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MODIFIED ISHIKAWA ITERATIVE SEQUENCES WITH ERRORS FOR ASYMPTOTICALLY SET-VALUED PSEUDOCONTRACTIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, some new convergence theorems of the modified Ishikawa and Mann iterative sequences with errors for asymptotically set-valued pseudocontractive mappings in uniformly smooth Banach spaces are given.

1. Introduction and preliminaries

Let E be a real Banach space, E^* be the topological dual space of E and $\langle \cdot, \cdot \rangle$ be the dual pair between E and E^*. Let $F(T)$ be the set of all fixed points of T and $J : E \to 2^{E^*}$ be the *normalized duality mapping* defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = ||x|| \cdot ||f||, \ ||f|| = ||x||\}, \ x \in E.$$

It is well known that $J(x) \neq \emptyset$ for all $x \in E$ and $D(J)$ (the domain of J)$= E$. If E is uniformly smooth, then J is single-valued and uniformly continuous on any bounded subset of E.

Definition 1.1. Let D be a nonempty subset of E and $T : D \to D$ be a mapping.

(1) The mapping T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$||T^n x - T^n y|| \leq k_n ||x - y||$$

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for all $x, y \in D$ and $n = 1, 2, \ldots$.

(2) The mapping T is said to be *pseudocontractive* if for any $x, y \in D$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le ||x-y||^2.$$

(3) The mapping T is said to be *asymptotically pseudocontractive* if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and for any $x, y \in D$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \le k_n ||x-y||^2$$

for all $n = 1, 2, \ldots$.

DEFINITION 1.2. Let D be a nonempty subset of E and $T : D \to 2^D$ be a set-valued mapping. T is said to be *asymptotically set-valued pseudocontractive* if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and, for any $x, y \in D$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle \xi - \eta, j(x-y)\rangle \le k_n ||x-y||^2$$

for all $\xi \in T^n x$, $\eta \in T^n y$ and $n = 1, 2, \ldots$.

The following proposition follows from Definition 1.1 immediately.

PROPOSITION 1.1. *Let D be a nonempty subset of E.*

(1) *If $T : D \to D$ is asymptotically nonexpansive, then T is an asymptotically pseudocontractive mapping.*

(2) *If $T : D \to D$ is pseudocontractive, then T is an asymptotically pseudocontractive mapping.*

Note that the converses of Proposition 1.1 (1) and (2) are not true as in the following examples:

EXAMPLE 1.1. [19] Let $E = \mathbb{R}$, $D = [0, 1]$ and define a mapping $T : D \to D$ by

$$Tx = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}$$

for all $x \in D$. It is easily to see that T is not Lipschitzian, and so it is not asymptotically nonexpansive. But, since T is monotonically decreasing and $T \circ T = I$, we have

$$(T^n x - T^n y)(x-y) = \begin{cases} |x-y|^2, & \text{if } n \text{ is even,} \\ (Tx - Ty)(x-y) \le 0 \le |x-y|^2, & \text{if } n \text{ is odd,} \end{cases}$$

and so T is an asymptotically pseudocontractive mapping with a constant sequence $\{1\}$.

EXAMPLE 1.2. Let $E = l^2$. Then E is a Hilbert space and J is an identity mapping. For any $x = (x_1, x_2, \dots, x_n, \dots) \in l^2$, define a mapping $T : l^2 \to l^2$ as follows:

$$Tx = (0, 4x_1, 0, 0, \dots, 0, \dots).$$

It is easy to see that T is an asymptotically pseudocontractive mapping. In fact, for any $x = (x_1, x_2, \dots, x_n, \dots) \in l^2$ and $y = (y_1, y_2, \dots, y_n, \dots) \in l^2$, we have

$$Tx = (0, 4x_1, 0, 0, \dots, 0, \dots), \quad Ty = (0, 4y_1, 0, 0, \dots, 0, \dots),$$

