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On the Conditional Dependence Structure of Multivariate Random Variables[1]

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Abstract

In this paper, we introduce a new notions of conditionally weak dependence and we study their properties, preservation of the conditionally weak independent and positive and negative quadrant dependent(CWQD) property under mixtures, limits, closure under convex combinations, and their interrelationships. Furthermore, we extend multivariate stochastic dependence to stronger conditions of dependence.

Keywords : CWQD; CSRTD; CRCSD; CRTDS; CTD2P.

1. Introduction

Lehmann(1966) introduced the concepts of positive(negative) dependence together with some other dependence concepts. Since then, a great many papers have been studied on the subject and its extensions, and numerous multivariate inequalities have been obtained. In other words, a great many papers have been devoted to various generalizations of Lehmann's concepts of finite-dimensional distributions and this results have been extended in several directions, see Karlin and Rinott(1980), Ebrahimi and Ghosh(1981) and Shaked(1982) and Sampson(1983) and Baek(1997). Furthermore, Brady and Singpurwalla(1990) introduced new conditionally versions of independent and positive and negative quadrant dependence concepts of random variables which were introduced below, namely positive dependence concepts(introduced by Ahmed(1978))

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These concepts are qualitative form of dependence which has led to many applications in applied probability, reliability and statistical inference such as analysis of variance, multivariate, hypothesis test, sequential testing.

Consider a system of two components with life lengths of random variables X_1 and X_2, operating in an environment which is characterized by an abstract (idealized and unobservable) parameter $\theta \in R$. Suppose that I_1, I_2 and I_3 partition R such that $I_1 \cup I_2 \cup I_3 = R$ and that when $\theta \in I_1$, the operating environment is classified as being "average" or normal whereas when $\theta \in I_2$ or $\theta \in I_3$ the operating environment is classified as being "mild" or harsh, respectively, then we can obtain the conditionally inequalities for system reliability. Certain kinds of conditionally dependence properties are useful concept in reliability theory and these results are of value as they help us to understand in what ways for dependence structures of random variables. Hence, we wish to investigate a new dependence concept weaker than conditionally quadrant dependent (introduced by Brady and Singpurwalla(1990)).

Definition1.1(1990). A sequence of random variables$X_1,\cdots,X_n$ is θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3 if (i), (ii), (iii) below hold,

(i) $(P(\bigcap_{i=1}^{n}(X_i > x_i) \mid \theta \in I_1) = \prod_{i=1}^{n} P(X_i > x_i \mid \theta \in I_1)$,

(ii) $P(\bigcap_{i=1}^{n}(X_i > x_i) \mid \theta \in I_2) \geqq \prod_{i=1}^{n} P(X_i > x_i \mid \theta \in I_2)$,

and

(iii) $P(\bigcap_{i=1}^{n}(X_i > x_i) \mid \theta \in I_3) \leqq \prod_{i=1}^{n} P(X_i > x_i \mid \theta \in I_3)$.

The importance of this paper lies in the fact that the notion introduced is weaker than the notion of conditionally independent and positive and negative quadrant dependent random variables and enjoys most of the properties and theoretical results of the latter notion. So, we introduce a new notion of conditionally weak independent and positive and negative quadrant dependence defined over multivariate random variables. This paper lays the foundation for a new concept in the theory by defining random dependence, proposing a property of random dependence and developing theorems based on this concept.

In section 2, we introduce a new notions of CWQD and some definitions of the conditionally stochastically right tail dependent(CSRTD), conditionally right corner set dependent(CRCSD), conditionally right tail dependent in sequence (CRTDS), conditionally totally dependent of order 2 in pairs (CTD2P) which were defined by Brady and Singpurwalla (1990) and we study their properties, the preservation of the conditionally weak independent and positive and negative dependent property under mixtures, limits, the closure under convex combination, and their interrelationships. Conclusions are given in section 3.

2. Definitions and Some Results

First, we start this section by stating the definitions of conditionally weak independent and positive and negative quadrant dependent(CWQD).

