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THE CRITICAL POINT EQUATION ON A FOUR DIMENSIONAL WARPED PRODUCT MANIFOLD

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ABSTRACT. On a compact oriented n-dimensional manifold (M^n, g), it has been conjectured that a metric g satisfying the critical point equation (2) should be Einstein. In this paper, we prove that if a manifold (M^4, g) is a 4-dimensional oriented compact warped product, then g can not be a solution of CPE with a non-zero solution function f.

1. Introduction

Let M be an n-dimensional compact orientable manifold. Denoting the space of all smooth Riemannian metrics on M by $\mathcal{R}M$, let $\mathcal{M}$ be the quotient of $\mathcal{R}M$ by the group of all diffeomorphisms of M. Then, for a given smooth structure $g \in \mathcal{M}$, its scalar curvature s_g is an element of the space of $C^\infty(M)$ functions, and the linearization of the scalar curvature is given by

$$s_g'(h) = -\Delta_g trh + \delta_g^*\delta_g h - g(h, r_g),$$

where Δ_g is the negative Laplacian of g, r_g its Ricci tensor, δ the divergence operator, and δ^* is the formal adjoint of δ. Also, the L^2-adjoint operator $s_g'^*$ of s_g' is given by

$$s_g'^*(f) = -g\Delta_g f + D_g df - fr_g \tag{1}$$

and the critical point equation, *denoted CPE hereafter*, is given by ([1])

$$z_g = s_g'^*(f), \tag{2}$$

where z_g is the traceless Ricci tensor, and f a function on M^n with vanishing mean value.

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For the case that the function f in CPE is trivial, it is clear that the metric g is Einstein. Therefore, all considerations in this paper are restricted to *a non-trivial function f only.* For the case that f is a smooth non-trivial function, the following statement has been conjectured ([1]):

CONJECTURE A. *If CPE holds for a non-trivial function f, then the metric g of the manifold M is Einstein.*

If this conjecture is true, it is known that (M, g) is isometric to a standard sphere S^n [10]. It turns out to be difficult to solve Conjecture A even with additional assumptions imposed on the metric. However, there are some partial answers to Conjecture A, such as those in [4] and [9] for example.

The purpose of the present paper is to prove the following Main Theorem as a partial (negative) answer to Conjecture A:

MAIN THEOREM. *Let (M, g) be a 4-dimensional oriented compact warped product given by $(M, g) = (B, \check{g}) \times_{\psi^2} (F, \hat{g})$ with $\psi > 0$. Then g can not be a solution of CPE with a non-zero solution function f.*

REMARK 1. Fisher and Marsden suggested the F-M conjecture in [3], stating that if a smooth function f satisfies $s_g'^*(f) = 0$, then a solution metric g is isometric to the standard sphere. For the relationships between Conjecture A and F-M conjecture may be found in [5]. However, counter-examples of F-M conjecture were found (e.g., see [8]), mostly warped product manifolds. Therefore, it naturally arises a question to ask whether there exists a warped product metric which constitutes a counter-example of Conjecture A.

Contrary to the case of F-M conjecture, our Main Theorem shows that no 4-dimensional warped products satisfy CPE unless they are isometric to standard spheres. Combining this result with that of 3-dimensional case in [6], we may conclude that the answer to the question in the previous paragraph is no for $n \leq 4$.

It is natural to ask a similar question for $n \geq 5$. However, the difficulty for answering the question increases greatly, and is beyond our current understanding of the problem.

REMARK 2. For a 4-dimensional warped product manifold $(M, g) = (B, \check{g}) \times_{\psi^2} (F, \hat{g})$ with $\psi > 0$, it is clear that B is complete if M is complete. If we consider the case when the 1-dimensional base space B is incomplete with $\psi \geq 0$ and the fiber F is Einstein, we may conclude that if g is a solution metric of (3) with a non-zero solution function f, then (M,g) is isometric to the standard sphere S^4(Proposition 11). It will be shown in Section 5.

