Commun. Korean Math. Soc. **21** (2006), No. 2, pp. 219–226

FACTOR RANK COMPARISONS OF MATRICES OVER TWO RELATED SEMIRINGS

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ABSTRACT. We consider matrices whose entries can be viewed as elements of both nonnegative integers and nonnegative rational numbers. We determine the differences of factor rank of matrices over the two semirings.

1. Introduction and preliminaries

A semiring is a binary system $(S, +, \times)$ such that $(S, +)$ is an Abelian monoid (identity 0), $(S, \times)$ is an Abelian monoid (identity 1), $\times$ distributes over $+$, $0 \times s = s \times 0 = 0$ for all s in S, and $1 \neq 0$. Usually S denotes the system and $\times$ is denoted by juxtaposition. Some examples of semiring which occur in combinatorics are Boolean algebra and the nonnegative integers with usual arithmatic. The concepts of matrix theory are defined over a semiring as over a field. Recently a number of authors have studied various problems of matrix theory over semirings(see [1]-[8]).

Let S be a semiring and $\mathbb{M}_{m,n}(S)$ be the set of $m \times n$ matrices with entries in S. If $A \in \mathbb{M}_{m,n}(S)$, then the *factor rank* [2, 8](or *semiring rank* in [3, 4]) of A over S is the smallest integer k such that A can be factored as $A = BC$ where $B \in \mathbb{M}_{m,k}(S)$ and $C \in \mathbb{M}_{k,n}(S)$. This factor rank is the same concept as rank when the semiring is a field. We denote the factor rank of a matrix A over S by $\phi_S(A)$.

In this paper, we compare the factor ranks when a matrix in $\mathbb{M}_{m,k}(Z^+)$ is considered as a matrix in $\mathbb{M}_{m,k}(Q^+)$ or $\mathbb{M}_{m,k}(R^+)$, where Z^+ , Q^+ and R^+ denote the semiring of nonnegative integers, nonnegative rational numbers and nonnegative real numbers, respectively.

For various factor rank comparisons, Beasley, Kirkland and Shader obtained some theorems in [2], and Beasley and Song also obtained some

Received September 28, 2005.
2000 Mathematics Subject Classification: 15A23, 15A36.
Key words and phrases: factor rank, semiring rank.

theorems in [4]. In this paper, we obtain two results on the comparisons of factor ranks of matrices over Z^+ and Q^+, respectively.

2. Factor rank comparisons of nonnegative matrices

LEMMA 2.1. *If K is a subsemiring of S, then $\phi_K(A) \geq \phi_S(A)$ for every matrix $A \in \mathbb{M}_{m,n}(K)$.*

PROOF. Assume that $\phi_K(A) = k$. Then there exist matrices $B \in \mathbb{M}_{m,k}(K)$ and $C \in \mathbb{M}_{k,n}(K)$ satisfying $A = BC$. Since B and C are in $\mathbb{M}_{m,k}(S)$ and $\mathbb{M}_{k,n}(S)$ respectively, we have $\phi_S(A) \leq k$. □

EXAMPLE 2.2. The inequality in Lemma 2.1 may be strict. For example, consider

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 3 & 7 \\ 5 & 1 & 14 \end{bmatrix}.$$

Then

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & \frac{3}{2} \end{bmatrix},$$

from which it follows that

$$\phi_{Q^+}(A) = 2. \tag{1}$$

Now, we claim that

$$\phi_{Z^+}(A) = 3. \tag{2}$$

To show (2), suppose that A can be factored over Z^+ as $[\boldsymbol{b_1} \mid \boldsymbol{b_2}]C$, where

$$\mathbf{b}_1 = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} \text{ and } \quad C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}.$$

Then the first column of A is a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$ over Z^+. That is,

$$\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} = c_{11}\mathbf{b}_1 + c_{21}\mathbf{b}_2.$$

Hence we may assume that

$$\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} \geq \mathbf{b}_1$$

(where the inequality holds entrywise). Thus the first entry b_{11} of $\mathbf{b}_1$ is zero. Then the first entry b_{12} of $\mathbf{b}_2$ cannot be zero, since if $b_{11} = b_{12} = 0$, then the first row of A would be a zero row. Thus c_{21} must be zero and hence $\mathbf{b}_1$ and $\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$ must equal.