Definition 2.1. A sequence of random variables $X_1,\cdots,X_n$ is θ conditionally weak independent and positive and negative quadrant dependent on I_1, I_2 and I_3 if

$$\text{(i)} \int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}(P(\bigcap_{i=1}^{n}(X_i> s_i)\,|\,\theta\in I_1)-\prod_{i=1}^{n}P(X_i> s_i\,|\theta\in I_1))ds_n\cdots ds_1=0,$$

$$\text{(ii)} \int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}(P(\bigcap_{i=1}^{n}(X_i> s_i)|\theta\in I_2)-\prod_{i=1}^{n}P(X_i> s_i\,|\theta\in I_2))ds_n\cdots ds_1\geqq 0,$$

and

$$\text{(iii)} \int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}(P(\bigcap_{i=1}^{n}(X_i> s_i)|\theta\in I_3)-\prod_{i=1}^{n}P(X_i> s_i\,|\theta\in I_3))ds_n\cdots ds_1\leqq 0.$$

The following illustrates example of CWQD.

Example 2.2. Let X_1, X_2 and X_3 be binary random variables with
$P\{X_i=1\}=p_i$ and assume $p_i+p_j\leq 1$. When we define as follows;
$P\{X_1=1,X_2=1,X_3=1\}$, $P\{X_1=1,\ X_2=1,\ X_3=0\}=p_1p_2-p_{123}$,
$P\{X_1=1,\ X_2=0,\ X_3=1\}=p_1p_3-p_{123}$, $P\{X_1=0,\ X_2=1,\ X_3=1\}=p_2p_3-p_{123}$,
$P\{X_1=1,\ X_2=0,\ X_3=0\}=p_1-p_1p_2-p_1p_3+p_{123}$,
$P\{X_1=0,\ X_2=1,\ X_3=0\}=p_2-p_2p_3-p_1p_2+p_{123}$,
$P\{X_1=0,\ X_2=0,\ X_3=1\}=p_3-p_1p_3-p_2p_3+p_{123}$,
$P\{X_1=0,\ X_2=0,\ X_3=0\}=1-p_1-p_2-p_3+p_1p_2+p_1p_3+p_2p_3-p_{123}$,
if $p_{123}=p_1p_2p_3=I_1$, $p_{123}\in I_2$, $p_{123}\in I_3$ where $I_1=p_1p_2p_3$, $I_2=(p_1p_2p_3,\min(p_1p_2,$ $p_2p_3,p_1p_3)]$, $I_3=[0,\ p_1p_2p_3)$ and $R=I_1\cup I_2\cup I_3$, then X_1, X_2 and X_3 are p_{123} cond itionally weak independent and positive and negative quadrant dependent on I_1, I_2, I_3.

Before introducing some results, let us present some basic properties of conditionally weak independent and positive and negative quadrant dependent random variables. It is not difficult to show that:

Property 1. Nondecreasing functions of a sequence of θ conditionally weak independent and positive and negative quadrant dependent random variables on I_1, I_2 and I_3 are θ conditionally weak independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Property 2. Any subset of θ conditionally weak independent and positive and negative quadrant dependent random variables on I_1, I_2 and I_3 are θ conditionally weak independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Property 3. A set of θ conditionally weak independent random variables are θ conditionally weak independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Property 4. The union of θ conditionally weak independent and positive and negative quadrant dependent random variables on I_1, I_2 and I_3 are θ conditionally weak independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Proof. Let $(X_1,\cdots,X_n)$ and $(Y_1,\cdots,Y_m)$ be independent random vectors each of which is θ conditionally weak independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Then

$$\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}\int_{y_1}^{\infty}\cdots\int_{y_m}^{\infty} P(\bigcap_{j=1}^{n}(X_j> s_j), \bigcap_{k=1}^{m}(Y_k> t_k) \mid \theta\in I_i)\, dt_m \cdots dt_1\, ds_n \cdots ds_1$$

$$= \int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}\int_{y_1}^{\infty}\cdots\int_{y_m}^{\infty} P(\bigcap_{j=1}^{n}(X_j> s_j) \mid \theta\in I_i)\cdot$$

