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Test for the Exponential Distribution Based on Multiply Type-II Censored Samples

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Abstract

In this paper, we develope three modified empirical distribution function type tests, the modified Cramer-von Mises test, the modified Anderson-Darling test, and the modified Kolmogorov-Smirnov test for the two-parameter exponential distribution with unknown parameters based on multiply Type-II censored samples.

For each test, Monte Carlo techniques are used to generate the critical values. The powers of these tests are also investigated under several alternative distributions.

Keywords : Anderson-Darling test; approximate maximum likelihood estimator; Cramer-von Mises test; exponential distribution; Kolmogorov-Smirnov test; multiply Type-II censored sample.

1. Introduction

The probability density function (PDF) and the cumulative distribution function (CDF) of two-parameter exponential distribution ($\mathrm{Exp}(\sigma,\theta)$) are given by

$$f_X(x;\theta,\sigma)=\frac{1}{\sigma}\exp\left(-\frac{x-\theta}{\sigma}\right), \quad x>\theta,\ \sigma>0 \tag{1.1}$$

and

$$F_X(x;\theta,\sigma)=1-\exp\left(-\frac{x-\theta}{\sigma}\right), \quad x>\theta,\ \sigma>0. \tag{1.2}$$

The problem of estimating parameters based on censored samples have been investigated by many authors. Especially, the approximate maximum likelihood estimating method was first developed by Balakrishnan (1989) for the purpose of providing the explicit estimators of the scale parameter in the Rayleigh distribution. Lin and Balakrishnan (2003) developed the exact prediction intervals for the failure times from the one-parameter and two-parameter exponential distributions based on doubly Type-II censored samples and they presented a

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computational algorithm for the determination of the exact percentage points of the pivotal quantities used in the construction of the prediction intervals.

Multiply Type-II censored sampling arises in life-testing experiments when the failure times of some units were not observed due to mechanical or experimental difficulties. Also, another multiply censored samples arise naturally when some units failed between two points of observation with exact times of failure of these units unobserved.

It has been noted that in most cases, the maximum likelihood method does not provide explicit estimators based on complete and censored samples. Especially, when the sample is multiply censored, the maximum likelihood method does not admit explicit solutions. Hence it is desirable to develop approximations to this maximum likelihood method which would provide us with estimators for the location and scale parameters that are explicit functions of order statistics.

Balasubramanian and Balakrishnan (1992) and Upadhyay et al. (1996) considered some estimations for the exponential distribution under multiply Type-II censoring. Kang (2003) proposed the approximate maximum likelihood estimators (AMLEs) of the location and the scale parameters of the two-parameter exponential distribution with multiply Type-II censoring. Recently, Kang and Lee (2005) derived the AMLEs of the scale and location parameters of the two-parameter exponential distribution based on multiply Type-II censored samples. They also obtained the moments of the proposed estimators.

Porter III et al. (1992) developed three modified Kolmogorov-Smirnov, Anderson-Darling, and Cramer-von Mises tests for the Pareto distribution with unknown parameters of the location and scale and known shape parameter based on the complete samples. Puig and Stephens (2000) studied some tests of fit for the Laplace distribution based on the empirical distribution function (EDF) statistics and the application of the Laplace distribution in the least absolute deviations regression.

In this paper, we propose three modified EDF type tests, including the modified Cramer-von Mises test, the modified Anderson-Darling test, and the modified Kolmogorov-Smirnov test for the exponential distribution with unknown parameters based on multiply Type-II censored samples using the AMLEs which were obtained by Kang and Lee (2005).

For each test, Monte Carlo techniques are used to generate the critical values. The powers of these tests are also investigated under normal, lognormal, beta, exponential distributions.

2. Approximate Maximum Likelihood Estimators

We assume that n items are put on a life test, but only a_1th,..., a_sth failures are observed, the rest are unobserved or missing, where $a_1, \dots, a_s$ are considered to be fixed. If this censoring arises, the scheme is known as multiply Type-II censoring scheme.

Let us assume that the following multiply Type-II censored sample from a sample of size n is

$$X_{a_1:n} \le X_{a_2:n} \le \cdots \le X_{a_s:n} \tag{2.1}$$

where $1 \le a_1 < a_2 < \cdots < a_s \le n$ and $X_{1:n}, \dots, X_{n:n}$ are order statistics of $X_1, \dots, X_n$.

