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DISCUSSION ON THE ANALYTIC SOLUTIONS OF THE SECOND-ORDER ITERATED DIFFERENTIAL EQUATION

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ABSTRACT. This paper is concerned with a second-order iterated differential equation of the form $c_0x''(z)+c_1x'(z)+c_2x(z) = x(az+bx(z)) + h(z)$ with the distinctive feature that the argument of the unknown function depends on the state. By constructing a convergent power series solution of an auxiliary equation, analytic solutions of the original equation are obtained.

1. Introduction

Eder [1] studies the equation $x'(z) = x(x(z))$ and analytic solutions are shown to exist by means of the Banach fixed point theorem. Further discussion is made in [3, 6] for existence of analytic solutions of equations $x'(z) = x(x^{[m]}(z))$ and $x''(z) = \sum_{j=0}^{m} p_j x^{[j]}(z)$, where $x^{[j]}(z)$ denote jth iterate of a complex function $x(z)$, i.e., $x^{[j]}(z) = x(x^{[j-1]}(z)), x^{[0]}(z) = z$, $p_j (j = 0, 1, \ldots, m)$ are all complex constant numbers.

A class of iterative differential equations above have quite a different form ordinary differential equations and iteration of the unknown function affects properties of solutions very much. The known theorems of existence and uniqueness for ordinary differential equations are not applicable.

In this paper, we will be concerned with a general class equation of the form

$$c_0x''(z) + c_1x'(z) + c_2x(z) = x(az + bx(z)) + h(z), \quad z \in \mathbf{C} \tag{1.1}$$

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with the initial condition $x(0) = 0, x'(0) = \alpha \neq 0$, where $x(z)$ denotes the unknown complex function, h is a given complex function, a, b and $c_i (i = 0, 1, 2)$ are all complex constant numbers.

Equation (1.1) includes a lot of important subjects as its special cases. In case $c_0 = c_2 = 0, c_1 = 1$ and $h(z) \equiv 0$, (1.1) changes into the equation $x'(z) = x(az + bx(z))$ (see [7]). In case $c_1 = c_2 = 0, c_0 = 1$ and $h(z) \equiv 0$, (1.1) into the equation $x''(z) = x(az + bx(z))$ (see [8]), when $a = 0$ and $b = 1$, such equation reduces to the second-order iterative differential equation $x''(z) = x(x(z))$ (see [5]). In case $c_0 = c_1 = c_2 = 0$, (1.1) changes into the iterative functional equation $x(az + bx(z)) + h(z) = 0$, in [4], a more general equation of the form $x(p(z) + bx(z)) = h(z)$ has been considered and existence of analytic solution is established.

In view of the general form of (1.1), it is expected that additional conditions are needed to guarantee the existence of nontrivial analytic solutions. The basic conditions that h is analytic in a neighborhood of the origin, $h(0) = 0$, $h'(0) = r \neq 0$, and $abc_0 \neq 0$ will be assumed throughout the rest of our discussions.

As in [4], we still reduce (1.1) with

$$x(z) = \frac{1}{b}[y(\beta y^{-1}(z)) - az], \tag{1.2}$$

called the generalized Schröder transformation sometimes, to the auxiliary equation

$$\begin{aligned} & c_0\left[\beta^2 y''(\beta z)y'(z) - \beta y''(z)y'(\beta z)\right] \\ & + c_1\left[\beta y'(\beta z) - ay'(z)\right](y'(z))^2 + c_2\left[y(\beta z) - ay(z)\right](y'(z))^3 \\ = & \left[y(\beta^2 z) - ay(\beta z)\right](y'(z))^3 + bh(y(z))(y'(z))^3, \end{aligned} \tag{1.3}$$

with the initial condition $y(0) = 0, y'(0) = \eta$, where y is the unknown complex function, $\beta \neq 0$ and $\beta = a + b\alpha$.

