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MULTIPLICITY RESULTS FOR A CLASS OF SECOND ORDER SUPERLINEAR DIFFERENCE SYSTEMS

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Abstract. Using Minimax principle and Linking theorem in critical point theory, we prove the existence of two nontrivial solutions for the following second order superlinear difference systems

$$(P)\begin{cases} \Delta^2 x(k-1)+g(k,y(k))=0, & k\in[1,T],\\ \Delta^2 y(k-1)+f(k,x(k))=0, & k\in[1,T],\\ x(0)=y(0)=0, x(T+1)=y(T+1)=0, \end{cases}$$

where T is a positive integer, [1,T] is the discrete interval $\{1,2,\ldots,T\}$, $\Delta x(k)=x(k+1)-x(k)$ is the forward difference operator and $\triangle^2 x(k)=\triangle(\triangle x(k))$.

1. Introduction

In this paper, we will investigate the following discrete boundary value problems

$$(P)\begin{cases} \Delta^2 x(k-1)+g(k,y(k))=0, & k\in[1,T],\\ \Delta^2 y(k-1)+f(k,x(k))=0, & k\in[1,T],\\ x(0)=y(0)=0, x(T+1)=y(T+1)=0, \end{cases}$$

where T is a positive integer, [1,T] is the discrete interval $\{1,2,\ldots,T\}$, $\Delta x(k)=x(k+1)-x(k)$ is the forward difference operator and $\triangle^2 x(k)=\triangle(\triangle x(k))$. Recently, difference problems via variational methods have been widely studied by various authors. Guo and Yu [4] prove the existence and multiplicity of periodic and subharmonic solutions for the second order superlinear difference equation

$$\Delta^2 y(k-1)+f(k,y(k))=0, \tag{1.1}$$

where $f\in C(R\times R,R)$ and $f(k+m,z)=f(k,z),\forall(k,z)\in Z\times R$. Agarwal, Perera, and O' Regan [2] discussed the discrete boundary value

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problem

$$\begin{cases} \Delta^2 y(k-1) + g(k, y(k)) = 0, k \in [1, T], \\ y(0) = y(T+1) = 0. \end{cases} \tag{1.2}$$

Thanks to Minimax principle and Mountain Pass Lemma in critical point theory, they proved the existence of two nonnegative solutions and at least one of which is positive.The aim of this paper is to generalize the single equation (1.2) to a case of the system (P) and prove the existence of two nontrivial solutions for the system (P). In this situation, some interesting features appear due to the coupling in the system (P). Here we assume the following conditions.

$(H1)$: $f, g \in C([1, T] \times R, R)$.

$(H2)$: $\lim_{z\to 0} \dfrac{f(k,z)}{z} = \lim_{z\to 0} \dfrac{g(k,z)}{z} = 0$, uniformly with respect to $k \in [1, T]$.

$(H3)$: There exist $R > 0, \beta > 2$, such that

$$0 < \beta F(k, z) \le z f(k, z), 0 < \beta G(k, z) \le z g(k, z), \forall |z| \ge R,$$

where $F(k, z) = \displaystyle\int_0^z f(k, s) ds, G(k, z) = \int_0^z g(k, s) ds.$

THEOREM 1.1. *If conditions $(H1)$, $(H2)$ and $(H3)$ hold, the system (P) possess two nontrivial solutions.*

REMARK 1.2. By assumption $(H3)$, there exist $c_1, c_2 > 0$ such that

$$F(k, z), G(k, z) \ge c_1 z^\beta - c_2. \tag{1.3}$$

Combining $(H3)$ and (1.3), we obtain

$$z f(k, z) \ge \beta F(k, z) \ge \beta c_1 z^\beta - \beta c_2, z g(k, z) \ge \beta G(k, z) \ge \beta c_1 z^\beta - \beta c_2.$$

So we have

$$\lim_{z\to\infty} \frac{f(k,z)}{z} = \lim_{z\to\infty} \frac{g(k,z)}{z} = \infty.$$

That is $f(k, z), g(k, z)$ are superlinear at ∞.

