Journal of Korean
Data & Information Science Society
2007, Vol. 18, No. 2, pp. 535 ~ 541

Prediction of MTBF Using the Modulated Power Law Process[1)]

Myung Hwan Na[2)] · Young Sook Son[3)]
Sang Hoo Yoon[4)] · Moon Ju Kim[5)]

Abstract

The Non-homogeneous Poisson process is probably the most popular model since it can model systems that are deteriorating or improving. The renewal process is a model that is often used to describe the random occurrence of events in time. But both these models are based on too restrictive assumptions on the effect of the repair action. The Modulated Power Law Process is a suitable model for describing the failure pattern of repairable systems when both renewal-type behavior and time trend are present. In this paper we propose maximum likelihood estimation of the next failure time after the system has experienced some failures, that is, Mean Time Between Failure for the MPLP model.

Keywords : Mean Time Between Failure, Modulated Power Law Process, NHPP, Numerical example

1. Introduction

Consider a complex system under the development and testing process. The system is tested until it fails. Then if the system fails, it is repaired or redesigned

---

1) This research was supported by the Program for the Training of Graduate Students in Regional Innovation which was conducted by the Ministry of Commerce Industry and Energy of the Korean Government.

2) Professor, Department of Statistics, Chonnam National University, Gwangju, 500-757, Korea,
E-mail : nmh@chonnam.ac.kr

3) Professor, Department of Statistics, Chonnam National University, Gwangju, 500-757, Korea

4) Ph. D. student, Department of Statistics, Chonnam National University, Gwangju, 500-757, Korea

5) Ph. D. student, Department of Statistics, Chonnam National University, Gwangju, 500-757, Korea

if necessary, and it is tested again until it fails. This process continues until we reach a desirable which would reflect the quality of the final design. In this case, while the development program is succeeding, one would expect a tendency toward longer times between failures, and such systems are said to be undergoing reliability growth. On the other hand, if a deteriorating system is given only minimal repairs when it fails, one would expect a tendency toward shorter times between failures as the system ages. A Nonhomogeneous Poisson Process(NHPP) can be used in both situations. If the intensity function is decreasing, it provides a model for reliability growth and if increasing, a model for deteriorating system.

For an NHPP, the probability of an event in a small interval of time, $(t,\ t+\Delta t)$ depends only on t and not on the previous pattern of events. The limit,

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{P(an\, event \in (t, t+\Delta t))}{\Delta}, \quad \Delta t > 0,$$

is defined the intensity function of an NHPP. If the Poisson process is used to model the failure times of a repairable system, then the fact that λ depends on t and not on the previous pattern of failures means that a failed unit is in exactly the same condition after a repair as it was just before the failure. For this reason, the NHPP is often called the same-as-old model when it is applied to a repairable system. The homogeneous Poisson Process (HPP), for which the intensity function is a constant and the times between failures are independent and identically exponentially distributed, is a special case of both.

If the intensity function has the fo

$$\lambda(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}, t > 0,$$

then the process is called the power law process. This process has been called the Weibull process. Typically, the parameters β and θ are unknown and must be estimated from data from one or more systems. Crow (1974) developed many of the inference properties of the power law process. Rigdon and Basu(1989) presented a review of properties of the power law process.

The renewal process is a model that is often used to describe the random occurrence of events in time. For a renewal process, the times between failure are independent and identically distributed(i.i.d). Since the times between failure are iid, a repaired unit is always brought to a like-new condition. For this reason, the renewal process cannot be used to model a system experiencing deterioration or reliability improvement. The renewal process has also been called the as-good-as-new model.

Unfortunately, NHPP and renewal process are based on too restrictive assumptions on the effect of the repair action. For these reasons, several authors have recently proposed point process models which incorporate both renewal type behavior and time trend. The Modulated Power Law Process(MPLP) is compromise between the Renewal process and the NHPP with power intensity function, since its failure probability at a given time t depends both on the age t of a system and on the distance of t from the last failure time. Thus the MPLP is a suitable model for describing the failure pattern of repairable systems when both renewal-type behavior and time trend are present.