Kang and Lee (2005) proposed several estimators for the scale parameter and location parameter of the two-parameter exponential distribution based on multiply Type-II censored sample (2.1).

For the location parameter θ, Kang and Lee (2005) proposed some estimators as follows;

$$\hat{\theta}_1 = X_{a_1:n} \tag{2.2}$$

$$\hat{\theta}_2 = \frac{1}{h(a_2) - h(a_1)} \left[h(a_2) X_{a_1:n} - h(a_1) X_{a_2:n} \right] \tag{2.3}$$

$$\hat{\theta}_3 = [1 - (s-1)d] X_{a_1:n} + d \sum_{j=2}^{s} X_{a_j:n} \tag{2.4}$$

where

$$h(a) = \sum_{j=1}^{a} (n-j+1)^{-1}$$

$$g(a) = \sum_{j=1}^{a} (n-j+1)^{-2}$$

$$v = (s-1)^2 \left[h^2(a_1) - g(a_1) \right] + \sum_{j=2}^{s} g(a_j) + 2 \sum_{j=1}^{s-1} (s-j) g(a_j) + \left[\sum_{j=2}^{s} h(a_j) \right]^2 - 2(s-1) h(a_1) \sum_{j=2}^{s} h(a_j)$$

and

$$d = \frac{h(a_1) \left[(s-1) h(a_1) - \sum_{j=2}^{s} h(a_j) \right]}{v}.$$

They showed that the estimator $\hat{\theta}_3$ is more efficient than the other estimators of the location parameter, and the values of the mean squared errors of $\hat{\theta}_1$ and $\hat{\theta}_2$ are almost same in most cases. Since the estimator $\hat{\theta}_1$ overestimate the location parameter θ, we modify $\hat{\theta}_1 = X_{a_1:n}$ by $\hat{\theta}_1 = X_{a_1:n} - h(a_1)$ and use this modified estimator $\hat{\theta}_1$ and the AMLEs $\hat{\theta}_2$ and $\hat{\theta}_3$ to test the exponential distribution based on multiply Type-II censored sample.

Kang and Lee (2005) also proposed several estimators of the scale parameter σ as follows;

$$\hat{\sigma}_{1i} = \frac{-B_{1i} + \sqrt{B_{1i}^2 - 4s\,C_{1i}}}{2s}, \quad i = 0, 1, 2, 3. \tag{2.5}$$

where

$$B_{1i} = (a_1 - 1)\alpha_1 X_{a_1:n} - (n - a_s) X_{a_s:n} - \sum_{j=1}^{s} X_{a_j:n}$$
$$+ \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\alpha_{1j} X_{a_j:n} - \alpha_{2j} X_{a_{j-1}:n})$$
$$- \left[(a_1 - 1)\alpha_1 - (n - a_s) - s + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\alpha_{1j} - \alpha_{2j}) \right] \hat{\theta}_i$$

$$C_{1i} = \sum_{j=2}^{s} (a_j - a_{j-1} - 1)[\beta_{1j}(X_{a_j:n} - \hat{\theta}_i)^2 + 2\gamma_{1j}(X_{a_j:n} - \hat{\theta}_i)(X_{a_{j-1}:n} - \hat{\theta}_i)$$
$$- \gamma_{2j}(X_{a_{j-1}:n} - \hat{\theta}_i)^2] + (a_1 - 1)\beta_1(X_{a_1:n} - \hat{\theta}_i)^2$$

$$f(z) = e^{-z}$$

$$p_i = \frac{i}{n+1}$$

$$\xi_{a_j} = F^{-1}(p_{a_j}) = -\ln(1 - p_{a_j}),$$

$$\alpha_1 = \frac{f(\xi_{a_1})}{p_{a_1}} \left[1 + \xi_{a_1} + \frac{f(\xi_{a_1})}{p_{a_1}} \xi_{a_1} \right]$$

$$\beta_1 = -\frac{f(\xi_{a_1})}{p_{a_1}} \left[1 + \frac{f(\xi_{a_1})}{p_{a_1}} \right]$$

$$\alpha_{1j} = \frac{f(\xi_{a_j})}{p_{a_j} - p_{a_{j-1}}} \left(1 + \xi_{a_j} + \frac{f(\xi_{a_j})\xi_{a_j} - f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_j} - p_{a_{j-1}}} \right)$$