REMARK 1.3. By assumption $(H1)$ and $(H3)$, we can obtain $f(k, 0) = g(k, 0) = 0$ and $(x(k), y(k)) = (0, 0)$ is a trivial solution of the system (P).

2. Preliminaries

Let the class H of functions $x : [0, T+1] \to R$ such that $x(0) = x(T+1) = 0$ is a T-dimensional linear space with inner product $\langle x, \varphi \rangle = \sum_{k=1}^{T} x(k)\varphi(k), \forall x, \varphi \in H$, and the corresponding norm by

$$\|x\|_H = \left(\sum_{k=1}^{T} x^2(k) \right)^{\frac{1}{2}}, \forall x \in H.$$

It is easy to obtain $(H, \langle , \rangle)$ is a T-dimensional Hilbert space and homomorphism with R^T. Consider E is Cartesin product of $H \times H$ and the natural inner product on E is given by

$$\langle (x,y), (\varphi, \psi) \rangle_E = \sum_{k=1}^{T} (x(k)\varphi(k) + y(k)\psi(k)), \forall \varphi, \psi \in H.$$

The norm of an element $(x,y) \in E$ is defined by $\|(x,y)\|_E = (\|x\|_H^2 + \|y\|_H^2)^{\frac{1}{2}}$. Define the functional $I : E \to R$ by

$$I(x,y) = \sum_{k=1}^{T+1} (\Delta x(k-1)\Delta y(k-1)) - \sum_{k=1}^{T} (F(k, x(k)) - G(k, y(k))). \tag{2.1}$$

By the assumption $(H1)$, the functional I is well defined and $I \in C^1(E, R)$ with

$$\begin{aligned} \langle I'(x,y), (\varphi, \psi) \rangle &= \sum_{k=1}^{T+1} \big(\Delta x(k-1)\Delta \psi(k-1) \\ &\quad + \Delta y(k-1)\Delta \varphi(k-1) \big) \\ &\quad - \sum_{k=1}^{T} (f(k, x(k))\varphi(k) + (g(k, y(k))\psi(k)) \\ &= - \sum_{k=1}^{T+1} (\Delta^2 x(k-1)\psi(k-1) \\ &\quad + \Delta^2 y(k-1)\varphi(k-1)) \\ &\quad - \sum_{k=1}^{T} (f(k, x(k))\varphi(k) + (g(k, y(k))\psi(k)). \end{aligned} \tag{2.2}$$

So solutions of the system (P) are precisely the critical points of the functional I.

DEFINITION 2.1. A functional $I \in C^1(E,R)$ is said to satisfy (PS) condition, if there exists $c_3 > 0$, every sequence $(x_n, y_n) \in E$ such that

$$I(x_n, y_n) \leq c_3, I'(x_n, y_n) \to 0 \text{ as } n \to \infty, \tag{2.3}$$

has a strongly convergent subsequence in E.

LEMMA 2.2. *Let $u_j, v_j \geq 0, j = 1, 2, \dots, T, p > 1, q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\sum_{j=1}^{T} u_j v_j \leq \left(\sum_{j=1}^{T} u_j^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{T} v_j^q\right)^{\frac{1}{q}}.$$

From Lemma 2.2, we can define another norm in H,

$$\|x\|_p = \left(\sum_{k=1}^{T} |x(k)|^p\right)^{\frac{1}{p}}, \forall x \in H,$$

and it is easy to obtain that there exist two positive constants $c_4, c_5 > 0$ such that

$$c_4\|x\|_p \leq \|x\|_H \leq c_5\|x\|_p. \tag{2.4}$$

LEMMA 2.3. (Minimax principle, Chang [3]) *Suppose $I \in C^1(E,R)$ is bounded above and satisfies (PS) condition, then $\sup_{u \in E} I(u) = c_0$ is attained at the critical point of I.*

Let $E^+ = \{(x,x) : x \in H\}$ and $E^- = \{(x,-x) : x \in H\}$. Then there exists direct decomposition $E = E^+ \oplus E^-$, where E^+, E^- are finite dimensional space.

