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Bayesian Hypothesis Testing for Intraclass Correlation Coefficient

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Abstract

In this paper, we consider a Bayesian model selection for the intraclass corr-elation coefficient in familiar data. In particular, we compare two nested models such as the independence and intraclass models using the reference prior. A criterion for testing is the Bayesian Reference Criterion by Bernardo (1999) and the Intrinsic Bayes Factor by Berger and Pericchi (1996). We provide numerical examples using simulation data sets for illustration.

Keywords : Intraclass correlation coefficient; Bayesian reference criterion; intrinsic Bayes factor.

1. Introduction

The intraclass correlation coefficient ρ is frequently used to measure the degree of intrafamily resemblance with respect to characteristics such as blood pleasure, cholesterol, weight, height, stature, lung capacity, and so forth. Statistical inference concerning ρ for a single-sample problem based on a normal distribution has been studied by several authors (Scheffe, 1959; Rao, 1973; Rosner, Donner and Hennekens, 1977, 1979; Donner and Bull, 1983; Srivastava, 1984; Konishi, 1985; Gokhale and SenGupta, 1986; Velu and Rao, 1990).

Gokhale and SenGupta (1986) considered the optimal tests for $H_0 : \rho = 0$ against or $H_1 : \rho > 0$, when some or none of the marginal parameters are known.

For testing the equality of two intraclass correlation coefficients based on two independent multinormal samples under equal family sizes, Donner and Bull (1983) discussed the likelihood ratio test. Konishi and Gupta (1989) proposed a modified likelihood ratio test and derived its asymptotic null distribution. They also discussed another test procedure based on a modification of Fisher's z-transformation following Konishi (1985).

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Huang and Sinha (1993) considered an optimum invariant test of equality of intraclass correlation coefficients under equal family sizes for more than two intraclass correlation coefficients based on independent samples from several multinormal distributions.

Young and Bhandary (1998) considered a test of equality of two intraclass correlation coefficients based on two independent multinormal samples under unequal family sizes.

However, Bayesian inference for ρ has received very little attention. We consider the Bayesian approach to hypothesis testing for ρ.

Bernardo (1999) introduced a new model selection criterion, called the Bayesian Reference Criterion (BRC). To decide whether or not some data x are compatible with the null hypothesis $\theta = \theta_0$, assuming that the data have been generated from the model $p_x(\cdot \mid \theta, \omega)$, $\theta \in \Theta$, $\omega \in \Omega$, Bernardo computes the posterior mean of the logarithmic discrepancy,

$$d_r(x, \theta_0) = \int\int \delta(\theta_0, \theta, \omega) \pi_\delta(\theta, \omega \mid x) d\theta d\omega,$$

where,

$$\delta(\theta_0, \theta, \omega) = \inf_{\omega_0 \in \Omega} \int p_x(y \mid \theta, \omega) \log \frac{p_x(y \mid \theta, \omega)}{p_x(y \mid \theta_0, \omega_0)} dy,$$

and $\pi_\delta(\theta, \omega \mid x)$ is the posterior distribution which corresponds to the reference prior $\pi_\delta(\theta, \omega)$ as introduced by Bernardo (1979) and further developed by Berger and Bernardo (1989, 1992). Bernardo proposed rejection of H_0 if and only if this posterior mean exceeds some specified number.

In Bayesian model selection or testing problems, the Bayes factor under proper priors has been shown very successful. In practice, however, limited information and time constraints often requires the use of noninformative priors. Such priors are typically improper and are defined only up to arbitrary constants which affects the values of Bayes factors. Berger and Pericchi (1996) introduced a new model selection and hypotheses testing criterion, called the Intrinsic Bayes Factor (IBF) using a data-splitting idea, which would eliminate the arbitrariness of improper priors.

The outline of the remaining selections is as follows. In Section 2, we derive expressions for the BRC and IBF to solve our problem. In Section 3, we provide numerical examples using simulation data sets for illustration.

2. Bayesian Hypothesis Testing for Intraclass Correlation Coefficient

Let $\boldsymbol{X}_i = (X_{i1}, \cdots, X_{ip})'$ be $p \times 1$ vector of observations from the ith family, $i = 1,\cdots,n$. The structure of the mean vector and the covariance matrix for the familiar data is given by (Rao, 1973) as

$$\boldsymbol{\mu} = \mu \mathbf{1}_p \text{ and } \Sigma = \sigma^2((1-\rho)\boldsymbol{I}_p + \rho \boldsymbol{J}_p),$$

where $\mathbf{1}_p$ is the $p \times 1$ vector of 1's, $\boldsymbol{I}_p$ is the $p \times p$ identity matrix and $\boldsymbol{J}_p$ is the $p \times p$ matrix containing only ones. Here $\mu\,(-\infty < \mu < \infty)$ is the common mean and $\sigma^2\,(\sigma^2 > 0)$ is the common variance of members of the family and ρ, called the intraclass correlation coefficient, is the coefficient of correlation among the members of the family and $-1/(p-1) \leqq \rho \leqq 1$. The parameter of interest is ρ. It is assumed that $\boldsymbol{X}_i \sim N_p(\boldsymbol{\mu}, \Sigma)$, $i = 1,\cdots,n$, where N_p represents the multivariate normal distribution.