Lakey and Rigdon (1992 a) have introduced the MPLP which is special case of the inhomogeneous gamma process introduced by Bean (1981). Lakey and Rigdon (1992 b) propose Maximum likelihood point estimators of the three parameters of MPLP. Black and Rigdon (1996 a) describe an algorithm for obtaining the MLEs of the three parameters and derive the asymptotic variance of these point estimates.

In reliability analysis, it is quite important to be able to determine the next failure time after the system has experienced some failures during the development and test process. In order words, the mean time to the next failure at the n-th observed failure time t_n, or the Mean Time Between Failures at t_n, MTBF(t_n), is of significant interest.

In this paper we derive Mean Time Between Failure (MTBF) for the MPLP model and propose estimation of Mean Time Between Failure (MTBF) at the n-th failure time. Numerical examples illustrate the estimation procedure.

2. Mean Time Between Failure for Modulated Power Law Process

Suppose that a system failure occurs not at every shock but at every k-th shock, where k is a positive integer and suppose that shocks occur according to the NHPP with intensity function:

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{P(a\,shock \in (t, t+\Delta t))}{\Delta t}, \quad t > 0.$$

If for example $k = 3$, then the system failure occur at every third shock. Thus, a failed and repaired system would be in better condition than it was in before the failure occurrence, since 3 other shocks must occur in order to observe the next failure. Even if the explanation given previously in terms of shocks does not carry over when k is not an integer, the MPLP can be still defined for any positive value of k.

Let $T_1 < T_2 < \cdots < T_n$ denote the first n failure times of a failure truncated MPLP. The conditional reliability function of T_i given T_{i-1}, is;

$$\begin{aligned} R(t_i|t_{i-1}) &= \Pr(T_i > t | T_{i-1} = t_{i-1}) \\ &= \Pr(no\ failure\ in\ the\ interval[t_{i-1}, t]) \\ &= \Pr(N(t) - N(t_{i-1}) \le k-1) \\ &= \sum_{j=0}^{k-1} \frac{(U(t) - U(t_{i-1}))^j}{j!} \exp[-\mathrm{U}(\mathrm{t}) + \mathrm{U}(\mathrm{t}_{i-1})], \end{aligned}$$

where $U(t) = \int_0^t \lambda(u)du$ is mean value function of NHPP. Thus the conditional pdf of T_i given T_{i-1} can be easily computed from the above conditional reliability function as follows;

$$f(t_i|t_{i-1}) = \frac{1}{\Gamma(k)} \frac{\beta}{\theta} \left(\frac{t_i}{\theta}\right)^{\beta-1} \left[\left(\frac{t_i}{\theta}\right)^{\beta} - \left(\frac{t_{i-1}}{\theta}\right)^{\beta}\right]^{k-1} \exp\left[-\left(\frac{t_i}{\theta}\right)^{\beta} + \left(\frac{t_{i-1}}{\theta}\right)^{\beta}\right]. \tag{1}$$

Then the joint pdf of failure times $T_1 < T_2 < \cdots < T_n$, is;

$$f(t_1, t_{2,\ldots,} t_n | \beta, \theta, k) = \frac{1}{(\Gamma(k))^n} \frac{\beta^n}{\theta^{n\beta k}} exp\left[-\left(\frac{t_n}{\theta}\right)^{\beta}\right] \prod_{i=1}^{n} t_i^{\beta-1} \prod_{i=1}^{n} (t_i^{\beta} - t_{i-1}^{\beta})^{k-1}.$$

When $k=1$ the MPLP reduces to the PLP, when $\beta=1$ the process becomes a Gamma renewal process, when $k=1$ and $\beta=1$ the MPLP reduces to the HPP. Thus β is a measure of the system improvement or deterioration over the system life, whereas k is a measure of the improvement of worsening introduced by the repair actions.