LEMMA 2.4. (Linking theorem, Rabinowitz [5]) *Let $\rho > r > 0$ and let $e_0 \in E^+$ be such that $\|e_0\|_E = r$. Define*

$$M = \{u = w + \lambda e_0 : \|u\|_E \leq \rho, \lambda \geq 0, w \in E^-\},$$

$$\partial M = \{u = w + \lambda e_0 : z \in E^-, \|u\|_E = \rho, \lambda \geq 0, \text{or } \|u\|_E < \rho, \lambda = 0\},$$

$$N = \{u \in E^+ : \|u\|_E = r\}.$$

Let $I \in C^1(E,R)$ be such that $\inf_N I > \max_{\partial M} I$. If I satisfies (PS) condition with

$$c = \inf_{y \in \Gamma} \max_{u \in \partial M} I(y(u)), \Gamma = \{y \in C(M,E) : \gamma|_{\partial M} = id\}.$$

Then c is a critical value of I.

3. Proof of Theorem 1.1

In this section, we will prove the main theorem by using Lemma 2.3 and Lemma 2.4.

Lemma 3.1. *If $f(k,z), g(k,z)$ satisfy assumption $(H3)$, then the functional $I(x,y)$ is bounded above in E.*

Proof. By (1.3), we have

$$\begin{aligned} I(x,y) &= \sum_{k=1}^{T+1}(\Delta x(k-1)\Delta y(k-1)) - \sum_{k=1}^{T}(F(k,x(k)) + G(k,x(k))) \\ &= \sum_{k=1}^{T+1}(x(k)-x(k-1))(y(k)-y(k-1)) \\ &\quad - \sum_{k=1}^{T}(F(k,x(k)) + G(k,x(k))) \\ &\le \frac{1}{2}\sum_{k=1}^{T+1}[(x(k)-x(k-1))^2 + (y(k)-y(k-1))^2] \\ &\quad - c_1\sum_{k=1}^{T}(|x(k)|^\beta + |y(k)|^\beta) + c_2T \\ &\le \sum_{k=1}^{T+1}[(x^2(k)+x^2(k-1)) + (y^2(k)+y^2(k-1))] \\ &\quad - c_1\sum_{k=1}^{T}(|x(k)|^\beta + |y(k)|^\beta) + c_2T \\ &\le 2\|(x,y)\|_E^2 - c_6\|(x,y)\|_E^\beta + c_7, \end{aligned}$$

where c_6, c_7 are constants. From $\beta > 2$, there exists $K > 0$ such that

$$I(x,y) \le K, \forall x,y \in E.$$

□

Lemma 3.2. *If $(H1)$ and $(H3)$ hold, the functional I satisfies (PS) condition.*

Proof. Let $\{x,y\} \subset E$ such that

$$I(x_n,y_n) \le c_3, I'(x_n,y_n) \to 0,\ n \to \infty.$$

From $I'(x_n, y_n) \to 0$, we have

$$\begin{aligned}\langle I'(x_n, y_n), (x_n, y_n)\rangle &= 2\sum_{k=1}^{T+1}(\Delta x_n(k-1)\Delta y_n(k-1)) \\ &\quad -\sum_{k=1}^{T}(f(k, x_n(k))x_n(k) + g(k, y_n(k))y_n(k)) \\ &\leq \varepsilon_n\|(x_n, y_n)\|_E.\end{aligned}$$

On the other hand

$$I(x_n, y_n) = \sum_{k=1}^{T+1}(\Delta x_n(k-1)\Delta y_n(k-1)) - \sum_{k=1}^{T}(F(k, x_n(k)) + G(k, y_n(k))).$$