In the present paper, our Main Theorem will be proved as follows. For a 4-dimensional warped product manifold $(M,g) = (B,\check{g}) \times_{\psi^2} (F,\hat{g})$, only the following 3 cases are possible with respect to the dimension of B and F:

Case 1. $\dim B = 1$, $\dim F = 3$
Case 2. $\dim B = 2$, $\dim F = 2$
Case 3. $\dim B = 3$, $\dim F = 1$.

For the case that B is complete, we prove a non-existence theorem (Main Theorem) in Section 3. For the case that B is not complete with $\dim B = 1$, a rigidity result is shown in Section 5.

2. Preliminaries

This section is a brief collection of notations and results, which are needed in our subsequent considerations.

Among the partial answers to the conjecture A given in section 1, the following three theorems, which hold on an n-dimensional manifold (M,g), are needed in the next sections.

THEOREM 1. *Let g be a solution metric of CPE. If g is conformally flat, then (M,g) is isometric to a standard sphere* [9].

THEOREM 2. *Let (g,f) be a solution of CPE. If f always takes values greater than or equal to -1, then (M,g) is isometric to a standard sphere* [4].

THEOREM 3. *Let g be a solution metric of CPE. If the metric g is Einstein, (M,g) is isometric to a standard sphere S^n* [10].

Furthermore, using (1) and (2) we have another useful representation of CPE on an n-dimensional manifold (M,g), which may be written as

$$(1+f)z_g = D_g df + \frac{s_g f}{n(n-1)} g. \tag{3}$$

Taking the trace of (3), we have $\Delta_g f = -\frac{s_g}{n-1} f$. Note that the scalar curvature s_g of the metric g satisfying CPE is assumed to be constant ([1]). Therefore we have $\int_M f = 0$, and hence f takes both positive and negative values.

Finally, on a 4-dimensional compact warped product manifold $(M, g) = (B, \check{g}) \times_{\psi^2} (F, \hat{g})$, let

$\check{r}, \check{s}$: the lifts to M of Ricci and scalar curvature of B, respectively
$\hat{r}, \hat{s}$: the Ricci and scalar curvature of F, respectively
X_i : a lifted horizontal orthonormal frame field, $i = 1, \ldots, \dim B$
U_j : a lifted vertical orthonormal frame field, $j = 1, \ldots, \dim F$
$\mathcal{V}$: the vertical distribution.

Then, the following two propositions hold on M for each of 3 cases mentioned in the last paragraph of section 1:

PROPOSITION 4. *We have* ([1])
Case 1. dim $B = 1$ *and* $\dim F = 3$;

$$r(X,X) = -\frac{3\psi''}{\psi}$$
$$r(U_i,U_j) = \hat{r}(U_i,U_j) + \langle U_i,U_j\rangle \left(-\frac{\psi''}{\psi} - 2\frac{\psi'^2}{\psi^2}\right)$$
$$s = \frac{\hat{s}}{\psi^2} - 6\left(\frac{\psi''}{\psi} + \frac{\psi'^2}{\psi^2}\right)$$

Case 2. dim $B = 2$ *and* $\dim F = 2$;

$$r(X_i,X_j) = \check{r}(\check{X}_i,\check{X}_j) - \frac{2}{\psi}\check{D}d\psi(\check{X}_i,\check{X}_j)$$
$$r(U_i,U_j) = \hat{r}(U_i,U_j) + \langle U_i,U_j\rangle \left(-\frac{\check{\Delta}\psi}{\psi} - \frac{|d\psi|^2}{\psi^2}\right)$$
$$s = \check{s} + \frac{\hat{s}}{\psi^2} - 4\frac{\check{\Delta}\psi}{\psi} - 2\frac{|d\psi|^2}{\psi^2}$$

Case 3. dim $B = 3$ *and* $\dim F = 1$;

$$r(X_i,X_j) = \check{r}(\check{X}_i,\check{X}_j) - \frac{1}{\psi}\check{D}d\psi(\check{X}_i,\check{X}_j)$$
$$r(U,U) = -\frac{\check{\Delta}\psi}{\psi}$$
$$s = \check{s} - 2\frac{\check{\Delta}\psi}{\psi}.$$

The next corollary follows from Proposition 4.