$$\beta_{1j} = -\frac{f(\xi_{a_j})}{p_{a_j} - p_{a_{j-1}}} \left(1 + \frac{f(\xi_{a_j})}{p_{a_j} - p_{a_{j-1}}} \right)$$

$$\gamma_{1j} = \frac{f(\xi_{a_j}) f(\xi_{a_{j-1}})}{(p_{a_j} - p_{a_{j-1}})^2}$$

$$\alpha_{2j} = \frac{f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \left(1 + \xi_{a_{j-1}} + \frac{f(\xi_{a_j})\xi_{a_j} - f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_j} - p_{a_{j-1}}} \right)$$

$$\beta_{2j} = -\frac{f(\xi_{a_j}) f(\xi_{a_{j-1}})}{(p_{a_j} - p_{a_{j-1}})^2} = -\gamma_{1j}$$

$$\gamma_{2j} = -\frac{f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \left(1 - \frac{f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \right)$$

and $\hat{\theta}_0 = \theta_0$ is known location parameter. They also proposed the other simple estimator which is the linear function of the available order statistics as follows;

$$\hat{\sigma}_{2i} = -\frac{B_{2i}}{A_2} \tag{2.6}$$

where

$$A_2 = s + (a_1 - 1)\alpha_2 + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)\alpha_{3j}$$

$$B_{2i} = (a_1 - 1)\beta_2 X_{a_1:n} - (n - a_s) X_{a_s:n} - \sum_{j=1}^{s} X_{a_j:n}$$
$$+ \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\beta_{3j} X_{a_j:n} + \gamma_{3j} X_{a_{j-1}:n})$$
$$- \left[(a_1 - 1)\beta_2 - (n - a_s) - s + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\beta_{3j} + \gamma_{3j}) \right] \hat{\theta}_i$$

$$\alpha_2 = \frac{f(\xi_{a_1})}{p_{a_1}} \xi_{a_1} \left[\frac{f(\xi_{a_1})}{p_{a_1}} \xi_{a_1} + \xi_{a_1} \right]$$

$$\beta_2 = \frac{f(\xi_{a_1})}{p_{a_1}} \left[1 - \xi_{a_1} - \frac{f(\xi_{a_1})}{p_{a_1}} \xi_{a_1} \right]$$

$$\alpha_{3j} = \frac{f(\xi_{a_j})\xi_{a_j}^{\ 2} - f(\xi_{a_{j-1}})\xi_{a_{j-1}}^{\ 2}}{p_{a_j} - p_{a_{j-1}}} + \left(\frac{f(\xi_{a_j})\xi_{a_j} - f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_j} - p_{a_{j-1}}} \right)^2$$

$$\beta_{3j} = \frac{f(\xi_{a_j})}{p_{a_j} - p_{a_{j-1}}} \left(1 - \xi_{a_j} - \frac{f(\xi_{a_j})\xi_{a_j} - f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_j} - p_{a_{j-1}}} \right)$$

and

$$\gamma_{3j} = -\frac{f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \left(1 - \xi_{a_{j-1}} - \frac{f(\xi_{a_j})\xi_{a_j} - f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_j} - p_{a_{j-1}}} \right)$$

Kang and Lee (2005) showed that the estimator $\hat{\sigma}_{20}$ is generally more efficient than the estimator $\hat{\sigma}_{10}$ for known location parameter, and the estimator $\hat{\sigma}_{23}$ is more efficient than the other estimators for unknown location parameter. We will use the estimators $\hat{\theta}_l$, $\hat{\sigma}_{kl}$, $k = 1,2$; $l = 0,1,2,3$ in the goodness-of-fit tests.

3. Goodness of fit tests

A well known empirical distribution function $F_n(x)$ is

$$F_n(x) = \frac{\#[X_i \le x]}{n}, \quad -\infty \le x \le \infty. \tag{3.1}$$

Now we consider well known three type tests, including Kolmogorov-Smirnov, Cramer-von Mises, and Anderson-Darling test as follows;

The Kolmogorov-Smirnov test statistics are defined by

$$D^+ = \sup_x [F_n(x) - F_0(x)]$$
$$D^- = \sup_x [F_0(x) - F_n(x)] \tag{3.2}$$
$$D = \max[D^+, D^-],$$

the Cramer-von Mises test statistic is generally defined by

$$W^2 = n\int_{-\infty}^{\infty} [F_n(x) - F_0(x)]^2 dF_0(x), \tag{3.3}$$

and the Anderson-Darling test statistic is defined by

$$A^2 = n\int_{-\infty}^{\infty} [F_n(x) - F_0(x)]^2 \psi(x)\, dF_0(x), \tag{3.4}$$

where $\psi(x) = F_0(x)[1 - F_0(x)]^{-1}$ and $F_0(x)$ is the CDF assumed under H_0.