$$T^n x = (0, 0, 0, \dots, 0, \dots), \quad T^n y = (0, 0, 0, \dots, 0, \dots)$$

for all $n = 2, 3, \dots$. It follows that

$$\begin{aligned} \langle Tx - Ty, x - y \rangle &= 4(x_1 - y_1)(x_2 - y_2) \\ &\le 2[(x_1 - y_1)^2 + (x_2 - y_2)^2] \\ &= 2\|x - y\|^2 \end{aligned}$$

and

$$\langle T^n x - T^n y, x - y \rangle = 0 \le \|x - y\|^2$$

for all $n = 2, 3, \dots$. Letting $k_1 = 2$ and $k_n = 1$ for all $n = 2, 3, \dots$, then $\lim_{n\to\infty} k_n = 1$ and so T is an asymptotically pseudocontractive mapping. However, T is not a pseudocontractive mapping. In fact, taking

$$x^0 = (2, 2, 0, 0, \dots, 0, \dots), \quad y^0 = (1, 1, 0, 0, \dots, 0, \dots),$$

then

$$\langle Tx^0 - Ty^0, x^0 - y^0 \rangle = 4 = 2\|x^0 - y^0\|^2.$$

This implies that T is not a pseudocontractive mapping.

The concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [10], which was closely related to the theory of fixed points of mappings in Banach spaces. An early fundamental result due to Goebel and Kirk [10] showed that, if E is a uniformly convex Banach space, D is a nonempty bounded closed convex subset of E and $T : D \to D$ is an asymptotically nonexpansive mapping, then

T has a fixed point in D. This result is a generalization of the corresponding results in Browder [3] and Kirk [16]. On the other hand, the concept of asymptotically pseudocontractive mappings was introduced by Schu [20].

The iterative approximation problems for nonexpansive mapping, asymptotically nonexpansive mappings and asymptotically pseudocontractive mapping were studied extensively by Browder [3], Goebel and Kirk [10], Kirk [16], Liu [17], Schu [20] and Xu [21, 22] in the setting of Hilbert spaces or uniformly convex Banach spaces.

Recently, Huang and Bai [11] introduced some new iterative methods for set-valued mappings and studied the convergence of Ishikawa and Mann iterative sequences with errors for set-valued strongly pseudocontractive mappings and set-valued strongly accretive mappings in Banach spaces. For some related works, we refer to [1], [4–7, 9], [8, 12–15], [23] and the references therein.

In this paper, we use a new approximation technique to study the convergence problems of modified Ishikawa and modified Mann iterative processes with errors for asymptotically set-valued pseudocontractive mappings in uniformly smooth Banach spaces.

Now, we introduce the modified iterative sequences with errors for set-valued mappings as follows:

DEFINITION 1.3. Let D be a nonempty convex subset of E, $T : D \to 2^D$ be a set-valued mapping and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $[0, 1]$ satisfying some conditions.

(1) The sequence $\{x_n\}$ defined by

$$\text{(1.1)} \qquad \begin{cases} x_0 \in D, \\ x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n \eta_n + \gamma_n u_n, \quad \exists\, \eta_n \in T^n y_n, \\ y_n = (1 - \beta_n - \delta_n)x_n + \beta_n \xi_n + \delta_n v_n, \quad \exists\, \xi_n \in T^n x_n \end{cases}$$

for $n = 0, 1, 2, \ldots$ is called the *modified Ishikawa iterative sequence with errors* for T, where $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in D.

(2) In (1), if $\beta_n = 0$ and $\delta_n = 0$ for $n = 0, 1, 2, \ldots$, then the sequence $\{x_n\}$ defined by

$$\text{(1.2)} \qquad \begin{cases} x_0 \in D, \\ x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n \xi_n + \gamma_n u_n, \quad \exists\, \xi_n \in T^n x_n \end{cases}$$

for $n = 0, 1, 2, \dots$ is called the *modified Mann iterative sequence with errors* for T.