COROLLARY 5. *If the scalar curvature s of M is constant, then the scalar curvature $\hat{s}$ of F is constant.*

Proof. Proposition 4 gives $\hat{s} = \psi^2 s + 6(\psi''\psi + \psi'^2)$ in the Case 1. Since $\hat{s}$ is a function on F, and the right-hand side of this equation is a function on B, $\hat{s}$ should be a constant function on F, i.e., F is of constant scalar curvature. In the Case 2, this corollary follows similarly, since we have $\hat{s} = \psi^2(s - \check{s}) + 4\psi\check{\Delta}\psi + 2|d\psi|^2$. Finally, we have $\hat{s} = 0$ in the Case 3. □

3. Proof of main theorem

This section is devoted to the proof of our Main Theorem. Throughout this section we assume that B is complete with $\psi > 0$. The proof of Main Theorem follows directly from the following lemmas. For the Case 1 ($\dim B = 1$), Lemma 6 and 7 give the proof. Lemma 8 and 9 give the proof of the Case 2 ($\dim B = 2$). For the remaining Case 3 ($\dim B = 3$), the proof follows from Lemma 10.

LEMMA 6. *Let $(M, g) = (B, \check{g}) \times_{\psi^2} (F, \hat{g})$ with the 1-dimensional complete base B. Suppose that f is a function of B only. Then g can not be a solution of* (3).

Proof. First note that

$$(4) \qquad \langle D_{U_i} df, U_i \rangle = \langle D_{U_i} \check{df}, U_i \rangle = \frac{\psi' f'}{\psi},$$

where we used the fact that $D_{U_i} \check{df} = \frac{\psi' \partial_t f}{\psi} U_i$ [1]. Now, from (3) we have

$$(1 + f) \left(\hat{r}(U_i, U_i) - \frac{\psi''}{\psi} - 2\frac{\psi'^2}{\psi^2} - \frac{s}{4} \right) = \frac{\psi'}{\psi} f' + \frac{sf}{12}.$$

Then it is easy to see that $\hat{r}(U_1, U_1) = \hat{r}(U_2, U_2) = \hat{r}(U_3, U_3)$, or $\hat{r}(\widehat{U}_i, \widehat{U}_i) = \frac{\hat{s}}{3}$ for all $1 \leq i \leq 3$, and $\hat{r}(U_i, U_j) = 0$ for any $i \neq j$. Thus F is an Einstein manifold. Since F is 3-dimensional, it follows that F should be of constant sectional curvature. In other words, F is isometric to S^3/Γ, with $\Gamma \subset SO(4)$. Hence we have $M = S^1 \times_{\psi^2} (S^3/\Gamma)$ with the metric g given by $g = dt^2 + \psi(t)^2 g_0$. This metric g is conformally flat, since g_0 is of constant curvature, c.f. [8]. Then, by Theorem 1, (M, g) is isometric to a standard sphere S^4. This is clearly a contradiction, since there is no nonvanishing function ψ with $S^4 = S^1 \times_{\psi^2} (S^3/\Gamma)$. □

LEMMA 7. *Let $(M, g) = (B, \check{g}) \times_{\psi^2} (F, \hat{g})$ with the 1-dimensional complete base B. Then g can not be a solution of* (3).

Proof. From (3), we have

$$(5) \qquad 0 = (1 + f) z(X, U_i) = \langle D_X df, U_i \rangle = X \langle df, U_i \rangle - \langle df, D_X U_i \rangle$$

for a lifted horizontal vector field X. Note that $U_i = \frac{1}{\psi} \widehat{U}_i$, where $\widehat{U}_i$ is a lift of vector field on F. Therefore

$$(6) \qquad D_X U_i = D_X \left(\frac{1}{\psi} \widehat{U}_i \right) = -\frac{X(\psi)}{\psi^2} \widehat{U}_i + \frac{1}{\psi} D_X \widehat{U}_i = 0,$$

where we used the fact that $D_X\widehat{U}_i = \frac{X(\psi)}{\psi}\widehat{U}_i$ [11] in the last equality. Now, substitution of (6) into (5) gives

$$XU_i(f) = X\langle df, U_i\rangle = 0. \tag{7}$$

Therefore $U_i(f) = \frac{1}{\psi}\widehat{U}_i(f)$ is a function of F only, and it is easy to see that f can be written as

