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# A NOTE ON A CHOQUET-DENY-TYPE THEOREM

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Abstract. We present a Choquet-Deny-type theorem for downward filtering convex sets of continuous functions and show that the Identity Korovkin cone of a downward filtering convex cone $S$ is exactly the uniform closure of $S$.

## 1. Introduction and preliminaries

The uniformly closed convex cones which are inf-lattices of continuous functions on a compact Hausdorff space have been characterized by Choquet and Deny [1].

Priestley [5] has proved a Choquet-Deny-type theorem for affine continuous functions on a Choquet simplex by replacing the inf-lattice condition with a filtering condition. Inspired by this paper, we define downward filtering subsets of the space of continuous functions and present a Choquet-Deny-type theorem which extends some previous results of the second author [6]. We do not treat the case of affine functions on a Choquet simplex.

As an application, we show that the Identity Korovkin cone of a downward filtering convex cone $S$ is exactly the uniform closure of $S$.

Let $X$ be a compact Hausdorff space and let $C(X;\mathbb{R})$ be the space of all continuous real-valued functions defined on $X$, endowed with the sup-norm

$$\|f\| = \sup\{|f(x)| : x \in X\}$$

for each $f \in C(X;\mathbb{R})$.

If $S$ is a subset of $C(X;\mathbb{R})$, then $\overline{S}$ denotes its uniform closure.

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The symbol $\triangle$ denotes the set of all *positive* linear forms on $C(X;\mathbb{R})$. Hence $\varphi \in \triangle$ means that $\varphi(f) \geq 0$ for all $f \in C(X;\mathbb{R})$ such that $f \geq 0$. Notice that every element of $\triangle$ is continuous.

Here the functions $f \wedge g$ and $f \vee g$ are defined by

$$\begin{aligned}(f \wedge g)(x) &= \inf\{f(x), g(x)\}\\ (f \vee g)(x) &= \sup\{f(x), g(x)\}\end{aligned}$$

for every $x \in X$ and $f, g \in C(X;\mathbb{R})$.

For any $f \in C(X;\mathbb{R})$, the function $f^+ = f \vee 0$ is called the *positive* part of $f$.

A subset $S$ of $C(X;\mathbb{R})$ is called an *inf-lattice*, if $f, g \in S$ implies $f \wedge g \in S$. On the other hand, a subset $S$ of $C(X;\mathbb{R})$ is a *convex cone* if $f, g \in S$ and $\lambda \geq 0$ imply $f + g \in S$ and $\lambda f \in S$.

DEFINITION 1.1. A subset $S$ of $C(X;\mathbb{R})$ is called *downward filtering* if for every $f \in C(X;\mathbb{R})$ and every finite family $\{g_1, \ldots, g_n\} \subset S$ with $f < g_1 \wedge \cdots \wedge g_n$, there exists $g \in S$ such that $f < g \leq g_1 \wedge \cdots \wedge g_n$.

EXAMPLE 1.2. Every inf-lattice in $C(X;\mathbb{R})$ is downward filtering.

EXAMPLE 1.3. Let $X$ be a compact Hausdorff space and $S$ be a uniformly dense subset of $C(X;\mathbb{R})$. Let $f_1$ and $f_2$ be any two elements of $C(X;\mathbb{R})$ with $f_1 < f_2$. Since $S$ is dense in $C(X;\mathbb{R})$ there exists $g \in S$ such that $f_1 < g < f_2$. Indeed, let

$$\varepsilon = \inf\{f_2(x) - f_1(x) : x \in X\}.$$

It follows from the compactness of $X$, that $\varepsilon > 0$. Let $h = (f_1 + f_2)/2$. Approximate $h$ within $\varepsilon/4$ by a function $g \in S$. Then $f_1 < g < f_2$ on $X$. Hence $S$ is a downward filtering subset of $C(X;\mathbb{R})$.