For the second column of A, we have

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = c_{12} \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} + c_{22}\mathbf{b}_2.$$

Then $c_{12} = 0$ from $1 = 5c_{12} + b_{32}c_{22}$. Therefore $\mathbf{b}_2$ must be $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$.

Now, consider the third column of A. Then we have

$$\begin{bmatrix} 3 \\ 7 \\ 14 \end{bmatrix} = c_{13} \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} + c_{23} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

But this is impossible since the nonnegative integer c_{23} must satisfy $3 = 0 \cdot c_{13} + 2 \cdot c_{23}$. Thus we see that $\phi_{Z^+}(A) = 3$, as required in (2).

LEMMA 2.3. ([1]) *Suppose that A is a $p \times q$ matrix over a semiring S. If*

$$A = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}, \quad \textit{then} \quad \phi_S(A) = \phi_S(B).$$

LEMMA 2.4. *If a matrix $A \in \mathbb{M}_{m,n}(Z^+)$ satisfies $\phi_{Q^+}(A) = 1$, then $\phi_{Z^+}(A) = 1$.*

PROOF. Suppose that $\phi_{Q^+}(A) = 1$. Then we can factor A as $A = \mathbf{b}\mathbf{c}^t$ where $\mathbf{b} \in \mathbb{M}_{m,1}(Q^+)$ and $\mathbf{c} \in \mathbb{M}_{n,1}(Q^+)$. Let b_i(*or* c_j) be an entry in $\mathbf{b}$ (or $\mathbf{c}$, respectively). If b_i(*or* c_j) is zero, then the ith row (or jth column, respectively) of A is zero, and hence the zero row (or zero column) does not change the factor rank of A over Z^+. Thus without loss of generality, we may assume that all entries of $\mathbf{b}$ and $\mathbf{c}$ are positive rational numbers. Then for a fixed positive entry b_i of $\mathbf{b}$, we have

$$\frac{b_k}{b_i} = \frac{b_k c_j}{b_i c_j} = \frac{a_{kj}}{a_{ij}}. \tag{3}$$

Since a_{kj} and a_{ij} are integers, $\frac{b_k}{b_i}$ is a rational number for each $k \in \{1, 2, \ldots, m\}$ and each $j \in \{1, 2, \ldots, n\}$. Write each $\frac{b_k}{b_i}$ in lowest term as $\frac{p_k}{q_k}$ with $p_k, q_k \in Z^+$ and let L be the least common multiple of the q_k's. For any k and j we have $\frac{p_k}{q_k} \cdot b_i c_j = a_{kj} \in Z^+$ by (3). Then each q_k divides $b_i c_j$ and hence L divides $b_i c_j$ for any j. Consequently, $(\frac{1}{L})(b_i \mathbf{c})$ is a vector with entries in Z^+, and $\frac{L}{b_i}\mathbf{b}$ is a vector with entries in Z^+ by the construction of L. Therefore A can be factored as $\{(\frac{1}{L})(b_i\mathbf{c})\}(\frac{L}{b_i}\mathbf{b})$ in Z^+, which shows that $\phi_{Z^+}(A) = 1$. □

Suppose that T is a subsemiring of S. Let $\Phi(T, S, m, n)$ denote the maximum integer k such that there exists a matrix in $\mathbb{M}_{m,n}(T)$ with factor rank k and for every $A \in \mathbb{M}_{m,n}(T)$ with $\phi_T(A) \le k$ we have $\phi_T(A) = \phi_S(A)$.

In the followings we obtain the value of $\Phi(Z^+, Q^+, m, n)$.