We consider an orthogonal transformation of the Helmert type applied to the $\boldsymbol{X}_i$. Let $\boldsymbol{Y}_i = (Y_{i1}, \cdots, Y_{ip})' = Q\boldsymbol{X}_i$. Under this orthogonal transformation, it is obvious that

$$\boldsymbol{Y}_i \sim N_p(\boldsymbol{\mu}^*, \Sigma^*),\ i = 1,\cdots,n, \tag{1}$$

where $\boldsymbol{\mu}^* = (\mu,0,\cdots,0)'$ and $\Sigma^* = \sigma^2 \mathrm{Diag}\{p^{-1}[1+(p-1)\rho], 1-p,\cdots,1-p\}$.

Suppose $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \cdots, \boldsymbol{Y}_n$ is a sequence of random vector the above intraclass model (1). The likelihood function is given by

$$f(\boldsymbol{y} \mid \rho,\sigma^2,\mu) \propto (\sigma^2)^{-\frac{np}{2}}(1-\rho)^{-\frac{n(p-1)}{2}}(1+(p-1)\rho)^{-\frac{n}{2}} \times \exp\left\{-\frac{1}{2\sigma^2}\left[\frac{\sum_{i=1}^{n}(y_{i1}-\mu)^2}{p^{-1}(1+(p-1)\rho)} + \frac{\sum_{i=1}^{n}\sum_{r=2}^{p} y_{ir}^2}{1-\rho}\right]\right\}.$$

2.1 BRC Approach

Now we consider the model $M_0 \equiv N_p(\boldsymbol{y}_i \mid \boldsymbol{\mu}_0^*, \Sigma_0^*)$, where $\boldsymbol{\mu}_0^* = (\mu_0,0,\cdots,0)'$ and $\Sigma_0^* = \sigma_0^2 \mathrm{Diag}\{p^{-1},1,\cdots,1\}$ for some $\mu_0 = \mu_0(\rho_0 = 0,\rho,\sigma^2,\mu)$, $\sigma_0^2 = \sigma_0^2(\rho_0 = 0,\rho,\sigma^2,\mu)$ to be specified, and the full model $M_1 \equiv N_p(\boldsymbol{y}_i \mid \boldsymbol{\mu}^*, \Sigma^*)$.

The logarithmic discrepancy between the assumed model and its closest approximation under the null is

$$\begin{aligned}\delta(\rho_0 = 0,\rho,\sigma^2,\mu) &= \inf_{\substack{\mu_0 \in (-\infty,\infty) \\ \sigma^2 \in (0,\infty)}} n \int N_p(\boldsymbol{y}_i \mid \boldsymbol{\mu}^*,\Sigma^*) \log \frac{N_p(\boldsymbol{y}_i \mid \boldsymbol{\mu}^*,\Sigma^*)}{N_p(\boldsymbol{y}_i \mid \boldsymbol{\mu}_0^*,\Sigma_0^*)} dy \\ &= \inf_{\substack{\mu_0 \in (-\infty,\infty) \\ \sigma^2 \in (0,\infty)}} n\left[\frac{p}{2}\log\frac{\sigma_0^2}{\sigma^2} - \frac{p-1}{2}\log(1-\rho)\right.\end{aligned}$$

$$-\frac{1}{2}\log(1+(p-1)\rho)-\frac{p}{2}+\frac{(\mu-\mu_0)^2+\sigma^2}{2\sigma_0^2 p^{-1}}\Bigg].$$

The infimum is attained at $\mu_0=\mu_0(\rho_0=0,\rho,\sigma^2,\mu)=\mu$ and $\sigma_0^2=\sigma_0^2(\rho_0=0,\rho,\sigma^2,\mu)=\sigma^2$ and, substituting, one has

$$\delta(\rho_0=0,\rho,\sigma^2,\mu)=-\frac{n(p-1)}{2}\log(1-\rho)-\frac{n}{2}\log(1+(p-1)\rho).$$

We now prove a theorem which establishes the convexity of δ as a function of ρ.