(3) In (1), if $\gamma_n = 0$ and $\delta_n = 0$ for $n = 0, 1, 2, \dots$, then the sequence $\{x_n\}$ defined by

$$\tag{1.3} \begin{cases} x_0 \in D, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \eta_n, & \exists\, \eta_n \in T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n \xi_n, & \exists\, \xi_n \in T^n x_n \end{cases}$$

for $n = 0, 1, 2, \dots$ is called the *modified Ishikawa iterative sequence* for T.

(4) In (3), if $\beta_n = 0$ for $n = 0, 1, 2, \dots$, then the sequence $\{x_n\}$ defined by

$$\tag{1.4} \begin{cases} x_0 \in D, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \xi_n, & \exists\, \xi_n \in T^n x_n \end{cases}$$

for $n = 0, 1, 2, \dots$ is called the *modified Mann iterative sequence* for T.

The following lemma plays an important role for our main results. It is actually Lemma 1 of Petryshyn [18], and even earlier, Asplund [2] proved a general result for single-valued duality mappings that can be used to derive this lemma. We include its proof for the sake of completeness.

LEMMA 1.1. *Let E be a real Banach space, $J : E \to 2^{E^*}$ be a normalized duality mapping. Then for all $x, y \in E$*

$$||x+y||^2 \le ||x||^2 + 2\langle y, j(x+y)\rangle$$

for all $j(x+y) \in J(x+y)$.

Proof. For any $x, y \in E$ and $j(x+y) \in J(x+y)$, we have

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, j(x+y)\rangle \\ &= \langle x, j(x+y)\rangle + \langle y, j(x+y)\rangle \\ &\le \frac{1}{2}(\|x\|^2 + \|j(x+y)\|^2) + \langle y, j(x+y)\rangle \\ &= \frac{1}{2}(\|x\|^2 + \|x+y\|^2) + \langle y, j(x+y)\rangle. \end{aligned}$$

This implies that

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

This completes the proof. □

2. Main results

Now, we give our main results of this paper.

THEOREM 2.1. *Let E be a real uniformly smooth Banach space, D be a nonempty bounded closed convex subset of E, and $T : D \to 2^D$ be an asymptotically set-valued pseudocontractive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be four real sequences in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n + \gamma_n \leq 1, \quad \beta_n + \delta_n \leq 1,$

(ii) $\alpha_n \to 0, \quad \beta_n \to 0, \quad \delta_n \to 0 \ (n \to \infty),$

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \sum_{n=0}^{\infty} \gamma_n < \infty.$

Let $x_0 \in D$ be any given point and $\{x_n\}$ be the modified Ishikawa iterative sequence with errors defined by (1.1). Then the sequence $\{x_n\}$ converges strongly to a fixed point q of T in D if and only if there exists a nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$(2.1) \qquad \langle s - q, J(y_n - q)\rangle \leq k_n||y_n - q||^2 - \phi(||y_n - q||)$$

for all $s \in T^n y_n$ and $n = 1, 2, \ldots$.

Proof. Since E is uniformly smooth, we know that the normalized duality mapping $J : E \to E^*$ is single-valued and uniformly continuous on any bounded subset of E.

Let $x_n \to q \in F(T)$. Since D is a bounded subset of E and $\{u_n\}$, $\{v_n\}$ are both bounded sequences in D, we know that $\{\xi_n\}$, $\{\eta_n\}$, $\{u_n\}$ and $\{v_n\}$ are all bounded. And, since $\beta_n \to 0$ and $\delta_n \to 0$, we have

$$y_n = (1 - \beta_n - \delta_n)x_n + \beta_n \xi_n + \delta_n v_n \to q, \quad (n \to \infty).$$

Letting

$$K = \sup_{n\geq 1}\{||y_n - q||\},$$

then $K < \infty$.