Taking $(H1)$ and $(H3)$ into account, we get

$$\begin{aligned}\varepsilon_n\|(x_n, y_n)\|_E + \beta c_3 &\geq \beta I(x_n, y_n) - \langle I'(x_n, y_n), (x_n, y_n)\rangle \\ &= (\beta-2)\sum_{k=1}^{T+1}(\Delta x_n(k-1)\Delta y_n(k-1)) \\ &\quad +\sum_{k=1}^{T}(f(k, x_n(k))x_n(k) + g(k, y_n(k))y_n(k)) \\ &\quad -\beta\sum_{k=1}^{T}(F(k, x_n(k)) + G(k, y_n(k))) \\ &\geq (\beta-2)\sum_{k=1}^{T+1}(\Delta x_n(k-1)\Delta y_n(k-1)) + \delta,\end{aligned}$$

where δ is a constant. By the equation (1.3) in [2], we can define another norm in H

$$\langle x, \varphi\rangle = \sum_{k=1}^{T+1}(\Delta x(k-1)\Delta\varphi(k-1)), \forall x, \varphi \in H.$$

Since H is a finite dimensional Hilbert space and all norms on finite dimensional space are equivalent, there exists $c_8 > 0$ such that

$$\sum_{k=1}^{T+1}(\Delta x_n(k-1)\Delta y_n(k-1)) > c_8\|(x, y)\|_E^2.$$

So $(x_n, y_n) \subset E$ is a bounded sequence. In a standard way, there exists a strong converge subsequence. □

LEMMA 3.3. *There exist $r > 0$ and $\alpha > 0$ such that $\inf_N I \geq \alpha > 0$.*

Proof. By $(H2)$, for given $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$F(k,z), G(k,z) \leq c_\varepsilon |z|^2, \forall |z| < r. \tag{3.1}$$

For any $u \in N$, there is $x \in H$ with $u = (x, x)$. From (3.1), we have

$$\begin{aligned} I(x,x) &= \sum_{k=1}^{T+1} (\Delta x(k-1))^2 - \sum_{k=1}^{T} (F(k, x(k)) + G(k, x(k))) \\ &\geq \frac{1}{2} x^T A x - 2c_\varepsilon \sum_{k=1}^{T} x^2(k), \end{aligned}$$

where $x = (x(1), x(2), \ldots, x(T))$ and

$$A = \left\{ \begin{array}{cccccc} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ 2 & -1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{array} \right\}_{T \times T}$$

It is obvious that A is a positive-definite symmetric matrix. Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_T$ denote the eigenvalue of the matrix A and $c_\varepsilon = \frac{1}{8}\lambda_1$. Then we obtain

$$I(x,x) \geq \frac{1}{2}\lambda_1 \|x\|_H^2 - 2\frac{1}{8}\lambda_1 \|x\|_H^2 = \frac{1}{4}\lambda_1 \|x\|_H^2.$$

Let $\alpha = \frac{1}{4}\lambda_1 r^2$. Then

$$I(u) = I(x,x) \geq \alpha > 0, \forall u = (x,x) \in N.$$

□

LEMMA 3.4. *There exist positive constant ρ, r and $\rho > r$ such that $\max_{\partial M} I = 0$.*

Proof. If $u \in \partial M$, then $u = w + \lambda e_0$ with either $\|u\|_E = \rho, \lambda \geq 0$ or $\|u\|_E < \rho, \lambda = 0$. Suppose $\lambda = 0$. Then we have $u \in E^-, u = (x, -x)$ and

$$I(x,-x) = \sum_{k=1}^{T+1} (\Delta x(k-1))^2 - \sum_{k=1}^{T} (F(k, x(k)) + G(k, x(-k))) \leq 0,$$

because $F(k,x(k))+G(k,x(-k))\geq 0$ by assumption $(H2)$. Let $e_0=(e,e)$. If $u=(x,-x)+\lambda(e,e)\in\partial M$ with $\|u\|_E<\rho$, then

$$\begin{aligned} I(u) &= \lambda^2\sum_{k=1}^{T+1}(\Delta e(k-1))^2-\sum_{k=1}^{T+1}(\Delta x(k-1))^2 \\ &\quad -\sum_{k=1}^{T}(F(k,\lambda e(k)+x(k))+G(k,\lambda e(k)-x(k))). \end{aligned}$$