Theorem 1. $\delta=\delta(\rho_0=0,\rho,\sigma^2,\mu)$ is a convex function with minimum at $\rho=0$.
Proof. We consider

$$t(\rho)=-(p-1)\log(1-\rho)-\log(1+(p-1)\rho),\quad -\frac{1}{p-1}\leqq\rho\leqq 1.$$

If suffices to show that $t(\rho)$ is a convex function of ρ with minimum at $\rho=0$. Then we have

$$\frac{\partial t(\rho)}{\partial\rho}=\frac{(p-1)p\rho}{(1-\rho)(1+(p-1)\rho)}=0 \text{ for } \rho=0$$

and

$$\frac{\partial^2 t(\rho)}{\partial\rho^2}=\frac{(p-1)p(1+(p-1)\rho^2)}{(1-\rho)^2(1+(p-1)\rho)^2}>0.$$

It implies that $t(\rho)$ is a convex function with minimum at $\rho=0$. □

Next we consider reference posterior of ρ when δ is the quantity of interest. The joint reference prior for (ρ,σ^2,μ) is given by (Kim, Kang and Lee, 2001) as

$$\pi_\rho(\rho,\sigma^2,\mu)\propto(\sigma^2)^{-1}(1-\rho)^{-1}(1+(p-1)\rho)^{-1}.$$

Moreover, δ is a piecewise invertible function of ρ, so the reference prior when δ is the quantity of interest is

$$\pi_\delta(\rho,\sigma^2,\mu)\propto(\sigma^2)^{-1}(1-\rho)^{-1}(1+(p-1)\rho)^{-1}.$$

It follows that the corresponding joint posterior for (ρ,σ^2,μ) given y is

$$\pi_\delta(\rho,\sigma^2,\mu\mid y)\propto(\sigma^2)^{-\frac{np}{2}-1}(1-\rho)^{-\frac{n(p-1)}{2}-1}(1+(p-1)\rho)^{-\frac{n}{2}-1}$$
$$\times\exp\left\{-\frac{1}{2\sigma^2}\left[\frac{\sum_{i=1}^{n}(y_{i1}-\mu)^2}{p^{-1}(1+(p-1)\rho)}+\frac{\sum_{i=1}^{n}\sum_{r=2}^{p}y_{ir}^2}{1-\rho}\right]\right\}.$$

Hence the reference posterior expectation of the logarithmic discrepancy is

$$
\begin{aligned}
d_r(y,\rho_0=0) &= \int_{-\frac{1}{p-1}}^{1}\int_0^{\infty}\int_{-\infty}^{\infty}\delta(\rho_0=0,\rho,\sigma^2,\mu)\pi_\delta(\rho,\sigma^2,\mu \mid y)d\mu d\sigma^2 d\rho \\
&\propto \int_{-\frac{1}{p-1}}^{1}\left(-\frac{n(p-1)}{2}\log(1-\rho)-\frac{n}{2}\log(1+(p-1)\rho)\right) \\
&\quad \times (1-\rho)^{-\frac{n(p-1)}{2}-1}(1+(p-1)\rho)^{-\frac{n+1}{2}} \\
&\quad \times \left[\frac{\sum_{i=1}^{n}(y_{i1}-\bar{y})^2}{p^{-1}(1+(p-1)\rho)}+\frac{\sum_{i=1}^{n}\sum_{r=2}^{p}y_{ir}^2}{1-\rho}\right]^{-\frac{np-1}{2}} d\rho,
\end{aligned}
$$

which involves the one-dimensional numerical integration.

2.2 IBF Approach

Now we develop IBF's for our example. Two models are under consideration : $M_0 \equiv N_p(y_i \mid \mu^*, V^*)$, where $\mu^*=(\mu,0,\cdots,0)'$ and $V^*=\sigma^2\mathrm{Diag}\{p^{-1},1,\cdots,1\}$ and $M_1 \equiv N_p(y_i \mid \mu^*,\Sigma^*)$.

Under the model M_0, the likelihood function is

$$
f_0(y \mid \sigma^2,\mu) \propto (\sigma^2)^{-\frac{np}{2}}\exp\left\{-\frac{1}{2\sigma^2}\left[p\sum_{i=1}^{n}(y_{i1}-\mu)^2+\sum_{i=1}^{n}\sum_{r=2}^{p}y_{ir}^2\right]\right\}.
$$

The reference prior for the parameters of model M_0 is

$$
\pi_0^N(\sigma^2,\mu)=(\sigma^2)^{-1}.
$$

Hence the marginal distribution $m_0^N(y)$ is determined by

$$
m_0^N(y)=\left\{p\sum_{i=1}^{n}(y_{i1}-\bar{y})^2+\sum_{i=1}^{n}\sum_{r=2}^{p}y_{ir}^2\right\}^{-\frac{1}{2}(np-1)}, \tag{2}
$$

where $\bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_{i1}$.