In view of (1.3), we have

$$\begin{aligned} & c_0\left[\frac{\beta y''(\beta z)y'(z) - y''(z)y'(\beta z)}{(y'(z))^2}\right] \\ & + \frac{c_1}{\beta}\left[\beta y'(\beta z) - ay'(z)\right] + \frac{c_2}{\beta}\left[y(\beta z) - ay(z)\right]y'(z) \\ = & \frac{1}{\beta}\left[y(\beta^2 z) - ay(\beta z)\right]y'(z) + \frac{b}{\beta}h(y(z))y'(z) \end{aligned}$$

or

$$c_0\left(\frac{y'(\beta z)}{y'(z)}\right)' = \frac{1}{\beta}\Big[y(\beta^2 z)y'(z) - (a+c_2)y(\beta z)y'(z) + ac_2 y(z)y'(z) + ac_1 y'(z) - \beta c_1 y'(\beta z) + bh(y(z))y'(z)\Big].$$

When $y'(0) = \eta \neq 0$, (1.3) is reduced equivalently to the integro-differential equation

(1.4)
$$c_0 y'(\beta z) = c_0 y'(z) + \frac{1}{\beta}y'(z)\int_0^z \Big[y(\beta^2 s)y'(s) - (a+c_2)y(\beta s)y'(s) + ac_2 y(s)y'(s) + ac_1 y'(s) - \beta c_1 y'(\beta s) + bh(y(s))y'(s)\Big]ds.$$

Firstly, we construct analytic solutions of (1.3) in the cases:

(C1) β is not on the unit circle in $\mathbf{C}$;

(C2) β lies on the unit circle in $\mathbf{C}$ but not a root of unity;

(C3) β is a unit root.

Finally, results on the auxiliary equation (1.3), we will prove (1.1) has analytic solutions in a neighborhood of the origin, and give its construction.

2. Analytic solution of the auxiliary equation in case (C1)

In this section, we discuss (1.3) in the case $|\beta| < 1$.

THEOREM 2.1. *Assume that* $0 < |\beta| < 1$. *Then for any* $\eta \in \mathbf{C}$, *the auxiliary equation* (1.3) *has an analytic solution* $y(z)$ *in a neighborhood of the origin such that* $y(0) = 0, y'(0) = \eta$.

Proof. Clearly, (1.3) has a trivial solution $y(z) \equiv c \in \mathbf{C}$, if $\eta = 0$. Assume $\eta \neq 0$. Under our assumption on h, let

$$h(z) = \sum_{n=1}^{\infty} h_n z^n, \quad h_1 = r. \tag{2.1}$$

We seek a solution of (1.3) in a power series of the form

$$y(z) = \sum_{n=1}^{\infty} y_n z^n, \quad y_1 = \eta \neq 0. \tag{2.2}$$

Introducing new functions (see [4])

$$H(z) = \rho h(\rho^{-1}z), Y(z) = \rho y(\rho^{-1}z),$$

we obtain from $h(0) = 0, y(0) = 0$ and $y'(0) = \eta \neq 0$ that $H(0) = 0, Y(0) = 0$ and $Y'(0) = y'(0) = \eta \neq 0$ respectively, and from (1.3) that

$$\begin{aligned} & C_0[\beta^2 Y''(\beta z)Y'(z) - \beta Y''(z)Y'(\beta z)] \\ & + C_1[\beta Y'(\beta z) - aY'(z)](Y'(z))^2 \\ & + c_2[Y(\beta z) - aY(z)](Y'(z))^3 \\ = & [Y(\beta^2 z) - aY(\beta z)][Y'(z)]^3 + bH(Y(z))(Y'(z))^3, \end{aligned}$$

where $C_0 = c_0\rho^2, C_1 = c_1\rho$, which is again an equation of the form (1.3). Here H is of the form

$$H(z) = \sum_{n=1}^{\infty} H_n z^n,$$

but $|H_n| = |h_n\rho^{1-n}| \leq 1$, for $n = 2, 3, \ldots$. So without loss of generality, we may assume that

$$|h_n| \leq 1, n = 2, 3, \ldots. \tag{2.3}$$

Inserting (2.1) and (2.2) into (1.4), we get
(2.4)