$$f = \psi b + c, \tag{8}$$

where b is a function on F and $c = c(t)$ is a function on B. Substituting (8) into $(1+f)z(X,X) = \langle D_X df, X\rangle + \frac{sf}{12}$, we have

$$(1+\psi b + c)(-3\frac{\psi''}{\psi} - \frac{s}{4}) = \psi'' b + c'' + \frac{s}{12}(\psi b + c), \tag{9}$$

where we used the fact that, from $\check{df} = bd\psi + dc$ and $D_X\widehat{df} = \mathcal{V}D_X\widehat{df}$,

$$\begin{aligned}\langle D_X df, X\rangle &= \langle D_X \check{df}, X\rangle + \langle D_X\widehat{df}, X\rangle = \langle D_X\check{df}, X\rangle \\ &= b\langle D_X d\psi, X\rangle + \langle D_X dc, X\rangle = \psi'' b + c''.\end{aligned}$$

Thus, the equation (9) can be rewritten as

$$b(-4\psi'' - \frac{s}{3}\psi) = (1+c)(\frac{3\psi''}{\psi} + \frac{s}{4}) + c'' + \frac{s}{12}c. \tag{10}$$

Note that both $-4\psi'' - \frac{s}{3}\psi$ and the right-hand side are functions of B only, while b is a function of F. Thus, in order that the equation (10) holds for any t, either b is constant or $-4\psi'' - \frac{s}{3}\psi$ has to be zero. If b is constant, f is a function of B only, and so g can not be a solution of (3) in virtue of Lemma 6. Now, we may assume that

$$\psi'' + \frac{s}{12}\psi = 0. \tag{11}$$

Since B is complete, ψ has to be defined on the whole of R. Moreover, since $B = S^1$, ψ has to be periodic. Therefore we may conclude that ψ is zero somewhere on B; if $\psi \neq 0$,

$$0 = \int_B \left(\frac{\psi'}{\psi}\right)' = \int_B \frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2} = -\int_B \left(\frac{s}{12} + \frac{\psi'^2}{\psi^2}\right) < 0$$

which is a contradiction. Hence the given warped product metric g can not be a solution of (3). □

LEMMA 8. *Let $(M,g) = (B,\check{g}) \times_{\psi^2} (F,\hat{g})$ with the 2-dimensional complete base B. Suppose that f is a function of B only. Then g can not be a solution of* (3).

A proof of Lemma 8 need another section and will be presented in the Section 4.

LEMMA 9. *Let $(M, g) = (B, \check{g}) \times_{\psi^2} (F, \hat{g})$ with the 2-dimensional complete base B. Then g can not be a solution of (3). In other words, Case 2 does not occur.*

Proof. In the beginning, the proof of Lemma 9 may go along the same line as the proof of Lemma 7, concluding that

$$f = \psi b + c, \tag{12}$$

where b is a function on F and c is a function on B. By Proposition 4 and $(1+f)\sum_{i=1}^{2} z(X_i, X_i) = \sum_{i=1}^{2}\langle D_{X_i} df, X_i\rangle + \frac{sf}{6}$, we have

$$(1+f)(\check{s} - \frac{2}{\psi}\check{\Delta}\psi - \frac{s}{2}) = \sum_{i=1}^{2}\langle D_{X_i} df, X_i\rangle + \frac{sf}{6}. \tag{13}$$

Now, substitution (12) into (13) gives

$$(1 + \psi b + c)(\check{s} - \frac{2}{\psi}\check{\Delta}\psi - \frac{s}{2}) = b\check{\Delta}\psi + \check{\Delta}c + \frac{s}{6}(\psi b + c), \tag{14}$$

where we used the fact that

$$\begin{aligned}\sum_{i=1}^{2}\langle D_{X_i} df, X_i\rangle &= \sum_{i=1}^{2}(\langle D_{X_i} \check{df}, X_i\rangle + \langle D_{X_i} \hat{df}, X_i\rangle) = \sum_{i=1}^{2}\langle D_{X_i} \check{df}, X_i\rangle \\ &= b\sum_{i=1}^{2}\langle D_{X_i} d\psi, X_i\rangle + \sum_{i=1}^{2}\langle D_{X_i} dc, X_i\rangle = b\check{\Delta}\psi + \check{\Delta}c\end{aligned}$$