LEMMA 2.5. *Suppose that T is a subsemiring of S. For some matrix $A \in \mathbb{M}_{p,q}(T)$, if $\phi_T(A) > \phi_S(A)$, then for all $m \ge p$ and $n \ge q$,*

$$\Phi(T, S, m, n) < \phi_T(A).$$

PROOF. It follows directly from the definition of $\Phi(T, S, m, n)$ and Lemma 2.3. □

THEOREM 2.6.

$$\Phi(Z^+, Q^+, m, n) = \begin{cases} 1 & \text{if } \min(m,n) = 1, \\ 2 & \text{otherwise.} \end{cases}$$

PROOF. If $\min(m, n) = 1$, then a matrix $A \in \mathbb{M}_{m,n}(Z^+)$ has factor rank 1 or 0. For each case, A has factor rank 1 or 0 respectively, as a matrix in $\mathbb{M}_{m,n}(Q^+)$. Thus $\Phi(Z^+, Q^+, m, n) = 1$ if $\min(m, n) = 1$.

If $\min(m, n) = 2$, then $\Phi(Z^+, Q^+, m, n) \le 2$.

Let $\min(m, n) \ge 3$. Then Example 2.2 shows that there exists a matrix $A \in \mathbb{M}_{3,3}(Z^+)$ such that $3 = \phi_{Z^+}(A) > \phi_{Q^+}(A) = 2$. And Lemma 2.5 shows that $\Phi(Z^+, Q^+, m, n) \le 2$ for $m \ge 3$ and $n \ge 3$. Thus we have

$$\Phi(Z^+, Q^+, m, n) \le 2 \tag{4}$$

for $\min(m, n) \ge 2$.

Suppose that $A \in \mathbb{M}_{m,n}(Z^+)$ satisfies $\phi_{Q^+}(A) = 1$ for $\min(m, n) \ge 2$. Then A has factor rank 1 over Z^+ by Lemma 2.4. Of course, if $\phi_{Z^+}(A) = 1$ then $\phi_{Q^+}(A) = 1$ by Lemma 2.1. Hence we have

$$\phi_{Z^+}(A) = 1 \quad \text{if and only if} \quad \phi_{Q^+}(A) = 1 \tag{5}$$

for any $A \in \mathbb{M}_{m,n}(Z^+)$.

Now suppose that $\phi_{Z^+}(A) = 2$ for $A \in \mathbb{M}_{m,n}(Z^+)$ with $\min(m,n) \geq 2$. Then $\phi_{Q^+}(A) \leq 2$ by Lemma 2.1. But $\phi_{Q^+}(A) \neq 1$ by (5). Thus $\phi_{Q^+}(A) = 2$. Therefore

$$\Phi(Z^+, Q^+, m, n) \geq 2. \tag{6}$$

From (4) and (6), we have $\Phi(Z^+, Q^+, m, n) = 2$ for $\min(m,n) \geq 2$. □

COROLLARY 2.7.

$$\Phi(Z^+, R^+, m, n) = \begin{cases} 1 & \text{if } \min(m,n) = 1, \\ 2 & \text{otherwise.} \end{cases}$$

PROOF. It is similar to the proof of Theorem 2.6. □

In [2] (Theorem 4.4), Beasley, Kirkland and Shader obtained a result for $\Phi(Z^+, R^+, m, n)$ as follows ;

$$\Phi(Z^+, R^+, m, n) = \begin{cases} 2 & \text{if } \min(m,n) = 2, \\ 1 & \text{otherwise.} \end{cases}$$

But we remark that this result is revised in the Corollary 2.7.

Now, we consider the matrices over the field Q of rational numbers and determine the value of $\Phi(Q^+, Q, m, n)$.

LEMMA 2.8. *If $A \in \mathbb{M}_{m.n}(Q^+)$, then $\phi_{Q^+}(A) = 1$ if and only if $\phi_Q(A) = 1$.*

PROOF. Suppose that $A \in \mathbb{M}_{m.n}(Q^+)$. If $\phi_Q(A) = 1$, then each column of A is a multiple of the first nonzero column of A by a rational number. Consequently, each column of A is a nonnegative multiple of that column by rational number, and hence $\phi_{Q^+}(A) = 1$ as well.