$$\begin{aligned} & c_0 \sum_{n=0}^{\infty} (n+1) y_{n+1} \beta^{n+1} z^n \\ = & c_0\beta \sum_{n=0}^{\infty} (n+1) y_{n+1} z^n \\ & + \sum_{n=2}^{\infty} \left[\sum_{k=1}^{n-1} \sum_{i=1}^{n-k} \frac{k(n-k+1-i)}{n-k+1} \beta^{2i} y_k y_i y_{n-k+1-i} \right] z^n \\ & - (a + c_2) \sum_{n=2}^{\infty} \left[\sum_{k=1}^{n-1} \sum_{i=1}^{n-k} \frac{k(n-k+1-i)}{n-k+1} \beta^{i} y_k y_i y_{n-k+1-i} \right] z^n \\ & + ac_2 \sum_{n=2}^{\infty} \left[\sum_{k=1}^{n-1} \sum_{i=1}^{n-k} \frac{k(n-k+1-i)}{n-k+1} y_k y_i y_{n-k+1-i} \right] z^n \\ & + ac_1 \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} k y_k y_{n-k+1} \right) z^n - c_1\beta \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \beta^{n-k} k y_k y_{n-k+1} \right) z^n \end{aligned}$$

$$+b\sum_{n=2}^{\infty}\left[\sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\frac{k(n-k+1-i)}{n-k+1}\right.$$
$$\left.\times\left(\sum_{\substack{L(m)=i\\ m=1,2,\dots,i}} h_m y_{l_1}\cdots y_{l_m}\right) y_k y_{n-k+1-i}\right] z^n,$$

where $L(m)=l_1+l_2+\cdots+l_m$, $l_i (i=1,2,\dots,m)$ are all positive integers. For the sake of convenience, we will set $L(m)=l_1+l_2+\cdots+l_m$ in latter discussions.

Comparing coefficients, we obtain

$$c_0y_1=c_0y_1, \quad \text{for} \quad n=0 \tag{2.5}$$

and

$$2c_0\beta(\beta-1)y_2=c_1(a-\beta)y_1^2, \quad \text{for} \quad n=1. \tag{2.6}$$

Generally, we have

(2.7)

$$\begin{aligned}
&(n+1)\beta c_0(\beta^n-1)y_{n+1}\\
&=\sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\frac{k(n-k+1-i)}{n-k+1}\beta^{2i}y_ky_iy_{n-k+1-i}\\
&\quad-(a+c_2)\sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\frac{k(n-k+1-i)}{n-k+1}\beta^{i}y_ky_iy_{n-k+1-i}\\
&\quad+ac_2\sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\frac{k(n-k+1-i)}{n-k+1}y_ky_iy_{n-k+1-i}\\
&\quad+ac_1\sum_{k=1}^{n}ky_ky_{n+1-k}-\beta c_1\sum_{k=1}^{n}\beta^{n-k}ky_ky_{n+1-k}\\
&\quad+b\sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\frac{k(n-k+1-i)}{n-k+1}\left(\sum_{\substack{L(m)=i\\ m=1,2,\dots,i}} h_m y_{l_1}\cdots y_{l_m}\right)y_ky_{n-k+1-i},\\
&\quad\text{for}\quad n=2,3,\dots.
\end{aligned}$$

In view of (2.5), then for arbitrary chosen $y_1=\eta\neq 0$, from (2.6) we can get y_2, so the sequence $\{y_n\}_{n=3}^{\infty}$ is successively determined by (2.7) in a unique manner. This implies that for (1.3), there exists a formal power series solution (2.2).

We need only to show that the resulting series (2.2) is converges in a neighborhood of the origin. First of all, note that

$$\lim_{n\to\infty} \frac{1}{\beta c_0(\beta^n - 1)} = -\frac{1}{\beta c_0}, (0 < |\beta| < 1).$$

Thus, there is some positive number M, such that

$$\left| \frac{1}{\beta c_0(\beta^n - 1)} \right| \le M, \quad \forall n \ge 1.$$

Let $N = \max\{|a + c_2| + |ac_2| + 1, |ac_1| + |c_1|, |b|\}$, and from (2.7), we have

$$\begin{aligned} |y_{n+1}| \le MN \Bigg[&\sum_{k=1}^{n-1} \sum_{i=1}^{n-k} |y_k||y_i||y_{n-k+1-i}| \\ &+ \sum_{k=1}^{n} |y_k||y_{n-k+1}| \\ &+ \sum_{k=1}^{n-1} \sum_{i=1}^{n-k} \left(\sum_{\substack{L(m)=i \\ m=1,2,\ldots,i}} |y_{l_1}| \cdots |y_{l_m}| \right) |y_k||y_{n-k+1-i}| \Bigg] \end{aligned}$$

for $n = 2, 3, \ldots$.