with $D_{X_i}\hat{df} = \mathcal{V} D_{X_i}\hat{df}$. It is easy to see that (14) can be rewritten as

$$b(\psi\check{s} - 3\check{\Delta}\psi - \frac{2s}{3}\psi) = \check{\Delta}c - (c+1)(\check{s} - \frac{2}{\psi}\check{\Delta}\psi - \frac{s}{2}) + \frac{s}{6}c. \tag{15}$$

Note that both $\psi\check{s} - 3\check{\Delta}\psi - \frac{2s}{3}\psi$ and the right-hand side are functions of B only, while b is a function of F only. Thus, in order that the equation (15) holds, either b is constant or $\psi\check{s} - 3\check{\Delta}\psi - \frac{2s}{3}\psi$ has to be zero. If b is constant, f is a function of B only, and so g can not be a solution of (3) in virtue of Lemma 8. Thus we may assume that

$$\check{s} = \frac{3\check{\Delta}\psi}{\psi} + \frac{2}{3}s \tag{16}$$

and

$$\check{\Delta}c - (c+1)(\check{s} - \frac{2}{\psi}\check{\Delta}\psi - \frac{s}{2}) + \frac{s}{6}c = 0. \tag{17}$$

Substitution of (16) into (17) gives

$$\psi\check{\Delta}c-(c+1)\check{\Delta}\psi-\frac{s}{6}\psi=0. \tag{18}$$

Now, integrating both sides of (18) over B, we have

$$\frac{s}{6}\int_B\psi=0,$$

where we used the fact that $\int_B\psi\check{\Delta}c=\int_B c\check{\Delta}\psi$. This equation together with the condition that $\psi>0$ leads to a contradiction. □

LEMMA 10. *Let $(M,g)=(B,\check{g})\times_{\psi^2}(F,\hat{g})$ with the 3-dimensional complete base B. Then g can not be a solution of (3). In other words, Case 3 does not occur.*

Proof. Suppose that the given warped product metric is a solution of CPE. From the definition of the Laplacian Δ, we have

$$\langle D_U df,U\rangle=\Delta f-\check{\Delta}f=-\frac{s}{3}f-\check{\Delta}f \tag{19}$$

and from CPE we also have

$$(1+f)\left(-\frac{\check{\Delta}\psi}{\psi}-\frac{s}{4}\right)=\langle D_U df,U\rangle+\frac{s}{12}f. \tag{20}$$

Hence, the following relation holds by combining (19) and (20):

$$(1+f)\left(-\frac{\check{\Delta}\psi}{\psi}-\frac{s}{4}\right)=-\check{\Delta}f-\frac{s}{4}f$$

which is equivalent to

$$(1+f)\check{\Delta}\psi+\frac{s}{4}\psi=\psi\check{\Delta}f. \tag{21}$$

Now, integrating both sides of (21) over B, we have

$$\int_B f\check{\Delta}\psi+\frac{s}{4}\int_B\psi=\int_B\psi\check{\Delta}f=\int_B f\check{\Delta}\psi$$

which implies that

$$\int_B\psi=0.$$

This equation together with the condition that $\psi>0$ leads to a contradiction. □

4. Proof of Lemma 8

This section is devoted to the proof of Lemma 8. Suppose that g is a solution of (3), namely CPE, and f is a function of B only. In the following two contentions, we shall prove that this assumption leads to a contradiction. We first consider the following two cases according to the values of f. In the first case that f always takes values greater than or equal to -1, the proof of the Lemma is completed since (M, g) is isometric to S^4 in virtue of Theorem 2. In the second case that f takes a value less than -1, this Lemma is proved in a series of two contentions under present conditions[1]. In this case, there exists a non-empty set $H = \{x \in B |\ f(x) = -1\}$. After investigating the analytic properties of the tensor $D_X X$ for a tangent vector field X to H in Contention 1, we prove in Contention 2 that the present conditions give a contradiction. Since this contradiction is obtained from the assumption that g is a solution of CPE, we may conclude that g can not be a solution of CPE, proving our Lemma.