Porter III et al. (1992) developed three modified EDF type tests, the Kolmogorov-Smirnov test, Anderson-Darling test, and Cramer-von Mises test for the Pareto distribution with unknown parameters of location and scale and known shape parameter with complete sample.

Let $U_i = F_0(X_i)$ then $U_i \sim \mathrm{U}(0,1)$. But if $F_0(x)$ contains unknown parameters, we denote the CDF $F_0(x)$ by $F_0(x;\theta_1, \dots \theta_h)$ and we must estimate the parameters from the sample and the estimated values are used in $F_0(x;\theta_1, \dots \theta_h)$ to make the transformation $U_{i:n} = F_0(X_{i:n};\theta_1, \dots \theta_h)$, for $i = 1, \dots, n$.

Then the Kolmogorov-Smirnov statistics are computed from

$$D^+ = \max_{1 \le i \le n}\left(\frac{i}{n} - u_{i:n}\right)$$
$$D^- = \max_{1 \le i \le n}\left(u_{i:n} - \frac{i}{n}\right) \tag{3.5}$$
$$D = \max[D^+, D^-],$$

the Anderson-Darling statistic is computed from

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln u_{i:n} + \ln(1 - u_{n+1-i:n})\right], \tag{3.6}$$

and the Cramer-von Mises statistic is computed from

$$W^2 = \sum_{i=1}^{n}\left(u_{i:n} - \frac{2i-1}{2n}\right)^2 + \frac{1}{12n}. \tag{3.7}$$

In the form given above, test statistics can only be used with complete samples, i.e. no censoring. Modification of the test statistics for censored samples and for composite hypothesis H_0 with unspecified parameters has been studied by Pettitt (1976) and Pettitt and Stephens (1976).

Now, we propose three EDF type modified Kolmogorov-Smirnov test, modified Anderson-Darling test, and modified Cramer-von Mises test for multiply Type-II censored samples with the AMLEs $\hat{\theta}_l$, $\hat{\sigma}_{kl}$, $k=1,2;\ l=0,1,2,3$.

The AMLEs $\hat{\sigma}_{kl}$ and $\hat{\theta}_l$ are used to obtain the hypothesized CDF as follows;

$$P_{a_j,k,l}=F(x_{a_j:n};\hat{\sigma}_{kl},\hat{\theta}_l),\quad j=1,\dots,s;\ k=1,2;\ l=0,1,2,3.$$

We propose the test statistics for testing the following hypotheses;

H_0 : the multiply Type-II censored sample comes from the exponential distribution with PDF (1.1).

H_1 : not H_0

The modified Kolmogorov-Smirnov test statistics based on multiply Type-II censored samples are given by

$$\begin{aligned} D_{k,l}^{+} &= \max_{1\le a_j\le s}\left(\frac{a_j}{n}-P_{a_j,k,l}\right)\\ D_{k,l}^{-} &= \max_{1\le a_j\le s}\left(P_{a_j,k,l}-\frac{a_j}{n}\right)\\ D_{k,l} &= \max\left[D_{k,l}^{+},\ D_{k,l}^{-}\right]. \end{aligned} \tag{3.8}$$

The modified Anderson-Darling test statistic based on multiply Type-II censored samples is given by

$$A_{k,l}^2=-s-\frac{1}{s}\sum_{j=1}^{s}(2a_j-1)\left[\ln P_{a_j,k,l}+\ln(1-P_{a_{s+1-j},k,l})\right]. \tag{3.9}$$

The modified Cramer-von Mises test statistic based on multiply Type-II censored samples is given by

$$W_{k,l}{}^2=\frac{1}{12n}+\sum_{j=1}^{s}\left(P_{a_j,k,l}-\frac{2a_j-1}{2n}\right)^2. \tag{3.10}$$

This procedure was repeated 10,000 times for the modified Kolmogorov-Smirnov, modified Anderson-Darling, and modified Cramer-von Mises goodness-of-fit tests for the two-parameter exponential distribution based on multiply Type-II censored samples, each sample size $n=20$ and 50, and $k=1,2$, and $l=0,1,2,3$. The 10,000 test statistics were then ordered for each of three tests. The 95th percentile were found and these critical values ($\alpha=0.05$) are given in <Table 1>.