In reliability analysis, one of important characteristics is the next failure time after the system has experienced some failures. In order words, the mean time to the next failure at the n-th observed failure time t_n. Now we derive Mean Time Between Failure (MTBF) for the MPLP model. Let $T_1 < T_2 < \cdots < T_n < \cdots$ be the successive system failure times. Then the MTBF at the n-th failure time t_n is defined as;

$$MTBF(t_n) = E[T_{n+1} - T_n | T_n = t_n].$$

In order to express the $MTBF(t_n)$ more explicitly, we need the conditional distribution of T_{n+1} given $T_n = t_n$. From (1), the conditional pdf of T_{n+1}, given $T_n = t_n$, is

$$f_{n+1}(t \mid t_1, \cdots, t_n) = \frac{1}{\Gamma(k)} \frac{\beta}{\theta} \left(\frac{t}{\theta}\right)^{\beta-1} \left[\left(\frac{t}{\theta}\right)^{\beta} - \left(\frac{t_n}{\theta}\right)^{\beta}\right]^{k-1} \exp\left[-\left(\frac{t}{\theta}\right)^{\beta} + \left(\frac{t_n}{\theta}\right)^{\beta}\right].$$

Then MTBF at $T_n = t_n$ for the MPLP is given by

$$\begin{aligned} MTBF(t_n) &= E[T_{n+1} - T_n | T_n = t_n] \\ &= \int_{t_n}^{\infty} t f_{n+1}(t|t_1,\cdots,t_n)dt - t_n \\ &= \int_{t_n}^{\infty} t \frac{1}{\Gamma(k)} \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}\left[\left(\frac{t}{\theta}\right)^{\beta} - \left(\frac{t_n}{\theta}\right)^{\beta}\right]^{k-1} \exp\left[-\left(\frac{t}{\theta}\right)^{\beta} + \left(\frac{t_n}{\theta}\right)^{\beta}\right]dt - t_n. \end{aligned}$$

Using integral by parts, we get

$$MTBF(t_n) = \frac{\beta}{\Gamma(k)}\int_{t_n}^{\infty}\left(\frac{t}{\theta}\right)^{\beta}\left[\left(\frac{t}{\theta}\right)^{\beta} - \left(\frac{t_n}{\theta}\right)^{\beta}\right]^{k-1}\exp\left[-\left(\frac{t}{\theta}\right)^{\beta} + \left(\frac{t_n}{\theta}\right)^{\beta}\right]dt - t_n.$$

Let $y = (t/\theta)^{\beta -} (t_n/\theta)^{\beta}$, then

$$MTBF(t_n) = \frac{\theta}{\Gamma(k)}\int_{0}^{\infty} y^{k-1}\left[y + \left(\frac{t_n}{\theta}\right)^{\beta}\right]^{\frac{1}{\beta}}\exp(-y)dy - t_n. \quad (2)$$

3. Maximum Likelihood Estimation

Let $t_1 < t_2 < \cdots < t_n$ denote the first n failure times of a failure truncated MPLP sample. Then the likelihood function, given failure times $t_1 < t_2 < \cdots < t_n$, is;

$$L(\beta,\theta,k|t_1,t_2,\cdots,t_n) = \frac{1}{(\Gamma(k))^n}\frac{\beta^n}{\theta^{n\beta k}} exp\left[-\left(\frac{t_n}{\theta}\right)^{\beta}\right]\prod_{i=1}^{n} t_i^{\beta-1}\prod_{i=1}^{n}(t_i^{\beta} - t_{i-1}^{\beta})^{k-1}.$$