CONTENTION 1. Under present conditions, we have $D_X X = -\frac{s}{12} df$ on H, where $H = \{x \in B |\ f(x) = -1\}$, $X \in TB$ is a tangent vector field to H, and $N \in TB$ is a normal vector field to H.

Proof. We may assume that H is non-empty; otherwise g is Einstein by Theorem 2. Also, we note in [4] that a point of H, which is a critical point of f, is a non-degenerate local minimum point of f, and that such non-degenerate critical points are isolated. Therefore, H is a set consisting of finite critical points of f, or hypersurfaces of M, or union of both.

Putting $W = |df|^2$, it was proved in [4] that W is constant in each component of H and does not vanish on H. Therefore, in a small tubular neighborhood of H, we may take orthonormal frame fields $\{X, N\}$, where $N = \frac{df}{W^{1/2}}$ on H. From (3), we have

$$Ddf = \frac{s}{12} g \tag{22}$$

on H. Then, it follows that on H we have

$$\frac{s}{12} = \langle D_X df, X\rangle = X\langle df, X\rangle - \langle df, D_X X\rangle = -\langle df, D_X X\rangle.$$

[1] In the proof of Contention 1 and 2, we assume that the metric g of warped product $(M, g) = (B, \check{g}) \times_{\psi^2} (F, \hat{g})$ is a solution of (3). We also suppose that $\dim B = 2$ and f takes a value less than -1. Hereafter, this situation will be described by the words "*present conditions*".

Hence, $D_X X = -\frac{s}{12}\frac{1}{W^{\frac{1}{2}}}N$ on H. □

CONTENTION 2. Under present conditions, we have a contradiction on H.

Proof. In virtue of Proposition 4, (3) may be reduced to

$$(1+f)(\check{s} - \frac{2}{\psi}\check{\Delta}\psi - \frac{s}{2}) = \check{\Delta}f + \frac{sf}{6} \tag{23}$$

$$(1+f)\left(\frac{\hat{s}}{\psi^2} - 2\frac{\check{\Delta}\psi}{\psi} - 2\frac{|d\psi|^2}{\psi^2} - \frac{s}{2}\right) = 2\frac{\langle d\psi, df\rangle}{\psi} + \frac{sf}{6} \tag{24}$$

since

$$\sum_{i=1}^{2} z(X_i, X_i) = \check{s} - \frac{2}{\psi}\check{\Delta}\psi - \frac{s}{2} \tag{25}$$

$$\sum_{i=1}^{2} z(U_i, U_i) = \frac{\hat{s}}{\psi^2} - 2\frac{\check{\Delta}\psi}{\psi} - 2\frac{|d\psi|^2}{\psi^2} - \frac{s}{2} \tag{26}$$

and $\check{g}(d\psi, df) = \langle d\psi, df\rangle$. Hence, using (23) and (24) we have

$$\check{\Delta}f = \frac{s}{6} \tag{27}$$

$$\frac{\langle d\psi, df\rangle}{\psi} = \frac{s}{12} \tag{28}$$

on H. The relation $\sum_{i=1}^{2} z(X_i, X_i) + \sum_{i=1}^{2} z(U_i, U_i) = 0$ gives

$$(1+f)(-\check{s} + \frac{2}{\psi}\check{\Delta}\psi + \frac{s}{2}) = 2\frac{\langle d\psi, df\rangle}{\psi} + \frac{sf}{6}. \tag{29}$$

Taking the Lie derivative of (29) with respect to df on H, we have

$$\begin{aligned} & W\left(-\check{s} + \frac{2}{\psi}\check{\Delta}\psi + \frac{s}{2}\right) \\ &= 2\frac{\langle D_{df}d\psi, df\rangle}{\psi} + 2\frac{\langle d\psi, D_{df}df\rangle}{\psi} - 2\frac{\langle d\psi, df\rangle^2}{\psi^2} + \frac{s}{6}W \\ &= 2\frac{\langle D_N d\psi, N\rangle}{\psi}W + \frac{s}{6}W, \end{aligned}$$

where

$$2\frac{\langle d\psi, D_{df}df\rangle}{\psi} = \frac{s}{6\psi}\langle df, d\psi\rangle = \frac{s^2}{72} = 2\frac{\langle d\psi, df\rangle^2}{\psi^2}$$

in virtue of (22) and (28). Therefore

$$-\check{s}+\frac{2}{\psi}\langle D_X d\psi, X\rangle+\frac{s}{3}=0, \tag{30}$$

where we used the fact that

$$\check{\Delta}\psi=\langle D_X d\psi, X\rangle+\langle D_N d\psi, N\rangle. \tag{31}$$