Under the model M_1, the likelihood function is

$$
\begin{aligned}
f_1(y \mid \rho,\sigma^2,\mu) &\propto (\sigma^2)^{-\frac{np}{2}}(1-\rho)^{-\frac{n(p-1)}{2}}(1+(p-1)\rho)^{-\frac{n}{2}} \\
&\quad \times \exp\left\{-\frac{1}{2\sigma^2}\left[\frac{\sum_{i=1}^{n}(y_{i1}-\mu)^2}{p^{-1}(1+(p-1)\rho)}+\frac{\sum_{i=1}^{n}\sum_{r=2}^{p}y_{ir}^2}{1-\rho}\right]\right\}.
\end{aligned}
$$

The reference prior for the parameters of model M_1 is

$$
\pi_1^N(\rho,\sigma^2,\mu)=(\sigma^2)^{-1}(1-\rho)^{-1}(1+(p-1)\rho)^{-1}.
$$

Hence the marginal distribution $m_1^N(y)$ is determined by

$$m_1^N(y) = \int_{-\frac{1}{p-1}}^{1} (1-\rho)^{-\frac{n(p-1)}{2}-1} (1+(p-1)\rho)^{-\frac{n+1}{2}} \times \left\{ \frac{\sum_{i=1}^{n}(y_{i1}-\bar{y})^2}{p^{-1}(1+(p-1)\rho)} + \frac{\sum_{i=1}^{n}\sum_{r=2}^{p} y_{ir}^2}{1-\rho} \right\}^{-\frac{1}{2}(np-1)} d\rho, \tag{3}$$

where $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_{i1}$.

Now, we find the $m_0^N(y(l))$, $m_1^N(y(l))$ of the minimal training samples of the model M_0, M_1, respectively. Since $m_1^N(y_i, y_j) = \infty$ for two distinct observations, but that training samples of three distinct observations are proper. Thus a training sample such as $y(l) = (y_i, y_j, y_k)$ is minimal, and indeed, then

$$m_0^N(y_i, y_j, y_k) = \left\{ \frac{(y_{i1}-\bar{y}_1)^2 + (y_{j1}-\bar{y}_1)^2 + (y_{k1}-\bar{y}_1)^2}{p^{-1}} + \sum_{r=2}^{p} y_{ir}^2 + \sum_{r=2}^{p} y_{jr}^2 + \sum_{r=2}^{p} y_{kr}^2 \right\}^{-\frac{1}{2}(3p-1)}, \tag{4}$$

$$m_1^N(y_i, y_j, y_k) = \int_{-\frac{1}{p-1}}^{1} (1-\rho)^{-\frac{1}{2}(3p+1)} (1+(p-1)\rho)^{-2} \times \left\{ \frac{(y_{i1}-\bar{y}_1)^2 + (y_{j1}-\bar{y}_1)^2 + (y_{k1}-\bar{y}_1)^2}{p^{-1}(1+(p-1)\rho)} + \frac{\sum_{r=2}^{p} y_{ir}^2 + \sum_{r=2}^{p} y_{jr}^2 + \sum_{r=2}^{p} y_{kr}^2}{1-\rho} \right\}^{-\frac{1}{2}(3p-1)} d\rho, \tag{5}$$

where $\bar{y}_1 = \frac{y_{i1}+y_{j1}+y_{k1}}{3}$.

Under our setting, we use the AIBF, GIBF and MIBF for our model selection criteria. The AIBF, GIBF and MIBF for M_1 to M_0 are respectively

$$B_{10}^{AI} = B_{10}^N \cdot \frac{1}{L}\sum_{l=1}^{L} B_{01}^N(y(l)),$$

$$B_{10}^{GI} = B_{10}^N \cdot \left(\prod_{l=1}^{L} B_{01}^N(y(l)) \right)^{\frac{1}{L}},$$

and

$$B_{10}^{MI} = B_{10}^N \cdot \mathrm{Med} B_{01}^N(y(l)).$$

Here $B_{10}^N = m_1^N(y)/m_0^N(y)$ and $B_{01}^N(y(l)) = m_0^N(y(l))/m_1^N(y(l))$, where $y(l) = (y_i, y_j, y_k)$, $m_0^N(y)$ and $m_1^N(y)$ are given in equation (2) and (3), respectively, and $m_0^N(y(l))$ and $m_1^N(y(l))$ are given in equation (4) and (5), respectively.