EXAMPLE 1.4. The set $S$ of all real concave and differentiable functions on $[0,1]$ is downward filtering. Indeed, given any $f \in C([0,1];\mathbb{R})$ and $g_1, \ldots, g_n \in S$ such that $f < h$, where $h = g_1 \wedge \cdots \wedge g_n$, there exists a concave Bernstein polynomial $B_m(h)$ such that $f < B_m(h) \leq g_1 \wedge \cdots \wedge g_n$. Recall that

$$B_m(h)(x) = \sum_{k=0}^{m} h\left(\frac{k}{m}\right) \binom{m}{k} x^k (1-x)^{m-k}.$$

Notice that $S$ is not an inf-lattice.

EXAMPLE 1.5. A real function $f$ on an ordered set $E$ is said to be increasing if $x, y \in E$ and $x \leq y$ imply $f(x) \leq f(y)$.

Let $[0,1]^n$ be the $n$-dimensional hypercube endowed with the order $x \leq y \Leftrightarrow y - x \in \mathbb{R}^n_+$, $x, y \in [0,1]^n$. The set of all increasing polynomials on $[0,1]^n$ is downward filtering. To prove it, we need the following result, which can be found in [3].

LEMMA 1.6. *Let $\phi$ be an upper semi-continuous real-valued function and $\psi$ a lower semi-continuous real-valued function on the compact ordered space $E$. Assume that*

$$\phi(x) < \psi(x) \text{ for } x \in E$$

*and that one of these functions is increasing. Then there is an increasing continuous real-valued function $h$ on $E$ such that*

$$\phi(x) < h(x) < \psi(x) \text{ for } x \in E.$$

PROPOSITION 1.7. *The set of all increasing polynomials on $[0,1]^n$ is a downward filtering subset of $C([0,1]^n;\mathbb{R})$.*

*Proof.* Let $S$ be the set of all increasing polynomials on $[0,1]^n$. Let $f \in C([0,1]^n;\mathbb{R})$ and $g_1,\ldots,g_l \in S$ such that $f < g_1 \wedge \cdots \wedge g_l$. By Lemma 1.6, there is an increasing continuous real-valued function $h$ on $[0,1]^n$ such that $f < h < g_1 \wedge \cdots \wedge g_l$. Since $[0,1]^n$ is compact and $S$ is uniformly dense in the space of increasing continuous real-valued functions on $[0,1]^n$, there is $g \in S$ such that $f < g < g_1 \wedge \cdots \wedge g_l$. Thus, $S$ is a downward filtering subset of $C([0,1]^n;\mathbb{R})$. □

## 2. Results

If $\varphi$ is a continuous linear form on $C(X;\mathbb{R})$ and $c \in \mathbb{R}$, the functional $\psi$ defined on $C(X;\mathbb{R})$ by

$$\psi(f) = \varphi(f) + c$$

for all $f \in C(X;\mathbb{R})$ is called a continuous *affine functional.* If $\varphi$ is positive, then $\psi$ is *monotone*, that is $f \leq g$ implies $\psi(f) \leq \psi(g)$ for each pair $f$ and $g$ of elements of $C(X;\mathbb{R})$. We denote by $\Gamma$ the set of all continuous monotone affine functionals on $C(X;\mathbb{R})$.

If $S$ is a subset of $C(X;\mathbb{R})$, we denote by $K^\wedge(S)$ the set of all $f \in C(X;\mathbb{R})$ such that $\psi(f) \leq f(x)$ for any pair $(x,\psi) \in X \times \Gamma$ such that $\psi(g) \leq g(x)$ for every $g \in S$.

We state the following result.

THEOREM 2.1. *Let $S$ be a nonempty convex subset of $C(X;\mathbb{R})$. If $S$ is downward filtering, then $\overline{S} = K^\wedge(S)$.*

To prove this theorem we use the arguments of Nachbin [4].

We shall denote by $P$ the set of all functions $f \in C(X;\mathbb{R})$ such that $f(x) \geq 0$ for all $x \in X$. Clearly, $P$ is a convex cone in $C(X;\mathbb{R})$.