If we define a power series $P(z) = \sum_{n=1}^{\infty} p_n z^n$ by

$$p_1 = |\eta|,\ p_2 = \frac{|c_1||a - \beta||\eta|^2}{2|c_0||\beta||\beta - 1|}$$

and

$$\begin{aligned} p_{n+1} = MN \Bigg[&\sum_{k=1}^{n-1} \sum_{i=1}^{n-k} p_k p_i p_{n-k+1-i} \\ &+ \sum_{k=1}^{n} p_k p_{n-k+1} \\ &+ \sum_{k=1}^{n-1} \sum_{i=1}^{n-k} \left(\sum_{\substack{L(m)=i \\ m=1,2,\ldots,i}} p_{l_1} \cdots p_{l_m} \right) p_k p_{n-k+1-i} \Bigg], n = 2, 3, \ldots, \end{aligned}$$

then it is easily seen that

$$|y_n| \le p_n,\ n = 1, 2, \ldots.$$

In other words,the series $P(z) = \sum_{n=1}^{\infty} p_n z^n$ is a majorant series of (2.2). Next, we show that $P(z)$ has a positive radius of convergence. Indeed, note that by formal calculation, we have

$$\begin{aligned}
P(z) &= |\eta|z + p_2 z^2 + \sum_{n=2}^{\infty} p_{n+1} z^{n+1} \\
&= |\eta|z - p_2 z^2 + MN\Bigg[\sum_{n=2}^{\infty}\Big(\sum_{k=1}^{n-1}\sum_{i=1}^{n-k} p_k p_i p_{n-k+1-i}\Big) z^{n+1} \\
&\quad + \sum_{n=2}^{\infty}\big(\sum_{k=1}^{n} p_k p_{n-k+1}\big) z^{n+1} \\
&\quad + \sum_{n=2}^{\infty}\Big(\sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\Big(\sum_{\substack{L(m)=i \\ m=1,2,\ldots,i}} p_{l_1}\cdots p_{l_m}\Big) p_k p_{n-k+1-i}\Big) z^{n+1}\Bigg] \\
&= |\eta|z - p_2 z^2 + MN\left[(P(z))^3 + (P(z))^2 - |\eta|^2 z^2 + \frac{(P(z))^3}{1-P(z)}\right],
\end{aligned}$$

i.e.,

$$P(z) = |\eta|z + p_2 z^2 + MN\left[(P(z))^3 + (P(z))^2 - |\eta|^2 z^2 + \frac{(P(z))^3}{1-P(z)}\right].$$

Consider now the implicit functional equation

$$\begin{aligned}
F(z,P) &= P(z) - |\eta|z - p_2 z^2 \\
&\quad - MN\left[(P(z))^3 + (P(z))^2 - |\eta|^2 z^2 + \frac{(P(z))^3}{1-P(z)}\right] \\
&= 0.
\end{aligned}$$

Since F is analytic on a disk with the origin as the center and $F(0,0) = 0, F'_P(0,0) = 1 \neq 0$, by the (analytic) implicit theorem [see 2, p.120], we see that $P = P(z)$ is analytic in a neighborhood of the origin and with a positive radius. This complete the proof. □

3. Analytic solution of the auxiliary equation in case (C2)

In the above result,we assume that $|\beta| < 1$. Next, we deal with the case (C2), i.e., β is on the unit circle, i.e., $|\beta| = 1$, but not a root of unity. We will need a preparatory result in Siegel [9].