3. Simulation Study

We consider the hypothesis testing problem for the intraclass correlation coefficient under the reference prior for several configurations of sample sizes n and dimensions p using BRC and IBF approaches. This is done numerically. For our simulation, we take $\mu=0$ and $\sigma^2=1$ without loss of generality.

First, to see the performance of tests by BRC for the hypotheses $H_0:\rho=0$ against $H_1:\rho\neq 0$, for fixed (ρ,σ^2,μ,p,n), we take 10,000 independent random samples of $\boldsymbol{Y}$ from the model (1). In particular, four samples of sizes 3, 5, 10 and 40 are taken and three dimensions 2, 3 and 5 are taken. <Table 1-3> give correspondence between the threshold value d^* of the test statistic $d_r(y,\rho_0=0)$, and 'type I' error probabilities, $P[d>d^*\mid H_0]$. Here, threshold value takes over the range of values from 1 to 9. For the cases presented in <Table 1-3>, we see that 'type I' error probabilities, $P[d>d^*\mid H_0]$ is smaller as sample size increases and $P[d>d^*\mid H_0]$ is larger as dimension increases.

Second, to see the performance of tests by IBF for the hypotheses $H_0:\rho=0$ against $H_1:\rho\neq 0$, of sizes 5, 10 and 40, and dimensions 2, 3 and 5 with 100 replications are simulated from the model (1). IBF results of tests are shown in <Table 4-6>. Mean values of B_{10} and powers by B_{10} in 100 replications are larger as simulated data are farther from the population of H_0 for each sample sizes and dimensions. These results meet our theoretical expectations. Among AIBF, GIBF and MIBF, there is not that much difference in powers, but generally GIBF gives the smallest powers. We can see that IBF results more agree with our theoretical expectation for testing as sample sizes and dimensions are larger.

<Table 1> Correspondence between the threshold value d^* of the test statistic, $d_r(y,\rho_0=0)$ and 'type I' error probabilities, $P[d>d^*\mid H_0]$ when $p=2$

d^*	$n=3$	$n=5$	$n=10$	$n=40$
1	0.5054	0.3836	0.3501	0.3265
2	0.1420	0.1025	0.0972	0.0902
2.5	0.0880	0.0567	0.0470	0.0459
3	0.0521	0.0345	0.0299	0.0247
4	0.0233	0.0134	0.0110	0.0086

<Table 1> (continued)

d^*	$n=3$	$n=5$	$n=10$	$n=40$
5	0.0097	0.0041	0.0042	0.0031
6	0.0048	0.0015	0.0010	0.0012
7	0.0023	0.0003	0.0000	0.0002
8	0.0008	0.0005	0.0000	0.0000
9	0.0006	0.0001	0.0001	0.0001

<Table 2> Correspondence between the threshold value d^* of the test statistic, $d_r(y, \rho_0 = 0)$, and 'type I' error probabilities, $P[d > d^* \mid H_0]$ when $p=3$

d^*	$n=3$	$n=5$	$n=10$	$n=40$
1	0.6842	0.4592	0.3769	0.3295
2	0.2078	0.1358	0.1002	0.0932
2.5	0.1212	0.0793	0.0605	0.0498
3	0.0795	0.0479	0.0334	0.0279
4	0.0309	0.0156	0.0128	0.0089
5	0.0141	0.0054	0.0028	0.0023
6	0.0045	0.0025	0.0018	0.0005
7	0.0032	0.0014	0.0005	0.0002
8	0.0014	0.0004	0.0001	0.0002
9	0.0012	0.0003	0.0000	0.0001

<Table 3> Correspondence between the threshold value d^* of the test statistic $d_r(y, \rho_0 = 0)$, and 'type I' error probabilities, $P[d > d^* \mid H_0]$ when $p=5$

d^*	$n=3$	$n=5$	$n=10$	$n=40$
1	1.0000	0.5645	0.4146	0.3469
2	0.3382	0.1852	0.1214	0.0926
2.5	0.2163	0.1131	0.0760	0.0544
3	0.1398	0.0769	0.0414	0.0291
4	0.0666	0.0270	0.0151	0.0096
5	0.0252	0.0114	0.0055	0.0033
6	0.0113	0.0043	0.0017	0.0011
7	0.0068	0.0021	0.0008	0.0007
8	0.0032	0.0007	0.0001	0.0001
9	0.0010	0.0001	0.0002	0.0000