Given a point $x \in X$, let

$$\begin{aligned} P_0(x) &= \{f \in P : f(x) = 0\}, \\ P_1(x) &= \{f \in P : f(x) = 1\}. \end{aligned}$$

LEMMA 2.2. *Let $\varphi : C(X;\mathbb{R}) \to \mathbb{R}$ be a non-zero continuous linear form with $\|\varphi\| \leq 1$, and let $x \in X$. Assume that $\varphi(f) \geq 0$ for all $f \in P_0(x)$. Then:*

(i) *$\varphi(h) \geq -1$ for all $h \in P_1(x)$.*
(ii) *$\varphi(f) \geq -f(x)$ for all $f \in P$.*

*Proof.* (i) Take $h \in P_1(x)$. Then, the function $g = h - (1 \wedge h)$ belongs to $P_0(x)$. Hence $\varphi(g) \geq 0$, and therefore $\varphi(h) \geq \varphi(1 \wedge h)$.

On the other hand, $\|1 \wedge h\| \leq 1$ and so $|\varphi(1 \wedge h)| \leq 1$. Hence $\varphi(1 \wedge h) \geq -1$ and therefore $\varphi(h) \geq -1$ for all $h \in P_1(x)$.

(ii) Take $f \in P$. If $f(x) = 0$, then $f \in P_0(x)$ and so $\varphi(f) \geq 0 = -f(x)$. If $f(x) > 0$, then $h = (f(x))^{-1} f$ belongs to $P_1(x)$, and by part (i), $\varphi(h) \geq -1$. Hence $\varphi(f) = \varphi(f(x)h) = f(x)\varphi(h) \geq -f(x)$. □

REMARK 2.3. If we denote by $\delta_x$ the evaluation functional $f \mapsto f(x)$, then part (ii) of Lemma 2.2 says that the linear form $\psi = \varphi + \delta_x$ is *positive*, that is, $\psi(f) \geq 0$ for all $f \in P$.

LEMMA 2.4. *If $S$ is a nonempty downward filtering subset of $C(X;\mathbb{R})$, then*

$$\overline{S} = \bigcap \{\overline{S - P_0(x)} : x \in X\}.$$

*Proof.* Since $0 \in P_0(x)$ for each $x \in X$, it follows that

$$\overline{S} \subset \bigcap \{\overline{S - P_0(x)} : x \in X\}.$$

Conversely, assume that $f \in \overline{S - P_0(x)}$ for all $x \in X$. Given $\varepsilon > 0$, for each $x \in X$, there exist $g_x \in S$ and $h_x \in P_0(x)$ such that

$$|g_x(t) - h_x(t) - f(t)| < \varepsilon/2$$

for all $t \in X$. Consider the set

$$V_x = \{t \in X : \; h_x(t) < \varepsilon/2\}.$$

Notice that $V_x$ is open and, since $h_x \in P_0(x)$, it follows that $x \in V_x$. Since $X$ is compact, there exist $x_1, \dots, x_n \in X$ such that $X = V_{x_1} \cup \cdots \cup V_{x_n}$. Define $g = g_{x_1} \wedge \cdots \wedge g_{x_n}$. Let $t \in X$ be given. For each index $1 \leq j \leq n$

$$g_{x_j}(t) \geq g_{x_j}(t) - h_{x_j}(t) > f(t) - \varepsilon/2.$$

Hence $g > f - \varepsilon$. Since $S$ is downward filtering, there exists $s \in S$ with $s \leq g$ and $s > f - \varepsilon$. On the other hand, given $t \in X$, there exists $j \in \{1, \dots, n\}$ such that $t \in V_{x_j}$, and then $h_{x_j}(t) < \varepsilon/2$ and $s(t) \leq g(t) \leq g_{x_j}(t)$ imply

$$s(t) - \varepsilon/2 \leq g_{x_j}(t) - \varepsilon/2 < g_{x_j}(t) - h_{x_j}(t) < f(t) + \varepsilon/2.$$