If $K = 0$, then $y_n = q$ for all $n = 1, 2, \ldots$. Hence (2.1) is true for all $n = 1, 2, \ldots$.

If $K > 0$, define

$$G_t = \{n \in \mathbb{N} : ||y_n - q|| \geq t\}, \ t \in (0, K),$$
$$G_K = \{n \in \mathbb{N} : ||y_n - q|| = K\},$$

where $\mathbb{N}$ is the set of all positive integers. Since $y_n \to q$, for any $t \in (0, K]$, there exists $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$,

$$||y_n - q|| < t.$$

This implies that, for each $t \in (0, K)$,

(a) G_t is a nonempty finite subset of $\mathbb{N}$,

(b) $G_{t_1} \subset G_{t_2}$ if $t_1 \geq t_2$ for $t_1, t_2 \in (0, K)$,

(c) $G_K = \bigcap_{t \in (0,K)} G_t$.

Since $T : D \to 2^D$ is asymptotically set-valued pseudocontractive, for any $q \in F(T)$ and for any y_n in D, we have

$$\langle s - q, J(y_n - q)\rangle \leq k_n ||y_n - q||^2 \tag{2.2}$$

for all $s \in T^n y_n$ and $n = 1, 2, \ldots$. By virtue of (2.2), we define a function

$$g(t) = \inf_{n \in G_t} \left\{ k_n ||y_n - q||^2 - \sup_{s \in T^n y_n} \{\langle s - q, J(y_n - q)\rangle\} \right\}, \quad t \in (0, K).$$

From (2.2) and the property (b), it follows

(d) $g(t) \geq 0$ for all $t \in (0, K)$,

(e) $g(t)$ is nondecreasing in $t \in (0, K)$.

Now, we define a function

$$\phi(t) = \begin{cases} 0, & \text{if } t = 0, \\ g(t), & \text{if } t \in (0, K), \\ \lim_{s \to K^-} g(s), & \text{if } t \in [K, \infty). \end{cases}$$

Then $\phi : [0, \infty) \to [0, \infty)$ is nondecreasing and $\phi(0) = 0$. For any $n \geq 1$, let $t_n = ||y_n - q||$.

(1) If $t_n = 0$, then $y_n = q$ and hence $\phi(||y_n - q||) = 0$. Thus we have

$$\langle s - q, J(y_n - q)\rangle = 0 = k_n ||y_n - q||^2 - \phi(||y_n - q||), \quad \forall s \in T^n y_n, \ \forall n \geq 1.$$

(2) If $t_n \in (0, K)$, then $n \in G_{t_n}$ and so we have

$$\begin{aligned} \phi(||y_n - q||) &= g(t_n) \\ &= \inf_{m \in G_{t_n}} \left\{ k_m ||y_m - q||^2 - \sup_{s \in T^m y_m} \{\langle s - q, J(y_m - q)\rangle\} \right\} \\ &\leq k_n ||y_n - q||^2 - \langle s - q, J(y_n - q)\rangle \end{aligned}$$

for all $s \in T^n y_n$.

(3) If $t_n = K$, then $n \in G_K = \cap_{s\in(0,K)} G_s$ and so we have

$$\begin{aligned}
&\phi(||y_n - q||) \\
&= \phi(t_n) = \lim_{s\to K^-} g(s) \\
&= \lim_{s\to K^-} \inf_{m\in G_s} \left\{ k_m ||y_m - q||^2 - \sup_{s\in T^m y_m} \{\langle s - q, J(y_m - q)\rangle\} \right\} \\
&\leq k_n ||y_n - q||^2 - \langle s - q, J(y_n - q)\rangle
\end{aligned}$$

for all $s \in T^n y_n$. Therefore, we proved the necessity.