<Table 4> The intrinsic Bayes factor's for simulated data when $p = 2$

ρ	n		AIBF	GIBF	MIBF
-0.9	5	Mean of B_{10}	10.50	5.260	6.410
		s.d. of B_{10}	23.80	5.970	6.590
		power by B_{10}	0.930	0.900	0.950
	10	Mean of B_{10}	0.175×10^4	956.0	0.106×10^4
		s.d. of B_{10}	0.420×10^4	0.240×10^4	0.252×10^4
		power by B_{10}	0.980	0.980	0.980
	40	Mean of B_{10}	0.326×10^{17}	0.186×10^{17}	0.222×10^{17}
		s.d. of B_{10}	0.255×10^{18}	0.134×10^{18}	0.161×10^{18}
		power by B_{10}	1.000	1.000	1.000
-0.5	5	Mean of B_{10}	1.570	1.180	1.450
		s.d. of B_{10}	2.750	1.750	1.940
		power by B_{10}	0.370	0.310	0.430
	10	Mean of B_{10}	5.910	3.850	4.750
		s.d. of B_{10}	23.50	13.90	15.30
		power by B_{10}	0.500	0.460	0.520
	40	Mean of B_{10}	0.235×10^5	0.155×10^5	0.203×10^5
		s.d. of B_{10}	0.214×10^6	0.140×10^6	0.183×10^6
		power by B_{10}	0.940	0.910	0.940
-0.1	5	Mean of B_{10}	1.050	0.878	1.020
		s.d. of B_{10}	1.540	1.090	1.310
		power by B_{10}	0.170	0.130	0.190
	10	Mean of B_{10}	0.709	0.563	0.736
		s.d. of B_{10}	0.907	0.645	0.754
		power by B_{10}	0.150	0.100	0.190
	40	Mean of B_{10}	0.637	0.505	0.697
		s.d. of B_{10}	1.720	1.340	1.870
		power by B_{10}	0.070	0.070	0.100
0.0	5	Mean of B_{10}	1.130	0.890	1.040
		s.d. of B_{10}	1.550	1.010	0.984
		power by B_{10}	0.190	0.150	0.240
	10	Mean of B_{10}	0.641	0.520	0.670
		s.d. of B_{10}	0.753	0.589	0.708
		power by B_{10}	0.080	0.050	0.120
	40	Mean of B_{10}	0.309	0.252	0.345
		s.d. of B_{10}	0.345	0.275	0.380
		power by B_{10}	0.060	0.020	0.070

<Table 4> (continued)

ρ	n		AIBF	GIBF	MIBF
0.1	5	Mean of B_{10}	1.100	0.882	1.010
		s.d. of B_{10}	1.280	0.886	0.789
		power by B_{10}	0.210	0.130	0.250
	10	Mean of B_{10}	1.300	0.852	1.070
		s.d. of B_{10}	5.900	3.070	3.480
		power by B_{10}	0.120	0.090	0.140
	40	Mean of B_{10}	0.521	0.408	0.575
		s.d. of B_{10}	1.650	1.200	1.780
		power by B_{10}	0.070	0.050	0.080
0.5	5	Mean of B_{10}	2.900	1.330	1.360
		s.d. of B_{10}	12.90	2.360	1.390
		power by B_{10}	0.460	0.350	0.440
	10	Mean of B_{10}	7.830	4.450	5.610
		s.d. of B_{10}	23.60	12.60	15.80
		power by B_{10}	0.570	0.460	0.590
	40	Mean of B_{10}	0.472×10^4	0.292×10^4	0.413×10^4
		s.d. of B_{10}	0.241×10^5	0.148×10^5	0.210×10^5
		power by B_{10}	0.900	0.880	0.900
0.9	5	Mean of B_{10}	34.90	8.570	7.770
		s.d. of B_{10}	99.00	13.20	11.30
		power by B_{10}	0.950	0.900	0.950
	10	Mean of B_{10}	0.105×10^5	0.244×10^4	0.224×10^4
		s.d. of B_{10}	0.454×10^5	0.929×10^4	0.811×10^4
		power by B_{10}	1.000	1.000	1.000
	40	Mean of B_{10}	0.295×10^{19}	0.696×10^{18}	0.714×10^{18}
		s.d. of B_{10}	0.292×10^{20}	0.686×10^{19}	0.703×10^{19}
		power by B_{10}	1.000	1.000	1.000