The probabilities of Type I error for the modified EDF type tests under exponential distribution (Exp(1,0)) are given in <Table 2>.

As expected, when underlying distribution is exponential, the probabilities of Type I error for three tests are close to the nominal significance level 0.05.

The powers of three modified tests with significance level 0.05 for the two-parameter exponential distribution based on multiply Type-II censored samples are investigated under 3 alternative distributions. These values are given in <Table 3>.

From <Table 3>, when the alternative distribution is beta distribution, the tests that use the estimator $\hat{\sigma}_{20}$ are more powerful than the tests that use the estimator $\hat{\sigma}_{10}$ for known location parameter, and the tests that use the modified estimator $\hat{\theta}_1$ are more powerful than the tests that use other estimators for unknown location parameter, and the modified Cramer-von Mises test and the modified Anderson-Darling test are more powerful than the modified Kolmogorov-Smirnov test except tests that use the estimator $\hat{\theta}_2$.

For normal alternative distribution, three tests are very good when the location parameter is known, and in most case the tests that use the estimator $\hat{\theta}_3$ are more powerful than the tests that use other estimators except the modified Kolmogorov-Smirnov test and small sample($n=20$) when the location parameter is unknown, and the modified Cramer-von Mises test is generally more powerful than the other tests, and the modified Anderson-Darling test is more powerful than the modified Kolmogorov-Smirnov test except tests that use the estimator $\hat{\theta}_2$.

For lognormal alternative distribution, all the tests show poor performance (The powers are less than or equal to 0.4339).

The power generally decrease as $m=n-s$ increase or sample size n decrease, where $m=n-s$ is the number of unobserved or missing data.

<Table 1> Critical values for the modified EDF type tests ($\alpha=0.05$)

n	m	a_j	The modified Kolmogorov-Smirnov test							
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{21}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	0.213	0.207	0.223	0.208	0.213	0.207	0.223	0.208
	2	1~18	0.205	0.199	0.216	0.201	0.205	0.199	0.216	0.201
		2~19	0.211	0.198	0.230	0.200	0.209	0.197	0.234	0.200
	5	3~17	0.202	0.183	0.245	0.188	0.199	0.182	0.240	0.187
50	0	1~50	0.143	0.142	0.147	0.142	0.143	0.142	0.147	0.142
	2	1~48	0.141	0.140	0.145	0.140	0.141	0.140	0.145	0.140
		2~49	0.143	0.139	0.150	0.139	0.142	0.139	0.151	0.139
	5	2~6 10~19 21~50	0.146	0.142	0.153	0.142	0.142	0.139	0.152	0.139

<Table 1> Continued

n	m	a_j	The modified Cramer-von Mises test							
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{21}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	0.216	0.198	0.250	0.198	0.216	0.198	0.250	0.198
	2	1~18	0.193	0.175	0.223	0.178	0.193	0.175	0.223	0.178
		2~19	0.197	0.168	0.254	0.170	0.202	0.168	0.271	0.170
	5	3~17	0.160	0.129	0.254	0.133	0.168	0.132	0.262	0.132
50	0	1~50	0.222	0.213	0.242	0.214	0.222	0.213	0.242	0.214
	2	1~48	0.212	0.203	0.233	0.204	0.212	0.203	0.233	0.204
		2~49	0.214	0.199	0.249	0.200	0.216	0.199	0.259	0.200
	5	2~6 10~19 21~50	0.203	0.184	0.234	0.184	0.197	0.183	0.230	0.183

n	m	a_j	The modified Anderson-Darling test							
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{21}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	1.300	1.088	1.602	1.085	1.300	1.088	1.602	1.085
	2	1~18	1.618	1.390	1.971	1.418	1.618	1.390	1.971	1.418
		2~19	4.963	4.651	5.849	4.663	4.988	4.657	5.860	4.663
	5	3~17	8.409	7.937	10.434	7.975	8.291	7.930	10.028	7.974
50	0	1~50	1.314	1.202	1.529	1.209	1.314	1.202	1.529	1.209
	2	1~48	1.491	1.375	1.699	1.390	1.491	1.375	1.699	1.390
		2~49	5.129	4.973	5.594	4.978	5.150	4.974	5.636	4.978
	5	2~6 10~19 21~50	14.307	14.256	14.565	14.257	14.604	14.483	14.966	14.473