Lemma 3.1. *Assume that* $|\alpha| = 1$, *α is not a root of unity, and* $\log|\alpha^n - 1|^{-1} \leq T\log n, n = 2,3,\ldots$ *for some positive constant T. Then* *there is a positive number* δ *such that* $|\alpha^n - 1|^{-1} < (2n)^\delta$ *for* $n = 1, 2, \ldots$. *Furthermore, the sequence* $\{d_n\}_{n=1}^\infty$ *defined by* $d_1 = 1$ *and*

$$d_n = \frac{1}{|\alpha^{n-1} - 1|} \max_{\substack{L(m)=n \\ 0<l_1\le\cdots\le l_m, m\ge 2}} \{d_{l_1} \cdots d_{l_m}\}, \quad n = 2, 3, \ldots,$$

will satisfy

$$d_n \le (2^{5\delta+1})^{n-1} n^{-2\delta}, \; n = 1, 2, \ldots.$$

THEOREM 3.1. *Assume that* $(C2)$ *holds. If there exists a constant* $T > 0$, *such that* $\log|\alpha^n - 1|^{-1} \le T \log n, (n = 2, 3, \ldots)$. *Then for any* $\eta \ne 0$, *the auxiliary* (1.3) *has an analytic solution* $y(z)$ *in a neighborhood of the origin, such that* $y(0) = 0$ *and* $y'(0) = \eta$.

Proof. As in the proof of Theorem 2.1, we seek a power series solution of (1.3) of the form (2.2). For chosen $y_1 = \eta$, using the same arguments as above we can determine a unique sequence $\{y_n\}_{n=2}^\infty$ by (2.6) and (2.7) recursively. From (2.3), (2.6), and (2.7), we have

(3.1)

$$|y_2| = |\beta - 1|^{-1} A,$$

$$|y_{n+1}| \le B|\beta^n - 1|^{-1} \Bigg[\sum_{k=1}^{n-1} \sum_{i=1}^{n-k} |y_k||y_i||y_{n-k+1-i}| + \sum_{k=1}^{n} |y_k||y_{n-k+1}| + \sum_{k=1}^{n-1} \sum_{i=1}^{n-k} \Bigg(\sum_{\substack{L(m)=i \\ m=1,2,\ldots,i}} |y_{l_1}| \cdots |y_{l_m}| \Bigg) |y_k||y_{n-k+1-i}| \Bigg],$$

$$n = 2, 3, \ldots,$$

where

$$A = \frac{|c_1||a - \beta||\eta|^2}{2|c_0|}, \quad B = |c_0|^{-1} N,$$

N is defined in the proof of Theorem 2.1.

To construct a power series $U(z) = \sum_{n=1}^\infty u_n z^n$, such that $u_1 = |\eta|$, $u_2 = |\beta - 1|^{-1} A$, and

$$u_{n+1} = B|\beta^n - 1|^{-1} \Bigg[\sum_{k=1}^{n-1} \sum_{i=1}^{n-k} u_k u_i u_{n-k+1-i} + \sum_{k=1}^{n} u_k u_{n-k+1} + \sum_{k=1}^{n-1} \sum_{i=1}^{n-k} \Bigg(\sum_{\substack{L(m)=i \\ m=1,2,\ldots,i}} u_{l_1} \cdots u_{l_m} \Bigg) u_k u_{n-k+1-i} \Bigg], \; n = 2, 3, \ldots.$$

Clearly,we have $|y_n| \le u_n$, $n = 1, 2, \ldots$. In other words, $U(z)$ is a majorant series of (2.2). We now only need to show that $U(z)$ has a positive radius of convergence. To see this, we define a sequence $\{v_n\}_{n=1}^{\infty}$ by $v_1 = |\eta|$, $v_2 = A$, and

$$v_{n+1} = B\Bigg[\sum_{k=1}^{n-1}\sum_{i=1}^{n-k} v_k v_i v_{n-k+1-i} + \sum_{k=1}^{n} v_k v_{n-k+1} + \sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\Bigg(\sum_{\substack{L(m)=i\\ m=1,2,\ldots,i}} v_{l_1}\cdots v_{l_m}\Bigg) v_k v_{n-k+1-i}\Bigg],\ n = 2, 3, \ldots.$$

Obviously, by induction we see that

$$u_n \le v_n d_n,\ n = 1, 2, \ldots, \tag{3.2}$$

where d_n is defined in Lemma 3.1.