Thus $s(t) < f(t) + \varepsilon$, and so

$$f - \varepsilon < s < f + \varepsilon.$$

Therefore $\|f - s\| < \varepsilon$, and this completes the proof. □

LEMMA 2.5. *If $S$ is a nonempty convex subset of $C(X;\mathbb{R})$, then*

$$K^{\wedge}(S) \subset \bigcap \{\overline{S - P_0(x)} : x \in X\}.$$

*Proof.* Let $f \in C(X;\mathbb{R})$ such that, for some point $x \in X$, $f \notin \overline{S - P_0(x)}$. Since $\overline{S - P_0(x)}$ is convex, by the Hahn-Banach separation theorem, there is a non-zero continuous linear form $\psi$ on $C(X;\mathbb{R})$ and a number $c \in \mathbb{R}$ such that

$$\psi(g - h) \leq c < \psi(f) \tag{2.1}$$

for all $g \in S$ and $h \in P_0(x)$. Without loss of generality we may assume $\|\psi\| \leq 1$. Let $g_0$ be some fixed element of $S$. For each $h \in P_0(x)$ and each $\lambda > 0$, we have $\psi(g_0 - \lambda h) \leq c$. Dividing by $\lambda > 0$ and letting $\lambda \to \infty$ we obtain

$$\psi(h) \geq 0 \text{ for each } h \in P_0(x). \tag{2.2}$$

By (ii) of Lemma 2.2, $\varphi := \psi + \delta_x$ is a positive linear form. Notice that $h = 0 \in P_0(x)$ and (2.1) imply $\psi(g) \leq c$ for all $g \in S$. Therefore $\varphi(g) - g(x) = \psi(g) \leq c$, and so

$$\varphi(g) - c \leq g(x) \text{ for all } g \in S. \tag{2.3}$$

On the other hand, (2.1) implies $\psi(f) > c$. Hence $\varphi(f) - f(x) = \psi(f) > c$, that is,

$$\varphi(f) - c > f(x). \tag{2.4}$$

Since $\varphi \in \triangle$ and so $\varphi - c \in \Gamma$, it follows from (2.3) and (2.4) that $f \notin K^{\wedge}(S)$. □

*Proof of Theorem 2.1.* Clearly $\overline{S} \subset K^{\wedge}(S)$. Since $S$ is convex, by Lemma 2.5 we obtain $K^{\wedge}(S) \subset \bigcap\{\overline{S - P_0(x)} : x \in X\}$. By Lemma 2.4, $\bigcap\{\overline{S - P_0(x)} : x \in X\} \subset \overline{S}$, because $S$ is downward filtering. Hence $\overline{S} = K^{\wedge}(S)$. □

In the following, we shall present a similar result for convex cones which are downward filtering subsets of $C(X;\mathbb{R})$.

If $S$ is a nonempty subset of $C(X;\mathbb{R})$, we denote by $K_0^{\wedge}(S)$ the set of all $f \in C(X;\mathbb{R})$ such that $\varphi(f) \leq f(x)$ for any pair $(x,\varphi) \in X \times \triangle$ such that $\varphi(g) \leq g(x)$ for all $g \in S$.

Clearly, $\overline{S} \subset K_0^{\wedge}(S)$ and $K^{\wedge}(S) \subset K_0^{\wedge}(S)$.

LEMMA 2.6. *If $S$ is a nonempty subset of $C(X;\mathbb{R})$ such that $\lambda S \subset S$ for all $\lambda \geq 0$, then $K_0^{\wedge}(S) \subset K^{\wedge}(S)$.*

*Proof.* Let $f \in K_0^{\wedge}(S)$. Let $(x,\psi) \in X \times \Gamma$ be such that $\psi(g) \leq g(x)$ for all $g \in S$. Now $\psi = \varphi + c$ for some $\varphi \in \triangle$ and $c \in \mathbb{R}$. Hence

$$\varphi(g) + c \leq g(x) \tag{2.5}$$

for all $g \in S$. Fix $g \in S$, arbitrarily. For each $\lambda > 0$, we have $\lambda g \in S$, and so $\lambda\varphi(g) + c \leq \lambda g(x)$. Dividing by $\lambda > 0$ and letting $\lambda \to \infty$ we get $\varphi(g) \leq g(x)$. Since $f \in K_0^{\wedge}(S)$, it follows that $\varphi(f) \leq f(x)$. Now $0 \in S$ and inequality (2.5) imply $c \leq 0$. Hence $\psi(f) = \varphi(f) + c \leq \varphi(f) \leq f(x)$, and so $f \in K^{\wedge}(S)$. □

THEOREM 2.7. *Let $S$ be a convex cone in $C(X;\mathbb{R})$. If $S$ is downward filtering, then $\overline{S} = K_0^{\wedge}(S)$.*

*Proof.* Apply Lemma 2.6 and Theorem 2.1. □

REMARK 2.8. Since every inf-lattice in $C(X;\mathbb{R})$ is downward filtering, Theorem 2.1 and Theorem 2.7 extend the previous results [6, Theorem 1] and [6, Theorem 2].