<Table 2> The probabilities of Type I error for the modified EDF type tests under exponential distribution (Exp(1,0)).

n	m	a_j	The modified Kolmogorov-Smirnov test							
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{21}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	0.0510	0.0487	0.0505	0.0476	0.0510	0.0487	0.0505	0.0476
	2	1~18	0.0507	0.0492	0.0541	0.0524	0.0507	0.0492	0.0541	0.0524
		2~19	0.0474	0.0456	0.0478	0.0451	0.0476	0.0454	0.0465	0.0451
	5	3~17	0.0506	0.0480	0.0519	0.0485	0.0516	0.0474	0.0529	0.0488
50	0	1~50	0.0442	0.0428	0.0440	0.0425	0.0442	0.0428	0.0440	0.0425
	2	1~48	0.0449	0.0448	0.0433	0.0447	0.0449	0.0448	0.0433	0.0447
		2~49	0.0444	0.0437	0.0450	0.0439	0.0453	0.0434	0.0435	0.0439
	5	2~6 10~19 21~50	0.0438	0.0429	0.0441	0.0429	0.0448	0.0424	0.0446	0.0428

n	m	a_j	The modified Cramer-von Mises test							
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{21}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	0.0498	0.0488	0.0522	0.0495	0.0498	0.0488	0.0522	0.0495
	2	1~18	0.0484	0.0472	0.0534	0.0465	0.0484	0.0472	0.0534	0.0465
		2~19	0.0444	0.0453	0.0449	0.0459	0.0458	0.0458	0.0456	0.0458
	5	3~17	0.0530	0.0488	0.0515	0.0494	0.0502	0.0478	0.0526	0.0488
50	0	1~50	0.0424	0.0424	0.0435	0.0436	0.0424	0.0424	0.0435	0.0436
	2	1~48	0.0446	0.0444	0.0462	0.0455	0.0446	0.0444	0.0462	0.0455
		2~49	0.0418	0.0439	0.0469	0.0458	0.0420	0.0439	0.0452	0.0458
	5	2~6 10~19 21~50	0.0420	0.0433	0.0475	0.0439	0.0417	0.0410	0.0454	0.0426

n	m	a_j	The modified Anderson-Darling test							
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{21}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	0.0509	0.0472	0.0512	0.0475	0.0509	0.0472	0.0512	0.0475
	2	1~18	0.0520	0.0526	0.0506	0.0534	0.0520	0.0526	0.0506	0.0534
		2~19	0.0483	0.0468	0.0489	0.0445	0.0482	0.0470	0.0482	0.0445
	5	3~17	0.0515	0.0510	0.0532	0.0497	0.0516	0.0500	0.0510	0.0495
50	0	1~50	0.0478	0.0426	0.0447	0.0429	0.0478	0.0426	0.0447	0.0429
	2	1~48	0.0437	0.0457	0.0451	0.0466	0.0437	0.0457	0.0451	0.0466
		2~49	0.0450	0.0449	0.0467	0.0443	0.0465	0.0455	0.0460	0.0443
	5	2~6 10~19 21~50	0.0474	0.0462	0.0468	0.0460	0.0465	0.0462	0.0487	0.0448

<Table 3> The powers of the modified EDF type tests for several alternative distributions.