$$P(\bigcap_{k=1}^{m}(Y_k> t_k) \mid \theta\in I_i)\, dt_m \cdots dt_1\, ds_n \cdots ds_1$$

$$= \int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}\int_{y_1}^{\infty}\cdots\int_{y_m}^{\infty} \prod_{j=1}^{n} P(X_j> s_j \mid \theta\in I_1)\cdot$$

$$\prod_{k=1}^{m} P(Y_k> t_k \mid \theta\in I_1)\, dt_m \cdots dt_1\, ds_n \cdots ds_1$$

$$\geq \int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}\int_{y_1}^{\infty}\cdots\int_{y_m}^{\infty} \prod_{j=1}^{n} P(X_j> s_j \mid \theta\in I_2)\cdot$$

$$\prod_{k=1}^{m} P(Y_k > t_k \mid \theta \in I_2)\, dt_m \cdots dt_1\, ds_n \cdots ds_1$$

$$\leq \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_m}^{\infty} \prod_{j=1}^{n} P(X_j > s_j \mid \theta \in I_3) \cdot$$

$$\prod_{k=1}^{m} P(Y_k > t_k \mid \theta \in I_3)\, dt_m \cdots dt_1\, ds_n \cdots ds_1$$

For proving the next theorem, we need the following definition, let $\underline{X} = (X_1, X_2, \cdots, X_n)$ and $\underline{Y} = (Y_1, Y_2, \cdots, Y_n)$.

Definition 2.3 (Brady and Singpurwalla, 1990). A random vector $\underline{Y}$ is θ conditionally stochastically right tail dependent(CSRTD) in the random vector $\underline{X}$ on I_1, I_2 and I_3 if $E(f(\underline{Y}) \mid \underline{X} > \underline{x}, \theta)$ is constant, increasing, and decreasing given $\theta \in I_1$, I_2 and I_3, respectively for any real valued increasing function f.

We give a set of sufficient conditions to preserve the conditionally weak quadrant dependent property under mixtures on I_1, I_2 and I_3, for $n \geq 3$.

Theorem 2.4. Let (a) X_j, $j = 1, 2, \cdots, m$ be θ conditionally weak independent and positive and negative quadrant dependent on I_1, I_2 and I_3, (b) Y_k, $k = 1, 2, \cdots, n$ be conditionally independent given $\underline{X}$ and θ and (c) Y_k be θ conditionally stochastically right tail dependent in $\underline{X}$ on I_1, I_2 and I_3, for all $k = 1, 2, \cdots, n$. Then $(\underline{X}, \underline{Y})$ is θ conditionally weak independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Proof.

$$\int_{x_1}^{\infty} \cdots \int_{x_m}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_n}^{\infty} P(\bigcap_{j=1}^{m} (X_j > s_j), \bigcap_{k=1}^{n} (Y_k > t_k) \mid \theta \in I_i)\, dt_n \cdots dt_1\, ds_m \cdots ds_1$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_m}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_m}^{\infty} P(\bigcap_{k=1}^{n} (Y_k > t_k) \mid \bigcap_{j=1}^{m} (X_j > s_j), \theta \in I_i) \cdot$$

$$P(\bigcap_{j=1}^{m} (X_j > s_j) \mid \theta \in I_i)\, dt_n \cdots dt_1\, ds_m \cdots ds_1$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_m}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_n}^{\infty} \prod_{k=1}^{n} P(Y_k > t_k \mid \bigcap_{j=1}^{m} (X_j > s_j), \theta \in I_i) \cdot$$

$$P(\bigcap_{j=1}^{m} (X_j > s_j) \mid \theta \in I_i)\, dt_n \cdots dt_1\, ds_m \cdots ds_1$$

using (b),

$$= \int_{x_1}^{\infty} \cdots \int_{x_m}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_n}^{\infty} \prod_{k=1}^{n} P(Y_k > t_k | \theta \in I_1) \cdot$$