On the other hand, in order to calculate $z(X,X)$ we take the Lie derivative of (3) with respect to df on H. Then

$$\begin{aligned} Wz(X,X) &= \langle D_{df}D_X df, X\rangle+\langle D_X df, D_{df}X\rangle+\frac{s}{12}W \\ &= W^{\frac{1}{2}}\langle D_N D_X df, X\rangle+|D_X df|^2+\frac{s}{12}W \\ &= W^{\frac{1}{2}}\langle D_N D_X df, X\rangle+\frac{s^2}{144}+\frac{s}{12}W \end{aligned} \tag{32}$$

on H, where we used the fact that W is constant on H, $D_{df}X = W^{-\frac{1}{2}}D_N X = W^{-\frac{1}{2}}D_X N = D_X df$, and $D_X df = \frac{s}{12}X$ on H. However, the relation $D_N D_X df = D_X D_N df + R(X,N)df$ gives

$$\begin{aligned} \langle D_N D_X df, X\rangle &= \langle D_X D_N df, X\rangle+\langle R(X,N)df, X\rangle \\ &= X\langle D_N df, X\rangle-\langle D_N df, D_X X\rangle-W^{\frac{1}{2}}K(X,N) \\ &= \frac{s}{12W^{\frac{1}{2}}}\langle D_N df, N\rangle-W^{\frac{1}{2}}\check{r}(N,N) \\ &= \frac{s^2}{144W^{\frac{1}{2}}}-W^{\frac{1}{2}}\check{r}(N,N), \end{aligned} \tag{33}$$

where we used the fact that $\langle D_N df, X\rangle=0$, $D_X X=-\frac{s}{12}\frac{1}{W^{1/2}}N$ on H, (22), and $\check{r}(N,N)=K(X,N)$. Now substituting (33) into (32), we have

$$z(X,X)=\frac{s^2}{72W}-\check{r}(N,N)+\frac{s}{12}. \tag{34}$$

Note that from Proposition 4 we have

$$z(X,X)=r(X,X)-\frac{s}{4}=\check{r}(X,X)-2\frac{\langle D_X d\psi, X\rangle}{\psi}-\frac{s}{4}. \tag{35}$$

Substituting (34) into (35) on H we have

$$\check{s}-2\frac{\langle D_X d\psi, X\rangle}{\psi}-\frac{s}{3}-\frac{s^2}{72W}=0. \tag{36}$$

The equation (36) clearly contradicts the equation (30). This contradiction comes from the assumption that g is a solution of CPE. Hence, g can not be a solution of CPE, proving our Lemma 8. □

5. A rigidity result

This section is devoted to the proof of the following rigidity result.

PROPOSITION 11. *Let M be a compact oriented 4-dimensional manifold containing an open dense subset U which is (for the induced metric) a warped product on a 1-dimensional basis with Einstein fibre F. If g is a solution metric of* (3), *then (M, g) is isometric to the standard sphere S^4.*

We take U as large as possible. If the base B is complete, then M is a warped product and Lemma 7 applies. If not, B is an open interval $(0, a)$ with $U = (0, a) \times F$. It is easy to see that M is the quotient of $[0, a] \times F$. Note that the following relation holds in order for M to be a complete manifold:

$$\psi(0) = \psi(a) = 0. \tag{37}$$

Note also that the following relation holds in order for M to be smooth (c.f. [7]):

$$\psi'(0) = -\psi'(a) = \sqrt{\frac{\hat{s}}{6}}. \tag{38}$$

Thus (37) and (38) become the initial conditions for the warping function ψ. On the other hand, in virtue of Proposition 4, ψ satisfies