Similarly to the proof of Theorem 2.1, it is easy to see that $V(z) = \sum_{n=1}^{\infty} v_n z^n$ satisfies the implicit functional equation

$$P(z, V) = V(z) - |\eta| z - Az^2 - B\big[(V(z))^3 + (V(z))^2 - |\eta|^2 z^2 + \frac{(V(z))^3}{1 - V(z)}\big] = 0.$$

Clearly, P is analytic in a neighborhood of the origin, $P(0,0) = 0$, $P'_V(0,0) = 1 \neq 0$, by the analytic implicit theorem [2, p.120], we see that $V = V(z)$ is analytic in a neighborhood of the origin and with a positive radius. Hence, there exists $R > 0$, such that $v_n \le R^n$ for $n = 1, 2, \ldots$. From (3.2), by Lemma 3.1, we now have

$$|u_n| \le R^n G^{n-1} n^{-2\delta},$$

where $G = 2^{5\delta+1}$, which show that $U(z)$ is convergent in $|z| < (RG)^{-1}$. The proof is complete. □

4. Analytic solution of the auxiliary equation in case (C3)

The following theorem is devoted to the case (C3). In the case, the constant β is not only on the circle, but also a root of unity.

THEOREM 4.1. *Assume that (C3) holds, more precisely, suppose that $\beta^p = 1$ for some $p \ge 2$ and $\beta^k \neq 1$ for all $1 \le k \le p - 1$. Let $\{y_n\}_{n=1}^{\infty}$ be* *determined recursively by*

$$y_1 = \eta \neq 0, \quad y_2 = \frac{c_1(a-\beta)y_1^2}{2c_0\beta(\beta-1)}$$

and

$$(n+1)\beta c_0(\beta^n - 1)y_{n+1} = \Xi(n,\beta),\ n = 2,3,\ldots,$$

where

$$\begin{aligned}\Xi(n,\beta) = {} & \sum_{k=1}^{n-1}\sum_{i=1}^{n-k} \frac{k(n-k+1-i)}{n-k+1}\beta^{2i} y_k y_i y_{n-k+1-i} \\ & - (a+c_2)\sum_{k=1}^{n-1}\sum_{i=1}^{n-k} \frac{k(n-k+1-i)}{n-k+1}\beta^{i} y_k y_i y_{n-k+1-i} \\ & + ac_2\sum_{k=1}^{n-1}\sum_{i=1}^{n-k} \frac{k(n-k+1-i)}{n-k+1} y_k y_i y_{n-k+1-i} \\ & + ac_1\sum_{k=1}^{n} k y_k y_{n+1-k} - \beta c_1 \sum_{k=1}^{n} \beta^{n-k} k y_k y_{n+1-k} \\ & + b\sum_{k=1}^{n-1}\sum_{i=1}^{n-k} \frac{k(n-k+1-i)}{n-k+1} \\ & \times \left(\sum_{\substack{L(m)=i \\ m=1,2,\ldots,i}} h_m y_{l_1}\cdots y_{l_m} \right) y_k y_{n-k+1-i}.\end{aligned}$$

If $\Xi(vp,\beta) = 0$ *for all* $v = 1,2,\ldots$, *then* (1.3) *has an analytic solution* $y(z)$ *in a neighborhood of the origin, such that* $y(0) = 0$, $y'(0) = \eta$.