Next, we have to prove the sufficiency. From Lemma 1.1, we have

(2.3)
$$\begin{aligned}
||x_{n+1} - q||^2 &= ||(1-\alpha_n-\gamma_n)(x_n - q) + \alpha_n(\eta_n - q) + \gamma_n(u_n - q)||^2 \\
&\leq (1-\alpha_n-\gamma_n)^2 ||x_n - q||^2 + 2\alpha_n\langle \eta_n - q, J(x_{n+1} - q)\rangle \\
&\quad + 2\gamma_n\langle u_n - q, J(x_{n+1} - q)\rangle \\
&= (1-\alpha_n-\gamma_n)^2 ||x_n - q||^2 \\
&\quad + 2\alpha_n\langle \eta_n - q, J(x_{n+1} - q) - J(y_n - q)\rangle \\
&\quad + 2\alpha_n\langle \eta_n - q, J(y_n - q)\rangle + 2\gamma_n\langle u_n - q, J(x_{n+1} - q)\rangle.
\end{aligned}$$

Now we consider the second term on the right side of (2.3). Since $\{\eta_n - y_n\}$, $\{x_n - \xi_n\}$, $\{x_n - v_n\}$ and $\{u_n - y_n\}$ are all bounded and

$$\begin{aligned}
&x_{n+1} - q - (y_n - q) \\
&= (1-\alpha_n-\gamma_n)(x_n - y_n) + \alpha_n(\eta_n - y_n) + \gamma_n(u_n - y_n) \\
&= (1-\alpha_n-\gamma_n)\{\beta_n(x_n - \xi_n) + \delta_n(x_n - v_n)\} \\
&\quad + \alpha_n(\eta_n - y_n) + \gamma_n(u_n - y_n),
\end{aligned}$$

we have $x_{n+1} - q - (y_n - q) \to \theta$ as $n \to \infty$. By the uniform continuity of J and the boundedness of $\{\eta_n - q\}$, we know that

(2.4) $$p_n := \langle \eta_n - q, J(x_{n+1} - q) - J(y_n - q)\rangle \to 0 \text{ (as } n \to \infty).$$

Substituting (2.4) and (2.1) into (2.3), we have

(2.5) $$\begin{aligned}
||x_{n+1} - q||^2 \leq (1-\alpha_n-\gamma_n)^2 ||x_n - q||^2 + 2\alpha_n p_n \\
+ 2\alpha_n\{k_n||y_n - q||^2 - \phi(||y_n - q||)\} + 2\gamma_n M,
\end{aligned}$$

where

$$M = \sup_{n\geq 0}\{||u_n - q||\cdot||x_{n+1} - q||\} < \infty.$$

Next we make an estimation for $||y_n - q||^2$. It follows from (1.1) and Lemma 1.1 that

(2.6)

$$\begin{aligned}||y_n - q||^2 &= ||(1-\beta_n-\delta_n)(x_n - q) + \beta_n(\xi_n - q) + \delta_n(v_n - q)||^2\\ &\leq (1-\beta_n-\delta_n)^2||x_n - q||^2 + 2\beta_n\langle \xi_n - q, J(y_n - q)\rangle\\ &\quad + 2\delta_n\langle v_n - q, J(y_n - q)\rangle\\ &\leq (1-\beta_n-\delta_n)^2||x_n - q||^2 + 2\beta_n M_1 + 2\delta_n M_2,\end{aligned}$$

where

$$M_1 = \sup_{n\geq 0}\{||\xi_n - q||\cdot||y_n - q||\} < \infty,$$

and

$$M_2 = \sup_{n\geq 0}\{||v_n - q||\cdot||y_n - q||\} < \infty.$$

Since $1-\alpha_n-\gamma_n \leq 1-\alpha_n$, substituting (2.6) into (2.5) and using $M_3 = \sup_{n\geq 0}||x_n - q||^2$ to simplify, we have