The converse follows from Lemma 2.1. □

LEMMA 2.9. *Let $A \in \mathbb{M}_{m.n}(Q^+)$ with $\min(m,n) \geq 2$. Then $\phi_Q(A) = 2$ if and only if $\phi_{Q^+}(A) = 2$.*

PROOF. If $\phi_{Q^+}(A) = 2$ then $\phi_Q(A) = 2$ by Lemmas 2.1 and 2.8. Suppose that $A \in \mathbb{M}_{m.n}(Q^+)$ and $\phi_Q(A) = 2$. We will show that $\phi_{Q^+}(A) = 2$ by using induction on the column n of A. For $n = 2$, $\phi_Q(A) = 2$ implies that $\phi_{Q^+}(A) \leq 2$. But Lemma 2.8 implies that $\phi_{Q^+}(A) = 2$.

Now, suppose that it holds for $n \geq 2$. Let $A \in \mathbb{M}_{m.n+1}(Q^+)$ and $\phi_Q(A) = 2$. Put $A = [\mathbf{a}_1|\mathbf{a}_2|\cdots|\mathbf{a}_n|\mathbf{a}_{n+1}]$ and let $A_1 = [\mathbf{a}_1|\mathbf{a}_2|\cdots|\mathbf{a}_n]$ be an $m \times n$ submatrix of A. If $\phi_Q(A_1) = 1$, then there exist one nonzero column $\mathbf{a}_j$ of A_1 such that $\mathbf{a}_i = \alpha_i \mathbf{a}_j$ with $\alpha_i \in Q^+$, for $i = 1, 2, \ldots, n$. Thus A can be factored as follows :

$$A = [\mathbf{a}_i \ \mathbf{a}_{n+1}] \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_{n-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since this factorization shows that $\phi_{Q^+}(A) \leq 2$. Hence we have $\phi_{Q^+}(A)$ $= 2$ by Lemma 2.8.

If $\phi_Q(A_1) = 2$, then $\phi_{Q^+}(A_1) = 2$ by the induction assumption. Hence we can factor A_1 as follows :

$$A_1 = B_1 C_1 = [\mathbf{b}_1 | \mathbf{b}_2] \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \end{bmatrix}, \tag{7}$$

where $B_1 \in \mathbb{M}_{m,2}(Q^+)$ and $C_1 \in \mathbb{M}_{2,n}(Q^+)$.

Then each column $\mathbf{a}_i$ in A_1 can be written as $\mathbf{a}_i = c_{1i}\mathbf{b}_1 + c_{2i}\mathbf{b}_2$ for $i = 1, 2, \ldots, n$.

Further, over a field Q, the two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are in the column space of A_1(and A) since they may be written as linear combinations of two linearly independent columns of A_1 over Q. Since $\phi_Q(A) = 2$, the column rank of A is also 2 over the field Q. Hence the three vectors $\mathbf{b}_1, \mathbf{b}_2$ and $\mathbf{a}_{n+1}$ in the column space of A are linearly dependent over Q. Thus there exist α, β and γ, not all zero, in Q such that

$$\alpha \mathbf{b}_1 + \beta \mathbf{b}_2 + \gamma \mathbf{a}_{n+1} = 0.$$

Since all the entries of $\mathbf{b}_1, \mathbf{b}_2$ and $\mathbf{a}_{n+1}$ are nonnegative rational numbers, one of α, β and γ is positive while another is negative. It follows that there are rational numbers $p, q \in Q^+$ such that one of the following holds:

(a) $\mathbf{b}_1 = p\mathbf{b}_2 + q\mathbf{a}_{n+1}$; (b) $\mathbf{b}_2 = p\mathbf{b}_1 + q\mathbf{a}_{n+1}$; (c) $\mathbf{a}_{n+1} = p\mathbf{b}_1 + q\mathbf{b}_2$.