(1) The beta alternative distribution (Beta(3,2))

n	m	a_j	The modified Kolmogorov-Smirnov test							
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{21}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	0.9938	0.7909	0.6983	0.6961	0.9938	0.7909	0.6983	0.6961
	2	1~18	0.9780	0.6610	0.5760	0.5595	0.9780	0.6610	0.5760	0.5595
		2~19	0.9859	0.5565	0.3334	0.4255	0.9851	0.6808	0.4029	0.4259
	5	3~17	0.9284	0.4106	0.0960	0.1799	0.9479	0.5811	0.1578	0.1828
50	0	1~50	1.0000	0.9993	0.9990	0.9989	1.0000	0.9993	0.9990	0.9989
	2	1~48	1.0000	0.9987	0.9975	0.9979	1.0000	0.9987	0.9975	0.9979
		2~49	1.0000	0.9960	0.9887	0.9926	1.0000	0.9984	0.9902	0.9926
	5	2~6 10~19 21~50	1.0000	0.9977	0.9805	0.9888	1.0000	0.9991	0.9932	0.9946

n	m	a_j	The modified Cramer-von Mises test							
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{21}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	1.0000	0.9837	0.9000	0.9315	1.0000	0.9837	0.9000	0.9315
	2	1~18	1.0000	0.9340	0.7961	0.8325	1.0000	0.9340	0.7961	0.8325
		2~19	0.9991	0.9178	0.5934	0.7526	1.0000	0.9881	0.6257	0.7526
	5	3~17	0.9819	0.8183	0.1761	0.4360	0.9975	0.9660	0.2912	0.4394
50	0	1~50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	2	1~48	1.0000	1.0000	0.9999	0.9999	1.0000	1.0000	0.9999	0.9999
		2~49	1.0000	0.9999	0.9992	0.9998	1.0000	1.0000	0.9992	0.9998
	5	2~6 10~19 21~50	1.0000	1.0000	0.9978	0.9994	1.0000	1.0000	0.9998	1.0000

n	m	a_j	The modified Anderson-Darling test							
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{21}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	1.0000	0.9908	0.8362	0.9273	1.0000	0.9908	0.8362	0.9273
	2	1~18	0.9966	0.8208	0.5359	0.6341	0.9966	0.8208	0.5359	0.6341
		2~19	0.9990	0.9581	0.2809	0.7261	0.9999	0.9949	0.3896	0.7265
	5	3~17	0.9510	0.8815	0.0062	0.2594	0.9856	0.9256	0.0297	0.2619
50	0	1~50	1.0000	1.0000	0.9999	1.0000	1.0000	1.0000	0.9999	1.0000
	2	1~48	1.0000	0.9999	0.9992	0.9997	1.0000	0.9999	0.9992	0.9997
		2~49	1.0000	1.0000	0.9964	0.9998	1.0000	1.0000	0.9967	0.9998
	5	2~6 10~19 21~50	1.0000	1.0000	0.9993	0.9998	1.0000	1.0000	1.0000	1.0000

(2) The normal alternative distribution (N(5,1))

n	m	a_j	The modified Kolmogorov-Smirnov test							
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{21}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	1.0000	0.6610	0.6985	0.6752	1.0000	0.6610	0.6985	0.6752
	2	1~18	1.0000	0.5645	0.6281	0.5925	1.0000	0.5645	0.6281	0.5925
		2~19	1.0000	0.3242	0.2949	0.3656	1.0000	0.3418	0.3768	0.3656
	5	3~17	1.0000	0.1112	0.0874	0.1482	1.0000	0.1339	0.1569	0.1518
50	0	1~50	1.0000	0.9973	0.9971	0.9975	1.0000	0.9973	0.9971	0.9975
	2	1~48	1.0000	0.9955	0.9958	0.9961	1.0000	0.9955	0.9958	0.9961
		2~49	1.0000	0.9832	0.9752	0.9869	1.0000	0.9844	0.9785	0.9869
	5	2~6 10~19 21~50	1.0000	0.9639	0.9582	0.9700	1.0000	0.9870	0.9817	0.9888

n	m	a_j	The modified Cramer-von Mises test							
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{21}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	1.0000	0.8797	0.8747	0.8936	1.0000	0.8797	0.8747	0.8936
	2	1~18	1.0000	0.8029	0.8061	0.8213	1.0000	0.8029	0.8061	0.8213
		2~19	1.0000	0.6237	0.5356	0.6839	1.0000	0.6373	0.5747	0.6840
	5	3~17	1.0000	0.3026	0.1742	0.3773	1.0000	0.3231	0.2903	0.3796
50	0	1~50	1.0000	0.9998	0.9998	0.9999	1.0000	0.9998	0.9998	0.9999
	2	1~48	1.0000	0.9997	0.9994	0.9997	1.0000	0.9997	0.9994	0.9997
		2~49	1.0000	0.9978	0.9956	0.9988	1.0000	0.9980	0.9958	0.9988
	5	2~6 10~19 21~50	1.0000	0.9946	0.9905	0.9962	1.0000	0.9986	0.9972	0.9992