(2.7)

$$\begin{aligned}&||x_{n+1} - q||^2\\ &\leq [(1-\alpha_n)^2 + 2\alpha_n k_n(1-\beta_n-\delta_n)^2]||x_n - q||^2\\ &\quad + 2\alpha_n(p_n + 2\beta_n k_n M_1 + 2\delta_n k_n M_2) - 2\alpha_n\phi(||y_n - q||) + 2\gamma_n M\\ &= ||x_n - q||^2 - \alpha_n\phi(||y_n - q||) - \alpha_n\{\phi(||y_n - q||) - (-2+\alpha_n+2k_n)M_3\\ &\quad -2(p_n + 2\beta_n k_n M_1 + 2\delta_n k_n M_2)\} + 2\gamma_n M.\end{aligned}$$

Let

$$\sigma = \inf_{n\geq 0}\Big\{||y_n - q||\Big\}.$$

Then, we know that $\sigma = 0$. Suppose the contrary. If $\sigma > 0$, then $||y_n - q|| \geq \sigma > 0$ for all $n \geq 0$. Hence $\phi(||y_n - q||) \geq \phi(\sigma) > 0$. From (2.7), it follows that

$$\begin{aligned}&||x_{n+1} - q||^2\\ &\leq ||x_n - q||^2 - \alpha_n\phi(\sigma)\\ &\quad - \alpha_n\{\phi(\sigma) - (-2+\alpha_n+2k_n)M_3\\ &\quad -2(p_n + 2\beta_n k_n M_1 + 2\delta_n k_n M_2)\} + 2\gamma_n M.\end{aligned} \tag{2.8}$$

Since $\alpha_n \to 0$, $\beta_n \to 0$, $\delta_n \to 0$, $p_n \to 0$ and $k_n \to 1$ as $n \to \infty$, there exists n_1 such that, for all $n \geq n_1$,

$$\phi(\sigma) - (-2 + \alpha_n + 2k_n)M_3 - 2(p_n + 2\beta_n k_n M_1 + 2\delta_n k_n M_2) > 0.$$

Hence, from (2.8), we have

$$||x_{n+1} - q||^2 \leq ||x_n - q||^2 - \alpha_n \phi(\sigma) + 2\gamma_n M \ (n \geq n_1),$$

that is,

$$\alpha_n \phi(\sigma) \leq ||x_n - q||^2 - ||x_{n+1} - q||^2 + 2\gamma_n M \ (n \geq n_1).$$

Therefore, for any $m \geq n_1$, we have

$$\begin{aligned}\sum_{n=n_1}^{m} \alpha_n \phi(\sigma) &\leq ||x_{n_1} - q||^2 - ||x_{m+1} - q||^2 + 2M \sum_{n=n_1}^{m} \gamma_n \\ &\leq ||x_{n_1} - q||^2 + 2M \sum_{n=n_1}^{m} \gamma_n.\end{aligned}$$

Letting $m \to \infty$, we have

$$\infty = \sum_{n=n_1}^{\infty} \alpha_n \phi(\sigma) \leq ||x_{n_1} - q||^2 + 2M \sum_{n=n_1}^{\infty} \gamma_n < \infty,$$

which is a contradiction and so $\sigma = 0$. Therefore, there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$y_{n_j} \to q, \quad (n_j \to \infty),$$

that is,

$$y_{n_j} = (1 - \beta_{n_j} - \delta_{n_j})x_{n_j} + \beta_{n_j}\xi_{n_j} + \delta_{n_j} v_{n_j} \to q, \quad (n_j \to \infty).$$