<Table 5> The intrinsic Bayes factor's for simulated data when $p=3$

ρ	n		AIBF	GIBF	MIBF
-0.4	5	Mean of B_{10}	6.900	4.990	6.290
		s.d. of B_{10}	15.20	11.20	13.70
		power by B_{10}	0.860	0.810	0.870
	10	Mean of B_{10}	327.0	185.0	240.0
		s.d. of B_{10}	0.147×10^{4}	774.0	0.103×10^{4}
		power by B_{10}	0.960	0.960	0.960
	40	Mean of B_{10}	0.834×10^{11}	0.525×10^{11}	0.661×10^{11}
		s.d. of B_{10}	0.793×10^{12}	0.499×10^{12}	0.625×10^{12}
		power by B_{10}	1.000	1.000	1.000
-0.1	5	Mean of B_{10}	1.530	1.180	1.430
		s.d. of B_{10}	4.230	3.490	4.110
		power by B_{10}	0.270	0.250	0.280
	10	Mean of B_{10}	0.925	0.717	1.000
		s.d. of B_{10}	1.730	1.420	1.890
		power by B_{10}	0.180	0.140	0.190
	40	Mean of B_{10}	1.770	1.330	1.980
		s.d. of B_{10}	6.060	4.710	6.800
		power by B_{10}	0.200	0.150	0.200
0.0	5	Mean of B_{10}	1.000	0.734	0.922
		s.d. of B_{10}	2.560	1.630	1.780
		power by B_{10}	0.160	0.100	0.150
	10	Mean of B_{10}	0.536	0.392	0.557
		s.d. of B_{10}	0.659	0.580	0.729
		power by B_{10}	0.090	0.060	0.090
	40	Mean of B_{10}	0.357	0.215	0.376
		s.d. of B_{10}	0.606	0.366	0.626
		power by B_{10}	0.070	0.030	0.080

<Table 5> (continued)

ρ	n		AIBF	GIBF	MIBF
0.1	5	Mean of B_{10}	0.928	0.653	0.845
		s.d. of B_{10}	1.260	0.989	1.350
		power by B_{10}	0.250	0.140	0.170
	10	Mean of B_{10}	0.811	0.393	0.642
		s.d. of B_{10}	1.290	0.499	0.789
		power by B_{10}	0.190	0.090	0.150
	40	Mean of B_{10}	1.550	0.694	1.230
		s.d. of B_{10}	9.630	3.990	6.920
		power by B_{10}	0.130	0.060	0.130
0.5	5	Mean of B_{10}	45.80	0.809	1.080
		s.d. of B_{10}	428.0	1.480	2.590
		power by B_{10}	0.370	0.140	0.180
	10	Mean of B_{10}	27.20	3.670	5.060
		s.d. of B_{10}	141.0	8.380	11.70
		power by B_{10}	0.690	0.480	0.560
	40	Mean of B_{10}	0.437×10^{12}	0.513×10^{11}	0.796×10^{11}
		s.d. of B_{10}	0.436×10^{13}	0.512×10^{12}	0.793×10^{12}
		power by B_{10}	1.000	1.000	1.000
0.9	5	Mean of B_{10}	0.163×10^{4}	43.60	55.20
		s.d. of B_{10}	0.932×10^{4}	293.0	378.0
		power by B_{10}	0.960	0.700	0.710
	10	Mean of B_{10}	0.401×10^{10}	0.125×10^{8}	0.106×10^{8}
		s.d. of B_{10}	0.275×10^{11}	0.867×10^{8}	0.680×10^{8}
		power by B_{10}	1.000	0.980	0.990
	40	Mean of B_{10}	0.119×10^{39}	0.688×10^{36}	0.559×10^{36}
		s.d. of B_{10}	0.118×10^{40}	0.682×10^{37}	0.552×10^{37}
		power by B_{10}	1.000	1.000	1.000

<Table 6> The intrinsic Bayes factor's for simulated data when $p = 5$

ρ	n		AIBF	GIBF	MIBF
-0.2	5	Mean of B_{10}	8.480	6.520	8.220
		s.d. of B_{10}	19.80	15.50	20.40
		power by B_{10}	0.810	0.720	0.800
	10	Mean of B_{10}	548.0	353.0	490.0
		s.d. of B_{10}	0.325×10^4	0.199×10^4	0.273×10^4
		power by B_{10}	0.990	0.960	1.000
	40	Mean of B_{10}	0.105×10^{10}	0.718×10^9	0.108×10^{10}
		s.d. of B_{10}	0.826×10^{10}	0.557×10^{10}	0.844×10^{10}
		power by B_{10}	1.000	1.000	1.000
-0.1	5	Mean of B_{10}	3.860	3.150	3.720
		s.d. of B_{10}	16.20	13.30	15.20
		power by B_{10}	0.290	0.260	0.310
	10	Mean of B_{10}	3.220	2.480	3.440
		s.d. of B_{10}	15.80	13.90	17.00
		power by B_{10}	0.280	0.210	0.300
	40	Mean of B_{10}	28.40	17.50	32.00
		s.d. of B_{10}	110.0	71.80	124.0
		power by B_{10}	0.570	0.470	0.580
0.0	5	Mean of B_{10}	0.574	0.401	0.515
		s.d. of B_{10}	0.878	0.796	0.945
		power by B_{10}	0.130	0.100	0.130
	10	Mean of B_{10}	0.620	0.398	0.638
		s.d. of B_{10}	1.610	1.400	1.780
		power by B_{10}	0.080	0.050	0.120
	40	Mean of B_{10}	0.188	0.045	0.142
		s.d. of B_{10}	0.196	0.054	0.148
		power by B_{10}	0.010	0.000	0.000

