. Math. (Szeged) **67** (2001), no. 3-4, 791–807.

[2] J. Anderson and C. Foias, *Properties which normal operators share with normal derivations and related operators*, Pacific J. Math. **61** (1975), no. 2, 313–325.

[3] F. F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, Lond. Math. Soc. Lecture Notes Series 2, 1971.

[4] ———, *Numerical ranges II*, Lond. Math. Soc. Lecture Notes Series 10, Cambridge University Press, New York-London, 1973.

[5] B. P. Duggal and R. E. Harte, *Range-kernel orthogonality and range closure of an elementary operator*, Monatsch. Math. **143** (2004), no. 3, 179–187.

[6] N. Dunford and J. T. Schwartz, *Linear Operators, Part I*, A Wiley-Interscience Publication, New York, 1968.

[7] M. R. Embry and M. Rosenblum, *Spectra, tensor products, and linear operator equations*, Pacific J. Math. **53** (1974), 95–107.

[8] H. G. Heuser, *Functional Analysis*, A Wiley-Interscience Publication, 1982.

[9] K. B. Laursen and M. M. Neumann, *An Introduction to Local Spectral Theory*, Lond. Math. Soc. Monographs (N. S.) 20, Oxford Univ. Press, 2000.

[10] M. Mbekhta, *Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux*, Glasgow Math. J. **29** (1987), no. 2, 159–175.

[11] A. M. Sinclair, *Eigenvalues in the boundary of the numerical range*, Pacific J. Math. **35** (1970), 231–234.

[12] A. Turnšek, *Orthogonality in C_p classes*, Monatsh. Math. **132** (2001), no. 4, 349–354.

JOSE ANGEL CANAVATI, CIMAT, APDO. POSTAL 402, GUANAJUATO, GTO, 36000, MEXICO
E-mail: canavati@cimat.mx

Slavisa V. Djordjević, Facultad de Ciencias Físico-Matemáticas, BUAP, Apdo. Postal 1152, Puebla, Pue. 72000, Mexico
E-mail: slavdj@fcfm.buap.mx

Bhagwati Prasad Duggal, 8 Redwood Grove, Northfield Avenue, London W15 4SZ, England, U.K.
E-mail: bpduggal@yahoo.co.uk