$$\prod_{j=1}^{m} P(X_j > s_j | \theta \in I_1) \, dt_n \cdots dt_1 \, ds_m \cdots ds_1$$

$$\geqq \int_{x_1}^{\infty} \cdots \int_{x_m}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_n}^{\infty} \prod_{k=1}^{n} P(Y_k > t_k | \theta \in I_2) \cdot$$

$$\prod_{j=1}^{m} P(X_j > s_j | \theta \in I_2) \, dt_n \cdots dt_1 \, ds_m \cdots ds_1$$

$$\leqq \int_{x_1}^{\infty} \cdots \int_{x_m}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_n}^{\infty} \prod_{k=1}^{n} P(Y_k > t_k | \theta \in I_3) \cdot$$

$$\prod_{j=1}^{m} P(X_j > s_j | \theta \in I_3) \, dt_n \cdots dt_1 \, ds_m \cdots ds_1$$

using (c) and (a).

The next theorem demonstrates the preservation of the conditionally weak independent and positive and negative dependent property under limits on I_1, I_2 and I_3.

Theorem 2.5. Let X_n be a sequence of θ conditionally weak independent and positive and negative quadrant dependent on I_1, I_2 and I_3, p-dimensional random vectors with distribution functions $H_n \to H$ weakly as $n \to \infty$, where H is the distribution function of a random vector $\underline{X} = (X_1, \cdots, X_p)$. Then $\underline{X}$ is θ conditionally weak independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Proof. For any $s_1, \cdots, s_p$ writing $\underline{X_n} = (X_{1n}, \cdots, X_{pn})$, $n \geqq 1$,

$$\int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} P(X_1 > s_1, X_2 > s_2, \cdots, X_p > s_p \mid \theta \in I_i) \, ds_p \cdots ds_1$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} \lim_{n \to \infty} P(X_{1n} > s_1, X_{2n} > s_2, \cdots, X_{pn} > s_p \mid \theta \in I_i) \, ds_p \cdots ds_1$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} \lim_{n \to \infty} \prod_{j=1}^{p} P(X_{jn} > s_j \mid \theta \in I_1) \, ds_p \cdots ds_1$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} \prod_{j=1}^{p} P(X_j > s_j \mid \theta \in I_1) \, ds_p \cdots ds_1$$

$$\geqq \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} \lim_{n \to \infty} \prod_{j=1}^{p} P(X_{jn} > s_j \mid \theta \in I_2) \, ds_p \cdots ds_1$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} \prod_{j=1}^{p} P(X_j > s_j \mid \theta \in I_2) \, ds_p \cdots ds_1$$

$$\leq \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} \lim_{n \to \infty} \prod_{j=1}^{p} P(X_{jn} > s_j \mid \theta \in I_3)\, ds_p \cdots ds_1$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} \prod_{j=1}^{p} P(X_j > s_j \mid \theta \in I_3)\, ds_p \cdots ds_1$$

The next theorem demonstrates the preservation of the conditionally weak dependent property under convex combinations on I_1, I_2 and I_3.

Theorem 2.6. Let H_1 and H_2 be two multivariate θ conditionally weak independent and positive and negative quadrant dependent distributions on I_1, I_2 and I_3 both having the same one-dimensional marginals. If $H_\alpha = \alpha H_1 + (1-\alpha) H_2$, $\alpha \in (0,1)$, then H_α is also θ conditionally weak independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Proof. By definition, the one-dimensional marginals of H_α are the same as those of H_1 and H_2. Also

$$\int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} P_{\overline{H_\alpha}}(X_1 > s_1, \cdots, X_p > s_p \mid \theta \in I_i)\, ds_p \cdots ds_1$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} [\alpha P_{\overline{H_1}}(X_1 > s_1, \cdots, X_p > s_p \mid \theta \in I_i)$$
$$+ (1-\alpha) P_{\overline{H_2}}(X_1 > s_1, \cdots, X_p > s_p \mid \theta \in I_i)]\, ds_p \cdots ds_1$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} [\alpha \prod_{j=1}^{p} P_{\overline{H_1}}(X_j > s_j \mid \theta \in I_1) + (1-\alpha) \prod_{j=1}^{p} P_{\overline{H_2}}(X_j > s_j \mid \theta \in I_1)]\, ds_p \cdots ds_1$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} \prod_{j=1}^{p} P_{\overline{H_\alpha}}(X_j > s_j \mid \theta \in I_1)\, ds_p \cdots ds_1$$