Since $\beta_n \to 0$, $\delta_n \to 0$ and $\{\xi_{n_j}\}$, $\{v_{n_j}\}$ are both bounded, we have

$$x_{n_j} \to q, \quad (n_j \to \infty). \tag{2.9}$$

Since $\alpha_n \to 0$, $\gamma_n \to 0$ and $\{\eta_{n_j}\}$, $\{u_{n_j}\}$ are both bounded, from (2.9), we have

$$x_{n_j+1} = (1 - \alpha_{n_j} - \gamma_{n_j})x_{n_j} + \alpha_{n_j}\eta_{n_j} + \gamma_{n_j} u_{n_j} \to q, \quad (n_j \to \infty)$$

and so

$$y_{n_j+1} = (1 - \beta_{n_j+1} - \delta_{n_j+1})x_{n_j+1} + \beta_{n_j+1}\xi_{n_j+1} + \delta_{n_j+1}v_{n_j+1} \to q, (n_j \to \infty).$$

By induction, we can prove that, for all $i \geq 0$, $x_{n_j+i} \to q$ and $y_{n_j+i} \to q$ as $n_j \to \infty$ for $i = 0, 1, 2, \ldots$, which implies that $x_n \to q$. This completes the proof. □

From Theorem 2.1 and Proposition 1.1, we can obtain the following theorems:

THEOREM 2.2. *Let E be a real uniformly smooth Banach space, D be a nonempty bounded closed convex subset of E, and $T : D \to 2^D$ be an asymptotically set-valued pseudocontractive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\gamma_n\}$ be two real sequences in $[0, 1]$ satisfying the following conditions:*

(i) $\alpha_n + \gamma_n \leq 1$,
(ii) $\alpha_n \to 0 \quad (n \to \infty)$,
(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \sum_{n=0}^{\infty} \gamma_n < \infty$.

Let $x_0 \in D$ be any given point and $\{x_n\}$ be the modified Mann iterative sequence with errors defined by (1.2). Then the sequence $\{x_n\}$ converges strongly to a fixed point q of T if and only if there exists a nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle s - q, J(x_n - q)\rangle \leq k_n||x_n - q||^2 - \phi(||x_n - q||) \tag{2.10}$$

for all $s \in T^n x_n$ and $n = 1, 2, \ldots$.

Proof. Taking $\beta_n = \delta_n = 0$ for all $n \geq 0$ in Theorem 2.1, then we have $y_n = x_n$ for all $n \geq 0$. Therefore, the conclusion of Theorem 2.2 follows from Theorem 2.1 immediately. □

The following theorem is the case of single valued mapping [5].

THEOREM 2.3. *Let E be a real uniformly smooth Banach space, D be a nonempty bounded closed convex subset of E, and $T : D \to D$ be an asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be four real sequences in $[0, 1]$ satisfying the conditions* (i)$\sim$(iii) *in Theorem*

2.1. Let $x_0 \in D$ be any given point and $\{x_n\}$ be the modified Ishikawa iterative sequence with errors defined by

$$(2.11) \qquad \begin{cases} x_0 \in D, \\ x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n T^n y_n + \gamma_n u_n, \\ y_n = (1 - \beta_n - \delta_n)x_n + \beta_n T^n x_n + \delta_n v_n \end{cases}$$

for $n = 0, 1, 2, \ldots$. Then the sequence $\{x_n\}$ converges strongly to $q \in F(T)$ if and only if there exists a nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$(2.12) \qquad \langle T^n y_n - q, J(y_n - q)\rangle \leq k_n ||y_n - q||^2 - \phi(||y_n - q||)$$

for $n = 1, 2, \ldots$.

We can easily prove the following theorem from the Proposition 1.1 and Theorem 2.3.

THEOREM 2.4. *Let E be a real uniformly smooth Banach space, D be a nonempty bounded closed convex subset of E, and $T : D \to D$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be four real sequences in $[0, 1]$ satisfying the conditions* (i)∼(iii) *in Theorem 2.1. Let $x_0 \in D$ be any given point and $\{x_n\}$ be the modified Ishikawa iterative sequence with errors defined by* (2.11). *Then the sequence $\{x_n\}$ converges strongly to $q \in F(T)$ if and only if there exists a nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ and $\phi(0) = 0$ such that* (2.12) *holds for $n = 1, 2, \ldots$.*

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