n	m	a_j	The modified Anderson-Darling test							
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{21}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	1.0000	0.8643	0.8056	0.8792	1.0000	0.8643	0.8056	0.8792
	2	1~18	1.0000	0.6388	0.5889	0.6668	1.0000	0.6388	0.5889	0.6668
		2~19	1.0000	0.5947	0.2601	0.6529	1.0000	0.6023	0.3690	0.6530
	5	3~17	1.0000	0.1846	0.0069	0.2247	1.0000	0.1998	0.0275	0.2276
50	0	1~50	0.9999	0.9998	0.9993	0.9999	0.9999	0.9998	0.9993	0.9999
	2	1~48	0.9999	0.9983	0.9973	0.9987	0.9999	0.9983	0.9973	0.9987
		2~49	1.0000	0.9976	0.9883	0.9983	1.0000	0.9976	0.9888	0.9983
	5	2~6 10~19 21~50	1.0000	0.9970	0.9937	0.9977	1.0000	0.9997	0.9987	0.9998

(3) The lognormal alternative distribution (LN(0,1))

n	m	a_j	The modified Kolmogorov-Smirnov test							
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{21}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	0.1454	0.1903	0.1523	0.1735	0.1454	0.1903	0.1523	0.1735
	2	1~18	0.0806	0.1045	0.0843	0.0936	0.0806	0.1045	0.0843	0.0936
		2~19	0.1010	0.1730	0.1182	0.1358	0.0977	0.1694	0.1049	0.1354
	5	3~17	0.0516	0.1321	0.0739	0.0877	0.0574	0.1248	0.0717	0.0874
50	0	1~50	0.2618	0.2871	0.2687	0.2816	0.2618	0.2871	0.2687	0.2816
	2	1~48	0.1849	0.1858	0.1754	0.1787	0.1849	0.1858	0.1754	0.1787
		2~49	0.2885	0.2546	0.2200	0.2330	0.2126	0.2506	0.2037	0.2330
	5	2~6 10~19 21~50	0.3614	0.3452	0.2978	0.3170	0.2634	0.3212	0.2583	0.3019

n	m	a_j	The modified Cramer-von Mises test							
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{21}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	0.1450	0.1632	0.1378	0.1494	0.1450	0.1632	0.1378	0.1494
	2	1~18	0.1136	0.0900	0.0787	0.0819	0.1136	0.0900	0.0787	0.0819
		2~19	0.1013	0.1427	0.1071	0.1151	0.1012	0.1367	0.0925	0.1147
	5	3~17	0.0619	0.1059	0.0798	0.0768	0.0779	0.0979	0.0662	0.0758
50	0	1~50	0.2958	0.2738	0.2574	0.2670	0.2958	0.2738	0.2574	0.2670
	2	1~48	0.2330	0.1786	0.1745	0.1737	0.2330	0.1786	0.1745	0.1737
		2~49	0.3279	0.2414	0.2238	0.2186	0.2412	0.2364	0.1931	0.2186
	5	2~6 10~19 21~50	0.4339	0.3652	0.3260	0.3376	0.2883	0.3187	0.2671	0.2973

n	m	a_j	The modified Anderson-Darling test							
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{21}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1~20	0.1349	0.1734	0.1163	0.1556	0.1349	0.1734	0.1163	0.1556
	2	1~18	0.0908	0.1141	0.0735	0.1014	0.0908	0.1141	0.0735	0.1014
		2~19	0.0758	0.1652	0.0863	0.1214	0.0781	0.1598	0.0810	0.1207
	5	3~17	0.0407	0.1527	0.0634	0.0972	0.0501	0.1441	0.0628	0.0972
50	0	1~50	0.3393	0.2720	0.2330	0.2648	0.3393	0.2720	0.2330	0.2648
	2	1~48	0.2938	0.2118	0.1824	0.2052	0.2938	0.2118	0.1824	0.2052
		2~49	0.3162	0.2491	0.1632	0.2219	0.2525	0.2453	0.1492	0.2219
	5	2~6 10~19 21~50	0.1016	0.1406	0.1172	0.1103	0.0853	0.1046	0.0786	0.0810

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