$$\geq \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} [\alpha \prod_{j=1}^{p} P_{\overline{H_1}}(X_j > s_j \mid \theta \in I_2) + (1-\alpha) \prod_{j=1}^{p} P_{\overline{H_2}}(X_j > s_j \mid \theta \in I_2)]\, ds_p \cdots ds_1$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} \prod_{j=1}^{p} P_{\overline{H_\alpha}}(X_j > s_j \mid \theta \in I_2)\, ds_p \cdots ds_1$$

$$\leq \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} [\alpha \prod_{j=1}^{p} P_{\overline{H_1}}(X_j > s_j \mid \theta \in I_3)\, ds_p \cdots ds_1$$
$$+ (1-\alpha) \prod_{j=1}^{p} P_{\overline{H_2}}(X_j > s_j \mid \theta \in I_3)]\, ds_p \cdots ds_1$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} \prod_{j=1}^{p} P_{\overline{H_\alpha}}(X_j > s_j \mid \theta \in I_3)\, ds_p \cdots ds_1.$$

Hence H_α is θ conditionally weak independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

We need a definition as follows for proving the Theorem 2.7.

Definition 2.7 (Brady and Singpurwalla, 1990). A sequence of random variables $X_1, X_2, \cdots, X_n$ is θ conditionally right corner set dependent(CRCSD) on I_1, I_2 and I_3 if

$P(\bigcap_{i=1}^{n}(X_i > x_i) \mid \bigcap_{i=1}^{n}(X_i > x_i'), \theta \in I_1)$ is constant in $x_1', \cdots, x_n'$ for every choices of $x_1, \cdots, x_n$,

$P(\bigcap_{i=1}^{n}(X_i > x_i) \mid \bigcap_{i=1}^{n}(X_i > x_i'), \theta \in I_2)$ is increasing in $x_1', \cdots, x_n'$ for every choices of $x_1, \cdots, x_n$,

and

$P(\bigcap_{i=1}^{n}(X_i > x_i) \mid \bigcap_{i=1}^{n}(X_i > x_i'), \theta \in I_3)$ is decreasing in $x_1', \cdots, x_n'$ for every choices of $x_1, \cdots, x_n$.

For $n = 2$, we say that X_1 and X_2 are θ right corner set dependent which is similar to the right corner set dependent condition discussed by Barlow and Proschan(1975).

The following theorem provides a characterization of CRCSD in the multivariate case.

Theorem 2.8. If $X_1, X_2, \cdots, X_n$ are CRCSD on I_1, I_2 and I_3 and $g_i : R \to R$ be a Boreal measurable strictly increasing function for each $i = 1, 2, \ldots, n$. Define $Y_i = g_i(X_i), i = 1, \cdots, n$. Then $Y_1, \cdots, Y_n$ are CRCSD on I_1, I_2 and I_3.

Proof. Let $P(\bigcap_{i=1}^{n}(X_i > x_i) \mid \bigcap_{i=1}^{n}(X_i > x_i'), \theta \in I_3)$ is decreasing in $x_1', \cdots, x_n'$ for all choices of $x_1, \cdots, x_n$ and for $i = 1, \ldots, n$, $y_i' = g_i(x_i')$ and $y_i = g_i(x_i)$.