$$6\psi''\psi + 6\psi'^2 + \psi^2 s - \hat{s} = 0 \tag{39}$$

with constants s and $\hat{s}$. For a solution ψ of (39), we observe that $\psi''(0) = 0$, since the differentiation of (39) gives $3\psi'''\psi + 9\psi''\psi' + s\psi\psi' = 0$. It is easy to see that $\psi_0 = \sqrt{\frac{2\hat{s}}{s}} \sin \sqrt{\frac{s}{12}} t$ with $a = \pi\sqrt{\frac{s}{12}}$ is a solution of the ordinary differential equation (39) with the initial conditions (37) and (38). The following lemma shows that the solution of (39) is unique.

LEMMA 12. *ψ_0 is the unique solution of* (39) *under the conditions* (37) *and* (38). *Therefore, $(M, g) = [0, a] \times_{\psi_0^2} S^3$.*

Proof. Let ψ be another solution of (39) with the initial conditions (37) and (38), and let $F = \frac{\psi}{\psi_0}$. It is easy to see that F is well-defined on $[0, a]$ with $F(0) = 1$, since $\psi_0(0) = \psi(0) = 0$ and $\psi_0'(0) = \psi'(0) = \sqrt{\frac{\hat{s}}{6}}$. Now, in order to prove our Lemma, we claim that $F \equiv 1$, implying that $\psi = \psi_0$. First we observe that $F'(0) = 0$ since $\psi'' = \psi_0''F + 2\psi_0'F' + \psi_0 F''$

and $\psi_0''(a) = \psi''(0) = 0$. Now, if we substitute $\psi = \psi_0 F$ into (39), F satisfies

$$F'' = -\frac{1}{\psi_0^2 F}\left(4\psi_0\psi_0' F F' + \psi_0^2 F'^2 + \frac{1}{6}\hat{s}(F^2 - 1)\right), \tag{40}$$

where we used the fact that ψ_0 satisfies (39). Define a function $\xi : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ by

$$\xi(t, y_1, y_2) = -\frac{1}{\psi_0^2 y_1}\left(4\psi_0\psi_0' y_1 y_2 + \psi_0^2 F'^2 + \frac{1}{6}\hat{s}(F^2 - 1)\right),$$

where ξ is obtained simply by replacing F and F' in (40) by y_1 and y_2 respectively. In order to prove the uniqueness of solutions of (40) satisfying $F(0) = 1$ and $F'(0) = 0$, it suffices to show that $\xi(t, y_1, y_2)$ is continuous with respect to t and is Lipschitz with respect to y_1 and y_2 in the rectangle $R = [0, a] \times [r, R_1] \times [-R_2, R_2]$ for all $0 < r < 1 < R_1$ and $R_2 > 0$. First, it is easy to see that ξ is continuous for t by letting $\xi(0, y_1(0), y_2(0)) = 0$. Secondly, ξ is Lipschitz, since ξ is smooth with respect to y_1, y_2 in R, and $|\xi(t, y_1(t), y_2(t)) - \xi(t, \bar{y}_1(t), \bar{y}_2(t))| < M(|y_1(t) - \bar{y}_1(t)| + |y_2(t) - \bar{y}_2(t)|)$ for some $M > 0$. Therefore, the ordinary differential equation (40) has the unique solution $F \equiv 1$, proving our claim. □

It is well known (see [1], Corollary 9.107) that the warped product $M = B \times_{\psi^2} F$ is Einstein if and only if $\hat{g}, \breve{g}, \psi$ satisfy that $(F, \hat{g})$ is Einstein, and

$$-\frac{\psi''}{\psi} - 3\frac{\psi'^2}{\psi^2} + \frac{\hat{s}}{3\psi^2} = \frac{s}{4} \tag{41}$$

and

$$-3\frac{\psi''}{\psi} = \frac{s}{4}. \tag{42}$$

It is an easy exercise for $\psi = \psi_0$ to satisfies (41), (42). Also note that F is Einstein from the assumption. Therefore we may conclude that the warped product U(or M) is Einstein. However, in virtue of Theorem 3, M must be isometric to a standard sphere.

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