In what follows, we shall give a description of the Identity Korovkin cone of convex cones which are downward filtering subsets of $C(X;\mathbb{R})$.

Let $S$ be a convex cone in $C(X;\mathbb{R})$. The *Identity Korovkin cone* of $S$, denoted by $\Im(S)$, consists of all functions $f \in C(X;\mathbb{R})$ such that $\lim_\alpha[T_\alpha(f) - f]^+ = 0$ for each equicontinuous net $\{T_\alpha\}_{\alpha\in A}$ of positive linear operators on $C(X;\mathbb{R})$ such that $\lim_\alpha[T_\alpha(g) - g]^+ = 0$ for all $g \in S$. For further information, see for example [2].

LEMMA 2.9. *If $S$ is a convex cone in $C(X;\mathbb{R})$, then $\Im(S) \subset K_0^{\wedge}(S)$.*

*Proof.* Let $f \notin K_0^{\wedge}(S)$, that is, there exist $x \in X$ and a positive linear form $\varphi$ on $C(X;\mathbb{R})$ such that $\varphi(g) \leq g(x)$ for all $g \in S$, but $\varphi(f) > f(x)$. Let $B$ be a basis of open neighborhoods of $x$ equipped with the partial ordering $\leq$ defined by $U \leq V$ if $V \subset U$. Thus, we can consider the net $\{T_V\}_{V \in B}$. It follows from Urysohn's Lemma that for every open neighborhood V of $x$ there exists a function $l_V \in C(X;\mathbb{R})$ such that

$$0 \leq l_V \leq 1, \; l_V(x) = 1, \text{ and } l_V(t) = 0 \text{ if } t \notin V.$$

Define the linear operator $T_V$ on $C(X;\mathbb{R})$ by

$$T_V(h) = h(1 - l_V) + \varphi(h) l_V$$

for every $h \in C(X;\mathbb{R})$.

Notice that $T_V$ is positive. Furthermore, $\{T_V\}_{V \in B}$ is equicontinuous because $\|T_V\| \leq 1 + \|\varphi\|$.

Let $V$ be any element of $B$. For every function $g \in S$, we consider the following cases:

*Case 1.* If $t \notin V$, then $T_V(g)(t) - g(t) = 0$.

*Case 2.* If $t \in V$, then

$$\begin{aligned} T_V(g)(t) - g(t) &= (\varphi(g) - g(t)) l_V(t) \\ &\leq (g(x) - g(t)) l_V(t) \\ &\leq |g(x) - g(t)| l_V(t) \\ &\leq |g(x) - g(t)|. \end{aligned}$$

Hence $T_V(g)(t) - g(t) < \varepsilon$ for every open neighborhood $V$ of $x$ such that $|g(x) - g(t)| < \varepsilon$ for all $t \in V$. It follows from Case 1 and Case 2 that for the collection $B'$ of such neighborhoods $V$, we have $[T_V(g)(t) - g(t)]^+ < \varepsilon$ for all $g \in S$ and for all $t \in X$. Hence $\lim_V [T_V(g) - g]^+ = 0$ for all $g \in S$. However,

$$T_V(f)(x) - f(x) = (\varphi(f) - f(x)) l_V(x) = \varphi(f) - f(x) > 0.$$

Thus, $\{[T_V(f) - f]^+\}_{V \in B'}$ does not converge to zero, that is, $f \notin \Im(S)$. □

We can now prove the following result.

THEOREM 2.10. *Let $S$ be a convex cone in $C(X;\mathbb{R})$. If $S$ is downward filtering, then $\Im(S) = \overline{S}$.*

*Proof.* It is easy to prove that $\overline{S} \subset \Im(S)$. Conversely, by Lemma 2.9 and Theorem 2.7 we obtain $\Im(S) \subset \overline{S}$. □

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