<Table 6> (continued)

ρ	n		AIBF	GIBF	MIBF
		Mean of B_{10}	0.488	0.223	0.306
	5	s.d. of B_{10}	0.557	0.337	0.404
		power by B_{10}	0.090	0.030	0.050
		Mean of B_{10}	0.845	0.089	0.182
0.1	10	s.d. of B_{10}	2.270	0.249	0.350
		power by B_{10}	0.110	0.010	0.020
		Mean of B_{10}	0.356×10^{5}	450.0	0.211×10^{4}
	40	s.d. of B_{10}	0.355×10^{6}	0.449×10^{4}	0.211×10^{5}
		power by B_{10}	0.280	0.120	0.180
		Mean of B_{10}	87.50	0.127	0.392
	5	s.d. of B_{10}	676.0	0.393	1.830
		power by B_{10}	0.410	0.010	0.050
		Mean of B_{10}	0.426×106	362.0	832.0
0.5	10	s.d. of B_{10}	0.259×10^{7}	0.194×10^{4}	0.523×10^{4}
		power by B_{10}	0.940	0.390	0.520
		Mean of B_{10}	0.601×10^{20}	0.113×10^{18}	0.218×10^{18}
	40	s.d. of B_{10}	0.555×10^{21}	0.107×10^{19}	0.205×10^{19}
		power by B_{10}	1.000	1.000	1.000
		Mean of B_{10}	0.886×10^{8}	124.0	42.90
	5	s.d. of B_{10}	0.863×10^{9}	885.0	166.0
		power by B_{10}	0.910	0.400	0.410
		Mean of B_{10}	0.358×10^{23}	0.315×10^{15}	0.146×10^{15}
0.9	10	s.d. of B_{10}	0.251×10^{24}	0.230×10^{16}	0.104×10^{16}
		power by B_{10}	1.000	1.000	1.000
		Mean of B_{10}	0.130×10^{82}	0.919×10^{74}	0.702×10^{74}
	40	s.d. of B_{10}	0.130×10^{83}	0.919×10^{75}	0.702×10^{75}
		power by B_{10}	1.000	1.000	1.000

4. Summary

The intraclass correlation coefficient ρ has a lengthy history of application in several different fields of research. Although the theory concerning ρ is well-established, its extension to Bayesian approach has received very *little* attention. The objective of this paper to find Bayesian method for comparing two nested models such as the intraclass and independence models based on the "reference" prior. The model comparison problem in this case amounts to testing the hypothesis $H_0 : \rho = 0$.

To decide whether or not some data y are compatible with the null hypothesis $\rho = 0$, assuming that the data have been generated from the model $N_p(\mu^*, \Sigma^*)$, where N_p represents the multivariate normal distribution and $\mu^* = (\mu, 0, \cdots, 0)'$ and $\Sigma^* = \sigma^2 \mathrm{Diag} p^{-1}[1 + (p-1)\rho], 1-p, \cdots, 1-p$, the present paper develops twofold.

The first is the "Bayesian Reference Criterion" as introduced by Bernardo (1999). We compute the posterior mean of the logarithmic discrepancy and rejection of H_0 if and only if this posterior mean exceeds some specified number d^*. (Bernardo suggested 2.5 or 5).

The second is the "Intrinsic Bayes Factor" as introduced by Berger and Pericchi (1996). We compute the Bayes factors using the minimal training sample to eliminate the arbitrariness of improper priors. But for a given data set, there will typically be many minimal training samples, the intrinsic Bayes factor will depend on choice of the minimal training sample. To eliminate this dependence and increase stability, Berger and Pericchi (1996) suggested to take the average of the intrinsic Bayes factor over all minimal training sample arithmetically (AIBF) or geometrically (GIBF) and use of median of the intrinsic Bayes factor over all minimal training sample (MIBF). We compute AIBF, GIBF, MIBF under consideration.

Numerical examples are given to illustrate our results.

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