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# STRONG COMMUTATIVITY PRESERVING MAPPINGS ON SEMIPRIME RINGS

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Abstract. Let $R$ be a semiprime ring and $f$ be an endomorphism on $R$. If $f$ is a strong commutativity preserving (simply, scp) map on a non-zero ideal $U$ of $R$, then $f$ is commuting on $U$.

A ring $R$ is said to be prime if $aRb = 0$ implies that either $a = 0$ or $b = 0$, and semiprime if $aRa = 0$ implies that $a = 0$ where $a$, $b \in R$. A prime ring is obviously semiprime. If $R$ is a ring and $S \subseteq R$, a mapping $f : R \to R$ is called strong commutativity preserving (simply, scp) on $S$ if $[x, y] = [f(x), f(y)]$ for all $x, y \in S$; and commuting on $S$ if $[f(x), x] = 0$ for all $x \in S$. For recent references on the commutativity in prime and semiprime rings, see [1] and [3]; and for scp maps see [2] and [4].

To prove the main result we need the following lemma which is of independent interest and can be used for further investigation.

Lemma 1. *If $R$ is a semiprime ring and $f$ is an endomorphism on $R$ which is scp on a non-zero right ideal $U$, then for all $x \in U$, $f(x) - x$ commutes with $[U, U]$.*

*Proof.* For all $x, y \in U$, we have $[x, xy] = [f(x), f(xy)]$. This implies that $x[x, y] = f(x)[x, y]$ and so

$$(f(x) - x)[x, y] = 0.$$

From $[x, yx] = [f(x), f(yx)]$ we can similarly show

$$[x, y](f(x) - x) = 0.$$

For all $x, y \in R$, replacing $y$ by $yr$, we get

$$(f(x) - x)y[x, r] = 0.$$

This implies that $(f(x) - x)U[x, r] = 0$ and so

$$(f(x) - x)UR[x, r] = 0.$$

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Since $R$ is semiprime, it must contain a family $\omega = \{P_\alpha | \alpha \in \lambda\}$ of prime ideals such that $\bigcap P_\alpha = 0$. If $P$ is a typical member of $\omega$ and $x \in U$, then from the last equation, we have $(f(x) - x)U \subseteq P$ or $[x, R] \subseteq P$. Suppose $\exists\ y \in U$ such that $[y, R] \not\subseteq P$. This implies that

$$(f(y) - y)U \subseteq P.$$

Let $z$ be any element of $U$ such that $[y + z, R] \subseteq P$. This means that $[z, R] \not\subseteq P$ and hence $(f(z) - z)U \subseteq P$. On the other hand if $[y + z, R] \not\subseteq P$, then $(f(y + z) - (y + z))U \subseteq P$. This implies that $(f(z) - z)U \subseteq P$. Thus we conclude that $(f(z) - z)U \subseteq P$ for all $z \in U$ and hence $(f(z) - z)[U, U] \subseteq P$ for all $z \in U$.

Since $P$ is arbitrary and $\bigcap P_\alpha = 0$, we have $(f(z) - z)[U, U] = \{0\}$ for all $z \in U$. Similarly we can show that

$$[U, U](f(z) - z) = \{0\}.$$

This implies that $(f(z) - z) \in C_R[U, U]$ for all $z \in U$. □

Now we can easily prove the following result:

THEOREM 1. *Let $R$ be a semiprime ring and $f$ be an endomorphism on $R$. If $f$ is scp on a non zero ideal $U$ of $R$, then $f$ is commuting on $U$.*

*Proof.* By above lemma and lemma 1 of [5], we have $(f(x) - x) \in C_R(U)$, for all $x \in U$. Thus we have $[f(x) - x, x] = 0$ for all $x \in U$. This implies that $[f(x), x] = 0$ for all $x \in U$ and hence $f$ is commuting on $U$. □

The following are two useful Corollaries of the preceding theorem.

COROLLARY 1. *Let $R$ be a semiprime ring and $f$ be an endomorphism on $R$. If $f$ is scp on $R$, then $f$ is commuting on $R$.*

COROLLARY 2. *Let $R$ be a prime ring and $f$ be an endomorphism on $R$. If $f$ is scp on $R$, then $R$ is a commutative integral domain.*

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