Proof. If $\Xi(vp,\beta) = 0$ for all $v = 1,2,\ldots$, for each v the corresponding y_{vp+1} in (2.7) has infinitely many choices in $\mathbf{C}$, for the case of convenience, we suppose that $y_{vp+1} = 0, v = 1,2,\ldots$, then (2.2) is the formal solution of (1.3) also. Now we need to prove that the power series (2.2) is convergent. Let

$$\Gamma = \max\left\{ \frac{1}{|\beta-1|}, \frac{1}{|\beta^2-1|}, \ldots, \frac{1}{|\beta^{p-1}-1|} \right\},$$

observe that $|\beta^n - 1|^{-1} \leq \Gamma$ for $n \neq vp$. It follows from (3.1) that

$$|y_{n+1}| \leq \Gamma B \left[\sum_{k=1}^{n-1}\sum_{i=1}^{n-k} |y_k||y_i||y_{n-k+1-i}| \right.$$

$$+\sum_{k=1}^{n}|y_k||y_{n-k+1}|$$
$$+\sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\left(\sum_{\substack{L(m)=i\\m=1,2,\dots,i}}|y_{l_1}|\cdots|y_{l_m}|\right)|y_k||y_{n-k+1-i}|\Bigg],$$

for all $n \neq vp, v = 1, 2, \dots$. In order to construct a majorant series, write $R(z) = \sum_{n=1}^{\infty} D_n z^n$, such that $D_1 = |\eta|$, $D_2 = \Gamma A$ and

$$D_{n+1} = \Gamma B\left[\sum_{k=1}^{n-1}\sum_{i=1}^{n-k} D_k D_i D_{n-k+1-i} + \sum_{k=1}^{n} D_k D_{n-k+1}\right.$$
$$\left.+\sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\left(\sum_{\substack{L(m)=i\\m=1,2,\dots,i}} D_{l_1}\cdots D_{l_m}\right) D_k D_{n-k+1-i}\right], n = 2, 3, \dots.$$

Furthermore, we have $|y_n| \leq D_n, n = 1, 2, \dots$.

In fact, $|y_1| = |\eta| = D_1$, for inductive proof we assume that $|y_j| \leq D_j$, for $j \leq n$. When $n = vp$, we have $|y_{n+1}| = 0 \leq D_{n+1}$; when $n \neq vp$, we have

$$|y_{n+1}| \leq \Gamma B\left[\sum_{k=1}^{n-1}\sum_{i=1}^{n-k}|y_k||y_i||y_{n-k+1-i}|\right.$$
$$+\sum_{k=1}^{n}|y_k||y_{n-k+1}|$$
$$\left.+\sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\left(\sum_{\substack{L(m)=i\\m=1,2,\dots,i}}|y_{l_1}|\cdots|y_{l_m}|\right)|y_k||y_{n-k+1-i}|\right]$$
$$\leq \Gamma B\left[\sum_{k=1}^{n-1}\sum_{i=1}^{n-k} D_k D_i D_{n-k+1-i} + \sum_{k=1}^{n} D_k D_{n-k+1}\right.$$
$$\left.+\sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\left(\sum_{\substack{L(m)=i\\m=1,2,\dots,i}} D_{l_1}\cdots D_{l_m}\right) D_k D_{n-k+1-i}\right]$$
$$= D_{n+1},$$

as required. Similarly to the proof of Theorem 3.1, it is easy to see that $R(z)$ satisfies the implicit equation

$$\begin{aligned}H(z,R) &= R(z) - |\eta|z - \Gamma A z^2 \\ &\quad - \Gamma B\left[(R(z))^3 + (R(z))^2 - |\eta|^2 z^2 + \frac{(R(z))^3}{1-R(z)}\right] = 0.\end{aligned}$$

Clearly, H is analytic in a neighborhood of the origin, $H(0,0) = 0$, $H'_R(0,0) = 1 \neq 0$, by the implicit theorem [2, p.120], we see that $R = R(z)$ is analytic in a neighborhood of the origin and with a positive radius. By convergence of the series of $R(z)$, we see that the series $y(z)$ is convergent in a neighborhood of the origin. The proof is complete. □

5. Analytic solutions for (1.1)

Having knowledges about the auxiliary equation (1.3), we are ready to give analytic solutions of (1.1).

THEOREM 5.1. *Suppose one of the conditions of Theorem 2.1, 3.1, and 4.1 is fulfilled. Then* (1.1) *has an analytic solution of the form*

$$x(z) = \frac{1}{b}[y(\beta y^{-1}(z)) - az]$$

in a neighborhood of the origin such that $x(0) = 0$, $x'(0) = \alpha$, *where* $y(z)$ *is an analytic solution of the auxiliary equation* (1.3).