By (1.3), there exist $c_1,c_2>0$ such that

$$F(k,\lambda e(k)+x(k))\geq c_1(\lambda e(k)+x(k))^\beta-c_2, \tag{3.2}$$

$$G(k,\lambda e(k)-x(k))\geq c_1(\lambda e(k)-x(k))^\beta-c_2. \tag{3.3}$$

From (2.4), (3.2) and (3.3), we obtain

$$\begin{aligned} I(u) &\leq \lambda^2r^2-c_1\sum_{k=1}^{T}\left[(\lambda e(k)+x(k))^\beta+(\lambda e(k)-x(k))^\beta\right]+2c_2T \\ &\leq \lambda^2r^2-c_1\left(\frac{1}{c_5}\right)^\beta\left[(\sum_{k=1}^{T}(\lambda e(k)+x(k))^2)^{\beta/2}\right. \\ &\quad \left.-c_1\left(\sum_{k=1}^{T}(\lambda e(k)-x(k))^2\right)^{\beta/2}\right]+2c_2T \\ &= \lambda^2r^2-2c_1\left(\frac{1}{c_5}\right)^\beta\left(\sum_{k=1}^{T}(\lambda^2e^2(k)+x^2(k))\right)^{\beta/2}+2c_2T \\ &\leq \lambda^2r^2-2c_1\left(\frac{1}{c_5}\right)^\beta(\lambda^2r^2+\|x(k)\|_H^2)^{\beta/2}+2c_2T \\ &\leq \lambda^2r^2-2c_1\left(\frac{1}{c_5}\right)^\beta(\lambda^2r^2+\|x(k)\|_H^\beta)+2c_2T \end{aligned}$$

Finally taking $\rho>\lambda>0$ and $\beta>2$, we have $\max_{\partial M}I=0$. From Lemma 3.3 and Lemma 3.4, the geometry of the Linking theorem is satisfied. □

The proof of theorem 1.1. In view of Lemma 3.1 and lemma 3.2, there exist $\overline{x},\overline{y}\in E$ such that $I(\overline{x},\overline{y})=c_0=\sup_{(x,y)\in E}I(x,y)$ is a critical value of the functional I by using Lemma 2.3 (Minimax principle). From Lemma 3.2, Lemma 3.3, Lemma 3.4, we can apply the Linking theorem (Lemma 2.4), it follows that the functional I has a critical value $c\geq$

$\alpha > 0$ and, hence there exist $(\widetilde{x}, \widetilde{y}) \in E$ and $I(\widetilde{x}, \widetilde{y}) = c$. Suppose $(\widetilde{x}, \widetilde{y}) \neq (\overline{x}, \overline{y})$, theorem 1.1 is valid. Suppose not. Then we have $c = I(\widetilde{x}, \widetilde{y}) = I(\overline{x}, \overline{y}) = c_0$, that is, $\sup_{(x,y)\in E} I(x,y) = \inf_{y\in\Gamma} \max_{u\in\partial M} I(\gamma(u))$. Choosing $\gamma = id$, we obtain $\sup_{u\in\partial M} I(u) = c_0$. Let $-e_0 \in E^+$. Then, similar to Lemma 3.4, there exist $\rho' > r > 0$ such that $I(u)|_{u\in\partial M'} \leq 0$, where $M' = \{u = w + \lambda(-e_0) : \|u\|_E \leq \rho', \lambda \geq 0, w \in E^-\}$. Using linking theorem again, there exists critical value $c^* = \inf_{y\in\Gamma'} \max_{u\in\partial M} I(\gamma(u))$, where $\Gamma' = \{\gamma' \in C(M', E) : \gamma'|_{\partial M} = id\}$. If $c^* \neq c_0$, theorem 1.1 is completed. If not, we have $c^* = c_0$ and $\sup_{u\in M} I(u) = c_0$. From $I(u)|_{u\in\partial M'} \leq 0$ and $I(u)|_{u\in\partial M} \leq 0$, the functional I attains maximal value in the interior of the set M and M', on the other hand, $(M \cap M') \subset E^-$ and $I(u) \leq 0, \forall u \in E^-$. These imply that there exists $(\widehat{x}, \widehat{y}) \in E, (\widehat{x}, \widehat{y}) \neq (\widetilde{x}, \widetilde{y})$ and $I(\widehat{x}, \widehat{y}) = c^* = c_0$. So the system (P) have two nontrivial solutions. □

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