Then for fixed j,

$P(\bigcap_{i=1}^{n}(X_i > x_i) \mid \bigcap_{i=1}^{n}(X_i > x_i'), \theta \in I_3)$ is decreasing in $x_1', \cdots, x_n'$ for all choices of x_j,

$\Leftrightarrow P(\bigcap_{i=1}^{n}(g(X_i) > g(x_i)) \mid \bigcap_{i=1}^{n}(g(X_i) > g(x_i')), \theta \in I_3)$ is decreasing in $g(x_1'), \cdots,$ $g(x_n')$ for all choices of $g(x_j)$,

$\Leftrightarrow P(\bigcap_{i=1}^{n}(Y_i > y_i) \mid \bigcap_{i=1}^{n}(Y_i > y_i'), \theta \in I_3)$ is decreasing in $y_1', \cdots, y_n'$ for all choices of y_j'.

Now letting $y_i' \to -\infty$ for all $i = 1, \cdots, j-1$, we obtain

$$P(Y_j > y_j \mid Y_{j-1} > y_{j-1}', \cdots, Y_1 > y_1', \theta \in I_3)$$

is decreasing in $y_1', \cdots, y_n'$ for all choices of y_j. Similarly, one handless the case for $\theta \in I_1$ and $\theta \in I_2$.

We now define the θ conditionally right tail dependent in sequence(CRTDS) on I_1, I_2 and I_3 for proving the Theorem 2.11.

Definition 2.9 (Brady and Singpurwalla, 1990). A sequence of random variables $X_1, X_2, \cdots, X_n$ is θ conditionally right tail dependent in sequence(CRTDS) on I_1, I_2 and I_3 if for $i = 2, \ldots, n$, $P(X_i > x_i \mid X_1 > x_1, X_2 > x_2, \cdots, X_{i-1} > x_{i-1}, \theta \in I_1)$ is constant in $x_1, x_2, \cdots, x_{i-1}$, $P(X_i > x_i \mid X_1 > x_1, X_2 > x_2, \cdots, X_{i-1} > x_{i-1}, \theta \in I_2)$ is increasing in $x_1, x_2, \cdots, x_{i-1}$, and
$P(X_i > x_i \mid X_1 > x_1, X_2 > x_2, \cdots, X_{i-1} > x_{i-1}, \theta \in I_3)$ is decreasing in $x_1, x_2, \cdots, x_{i-1}$.

For $n = 2$, we say that X_1 and X_2 are θ right tail dependent condition which is similar to the right tail dependent condition discussed by Barlow and Proschan (1975).

Brady and Singpurwalla(1990) have introduced another notions of dependence by generalizing the idea of positive regression dependence of Lehmann(1966).

Definition 2.10 (Brady and Singpurwalla, 1990). A sequence of random variables $X_1, X_2, \cdots, X_n$ with density function $f : R^n \to [0, \infty]$ is θ conditionally totally dependent of order 2 in pairs (CTD2P) on I_1, I_2 and I_3 if for any pair (x_i, x_j), $i \neq j$, $f(x_1, x_2, \cdots, x_n)$ considered as a function of (x_i, x_j) with the other arguments held fixed satisfies

$$f(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n \mid \theta \in I_1) f(x_1, \cdots, x_i', \cdots, x_j', \cdots, x_n \mid \theta \in I_1) \tag{2.1}$$
$$= f(x_1, \cdots, x_i', \cdots, x_j, \cdots, x_n \mid \theta \in I_1) f(x_1, \cdots, x_i, \cdots, x_j', \cdots, x_n \mid \theta \in I_1),$$

$$f(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n \mid \theta \in I_2) f(x_1, \cdots, x_i', \cdots, x_j', \cdots, x_n \mid \theta \in I_2) \tag{2.2}$$
$$\geq f(x_1, \cdots, x_i', \cdots, x_j, \cdots, x_n \mid \theta \in I_2) f(x_1, \cdots, x_i, \cdots, x_j', \cdots, x_n \mid \theta \in I_2),$$

and

$$f(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n \mid \theta \in I_3) f(x_1, \cdots, x_i', \cdots, x_j', \cdots, x_n \mid \theta \in I_3) \tag{2.3}$$
$$\leq f(x_1, \cdots, x_i', \cdots, x_j, \cdots, x_n \mid \theta \in I_3) f(x_1, \cdots, x_i, \cdots, x_j', \cdots, x_n \mid \theta \in I_3)$$

for every choice of $x_i < x_i'$ and $x_j < x_j'$.