Proof. By Theorem 2.1, 3.1, and 4.1, we can find an analytic solution $y(z)$ of the auxiliary equation (1.3) in the form of (2.2) such that $y(0) = 0$, $y'(0) = \eta \neq 0$. Clearly, the inverse $y^{-1}(z)$ exists and is analytic in a neighborhood of the origin. From (1.2), it is easy to see that

$$\begin{aligned}x'(z) &= \frac{\beta y'(\beta y^{-1}(z)) - a y'(y^{-1}(z))}{b y'(y^{-1}(z))}, \\ x''(z) &= \frac{\beta^2 y''(\beta y^{-1}(z)) y'(y^{-1}(z)) - \beta y''(y^{-1}(z)) y'(\beta y^{-1}(z))}{b(y'(y^{-1}(z)))^3},\end{aligned}$$

and from (1.3), so we have

$$\begin{aligned}&c_0 x''(z) + c_1 x'(z) + c_2 x(z) \\ &= c_0\left[\frac{\beta^2 y''(\beta y^{-1}(z)) y'(y^{-1}(z)) - \beta y''(y^{-1}(z)) y'(\beta y^{-1}(z))}{b(y'(y^{-1}(z)))^3}\right] \\ &\quad + c_1\left[\frac{\beta y'(\beta y^{-1}(z)) - a y'(y^{-1}(z))}{b y'(y^{-1}(z))}\right] + c_2\frac{1}{b}[y(\beta y^{-1}(z)) - az]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{b[y'(y^{-1}(z))]^3}\{c_0[\beta^2 y''(\beta y^{-1}(z))y'(y^{-1}(z)) \\
&\quad - \beta y''(y^{-1}(z))y'(\beta y^{-1}(z))] \\
&\quad + c_1[\beta y'(\beta y^{-1}(z)) - a y'(y^{-1}(z))][(y'(y^{-1}(z))]^2 \\
&\quad + c_2[y(\beta y^{-1}(z)) - az][(y'(y^{-1}(z))]^3\} \\
&= \frac{1}{b[y'(y^{-1}(z))]^3}\{[y(\beta^2 y^{-1}(z)) - a y(\beta y^{-1}(z))][y'(y^{-1}(z))]^3 \\
&\quad + bh(y(y^{-1}(z)))[y'(y^{-1}(z))]^3\} \\
&= \frac{1}{b}\{[y(\beta^2 y^{-1}(z)) - a y(\beta y^{-1}(z))] + bh(y(y^{-1}(z)))\} \\
&= \frac{1}{b}[y(\beta y^{-1}(y(\beta y^{-1}(z)))) - a y(\beta y^{-1}(z))] + h(z) \\
&= x(az + bx(z)) + h(z).
\end{aligned}
$$

Due to $y(0) = 0$, $y'(0) = \eta \neq 0$, and in view of the above assumption, then

$$x(0) = \frac{1}{b}[y(\beta y^{-1}(0)) - a0] = \frac{1}{b}[y(0) - 0] = 0,$$

and

$$x'(0) = \frac{\beta y'(0) - a y'(0)}{b y'(0)} = \frac{\beta\eta - a\eta}{b\eta} = \frac{\beta - a}{b} = \alpha.$$

The proof is complete. □

We remark that under our assumption, the (1.1) has an analytic solution of nontrivial in a neighborhood of the origin, and once existence guaranteed, it may be possible to expand (1.1) in series form and seek the desired solution instead of first finding a solution of (1.3). In fact, let

$$x(z) = x'(0)z + \frac{x''(0)}{2!}z^2 + \frac{x'''(0)}{3!}z^3 + \cdots,$$

we have $x'(0) = \alpha = \frac{\beta - a}{b} \neq 0$, and from (1.1), we also have $x''(0) = -\frac{c_1}{c_0}x'(0) = -\frac{c_1(\beta-a)}{bc_0} \neq 0$ (if $c_1 \neq 0$), etc.

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