Let us use (7) for the factorization of A in each case. For the case (a), we have

$$\begin{aligned} A &= [\mathbf{a}_1 | \mathbf{a}_2 | \cdots | \mathbf{a}_n | \mathbf{a}_{n+1}] \\ &= [\mathbf{b}_1 | \mathbf{b}_2] \begin{bmatrix} c_{11} & \cdots & c_{1n} & \frac{1}{q} \\ c_{21} & \cdots & c_{2n} & \frac{-p}{q} \end{bmatrix} \\ &= [p\mathbf{b}_2 + q\mathbf{a}_{n+1} | \mathbf{b}_2] \begin{bmatrix} c_{11} & \cdots & c_{1n} & \frac{1}{q} \\ c_{21} & \cdots & c_{2n} & \frac{-p}{q} \end{bmatrix} \\ &= [\mathbf{a}_{n+1} | \mathbf{b}_2] \begin{bmatrix} qc_{11} & qc_{12} & \cdots & c_{1n} & 1 \\ pc_{11} + c_{21} & pc_{12} + c_{22} & \cdots & pc_{1n} + c_{2n} & 0 \end{bmatrix}. \end{aligned}$$

For the case (b), we have

$$\begin{aligned} A &= [\mathbf{a}_1|\mathbf{a}_2|\cdots|\mathbf{a}_n|\mathbf{a}_{n+1}] \\ &= [\mathbf{b}_1|p\mathbf{b}_1 + q\mathbf{a}_{n+1}] \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} & \frac{-p}{q} \\ c_{21} & c_{22} & \cdots & c_{2n} & \frac{1}{q} \end{bmatrix} \\ &= [\mathbf{b}_1|\mathbf{a}_{n+1}] \begin{bmatrix} c_{11}+pc_{21} & c_{12}+pc_{22} & \cdots & c_{1n}+pc_{2n} & 0 \\ qc_{21} & qc_{22} & \cdots & qc_{2n} & 1 \end{bmatrix}. \end{aligned}$$

For the case (c), we have

$$\begin{aligned} A &= [\mathbf{a}_1|\mathbf{a}_2|\cdots|\mathbf{a}_n|\mathbf{a}_{n+1}] \\ &= [\mathbf{b}_1|\mathbf{b}_2] \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} & p \\ c_{21} & c_{22} & \cdots & c_{2n} & q \end{bmatrix}. \end{aligned}$$

Thus we have factored A as the product of both a matrix in $\mathbb{M}_{m,2}(Q^+)$ and a matrix in $\mathbb{M}_{2,n+1}(Q^+)$ in each case. Therefore $\phi_{Q^+}(A) = 2$. □

EXAMPLE 2.10. Let $M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \in \mathbb{M}_{4,4}(Q^+)$. Then we have

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Thus $\phi_Q(M) = 3$. Since $\phi_{R^+}(M) = 4$ from [2](Example 4.3), we have $\phi_{Q^+}(M) = 4$ by Lemma 2.1.

Now, we have the following factor rank comparison Theorem.

THEOREM 2.11.

$$\Phi(Q^+, Q, m, n) = \begin{cases} \min\{m,n\} & \text{if } \min(m,n) \leq 3, \\ 2 & \text{otherwise.} \end{cases}$$

PROOF. If $\min\{m,n\} \leq 2$, then $\Phi(Q^+, Q, m, n) = \min\{m,n\}$ by Lemmas 2.8 and 2.9.

Let $\min\{m,n\} = 3$. If $\phi_Q(A) = 3$ for an $m \times n$ matrix $A \in \mathbb{M}_{m,n}(Q^+)$, then $\phi_{Q^+}(A) \geq 3$. But we have $\phi_{Q^+}(A) \leq \min\{m,n\} = 3$.
Hence $\Phi(Q^+, Q, m, n) = 3$ from Lemmas 2.8 and 2.9.

If $\min\{m,n\} \geq 4$, then Example 2.10 shows that $\Phi(Q^+, Q, m, n) = 2$. □

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