For $n=2$, we say that X_1 and X_2 are θ conditionally totally dependent of order 2(CTD2) on I_1, I_2 and I_3.

The next theorem proves a CRTDS property for a sequence of random variables when the tail of the distribution function satisfies properties similar to (2.1), (2.2) and (2.3).

Theorem 2.11. Let (a)$f(x_1,\cdots,x_n)$ denote the joint p.d.f. of $(X_1, X_2, \cdots, X_n)$ satisfying (2.1), (2.2), and (2.3) in every pair of arguments when the remaining arguments are hold fixed. Assumed that (b) all the marginals $f_k(x_1,\cdots,x_k), 1 \le k < n$ satisfy analogous version of (2.1), (2.2) and (2.3) for every pair of arguments when the remaining arguments are held fixed. Then $(X_1, X_2, \cdots, X_n)$ is θ conditionally right tail dependent in sequence on I_1, I_2 and I_3.

Proof. Fix $x_3, \cdots, x_n$ each at $-\infty$. Then $f_2(x_1, x_2)$ satisfies an analogous version of Definition 2.9 in $-\infty < x_1, x_2 < \infty$, so that (X_1, X_2) is θ conditionally totally dependent of order 2(CTD2). Again for fixed x_2, $f_3(x_1, x_2, x_3)$ satisfies an analogous version of Definition 2.9 in $-\infty < x_1, x_3 < \infty$. Hence, for fixed x_2, $P(X_3 > x_3 \mid X_1 > x_1, X_2 > x_2, \theta \in I_3)$ is decreasing in x_1 for all x_3. By symmetry, $P(X_3 > x_3 \mid X_1 > x_1, X_2 > x_2, \theta \in I_3)$ is decreasing in x_2 for all x_3. It follows $P(X_3 > x_3 \mid X_1 > x_1, X_2 > x_2, \theta \in I_3)$ is decreasing in x_1, x_2 for all choices of x_3, Repetition of this argument yields the desired result that $P(X_i > x_i \mid X_1 > x_1, \cdots, X_{i-1} > x_{i-1}, \theta \in I_3)$ is decreasing in $x_1, \cdots, x_{i-1}$ for each $i = 2, \cdots, n$. Similarly, one handless the case for $\theta \in I_1$ and $\theta \in I_2$.

Finally, we now show that conditionally right tail dependent in sequence implies conditionally weak independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Theorem 2.12. Let $X_1, X_2, \cdots, X_n$ be θ conditionally right tail dependent in sequence on I_1, I_2 and I_3, then $X_1, X_2, \cdots, X_n$ are θ conditionally weak independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Proof. $\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} P(\bigcap_{j=1}^{n} (X_j > s_j) \mid \theta \in I_i) ds_n \cdots ds_1$

$$= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} P(X_1 > s_1 \mid \theta \in I_i) \prod_{k=2}^{n} P(X_k > s_k \mid \bigcap_{j=1}^{k-1} X_j > s_j, \theta \in I_i) ds_n \cdots ds_1$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \prod_{k=1}^{n} P(X_k > s_k \mid \theta \in I_1) ds_n \cdots ds_1$$

$$\geq \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \prod_{k=1}^{n} P(X_k > s_k \mid \theta \in I_2) ds_n \cdots ds_1$$

$$\leq \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \prod_{k=1}^{n} P(X_k > s_k \mid \theta \in I_3) ds_n \cdots ds_1$$

taking $s_j \rightarrow -\infty\ (j = 1, \cdots, k-1)$.

3. Conclusions

In this paper, we introduce a new weak concept in the theory of dependent and independent probability and in the areas in which positive and negative and independence are applied, such as reliability theory. Although this concept is called "new", it is really fundamental to the theory of the relationship between random variables. In section 2, we have further extended this theory to weaker theory of dependence similar to those in the literature of positive and negative and independent dependence and developed theorems which relate these theory.

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