Thus the log-likelihood function corresponding to the MPLP model is determined by

$$\begin{aligned} l(\theta,\beta,k|t_1,\cdots,t_n) &= \ln L(\beta,\theta,k|t_1,t_2,\cdots,t_n) \\ &= -n\ln\Gamma(k) + n\ln\beta - n\beta k\ln\theta - \left(\frac{t_n}{\theta}\right)^{\beta} \\ &\quad + (\beta-1)\sum_{i=1}^{n}\ln t_{i+}(k-1)\sum_{i=1}^{n}\ln(t_i^{\beta} - t_{i-1}^{\beta}). \end{aligned}$$

Using partial derivative, we obtain the likelihood equations

$$\frac{\partial}{\partial\theta}l = -\frac{n\beta k}{\theta} + \frac{\beta}{\theta}\left(\frac{t_n}{\theta}\right)^{\beta}$$
$$\frac{\partial}{\partial\beta}l = \frac{n}{\beta} - nk\ln\theta - \left(\frac{t_n}{\theta}\right)^{\beta}\ln\frac{t_n}{\theta} + \sum_{i=1}^{n}\ln t_i + (k-1)\sum_{i=1}^{n}\frac{t_i^{\beta}\ln t_i - t_{i-1}^{\beta}\ln t_{i-1}}{t_i^{\beta} - t_{i-1}^{\beta}}$$
$$\frac{\partial}{\partial k}l = -n\frac{\Gamma'(k)}{\Gamma(k)} - n\beta\ln\theta + \sum_{i=1}^{n}\ln(t_i^{\beta} - t_{i-1}^{\beta}).$$

Let $\boldsymbol{\theta} = (\theta,\beta,k)'$ denote the 3-dimensional column vector with parameters, $S = S(\boldsymbol{\theta})$ denote the score function of $l(\theta,\beta,k)$, that is, the 3-dimensional column vector with entries $\partial l/\partial\theta, \partial l/\partial l/\beta, \partial l/\partial k$ and let $H = H(\boldsymbol{\theta})$ denote the Hessian of $l(\theta,\beta,k)$, the 3×3 matrix with entries $\partial^2 l/\partial\theta^2, \partial^2 l/\partial\theta\beta, \partial^2 l/\partial\theta\partial k, \partial^2 l/\partial\beta^2, \partial^2 l/\partial\beta\partial k, \partial^2 l/\partial k^2$. The maximum likelihood equation is $S(\boldsymbol{\theta}) = 0$. We use Newton-Raphson method with step-halving for computing $\hat{\boldsymbol{\theta}}$ to start with an initial guess $\hat{\boldsymbol{\theta}}^{(0)}$ and iteratively determine $\hat{\boldsymbol{\theta}}^{(m+1)}$ by the formula

$$\hat{\boldsymbol{\theta}}^{(m+1)} = \hat{\boldsymbol{\theta}}^{(m)} - \frac{1}{2^M}H^{-1}(\hat{\boldsymbol{\theta}}^{(m)})S(\hat{\boldsymbol{\theta}}^{(m)})$$

where M is the smallest nonnegative integer such that

$$l\left\{\hat{\boldsymbol{\theta}}^{(m)} - \frac{1}{2^M}H^{-1}(\hat{\boldsymbol{\theta}}^{(m)})S(\hat{\boldsymbol{\theta}}^{(m)})\right\} \geq l\left\{\hat{\boldsymbol{\theta}}^{(m)} - \frac{1}{2^{M+1}}H^{-1}(\hat{\boldsymbol{\theta}}^{(m)})S(\hat{\boldsymbol{\theta}}^{(m)})\right\}.$$

We stop the iterations when $l(\hat{\boldsymbol{\theta}}^{(m+1)}) - l(\hat{\boldsymbol{\theta}}^{(m)}) \leq 10^{-6}$.

We now propose an estimator of $MTBF(t_n)$ from (2):

$$\widehat{MTBF}(t_n) = \frac{\hat{\theta}}{\Gamma(\hat{k})}\int_0^{\infty} y^{\hat{k}-1}\left[y + \left(\frac{t_n}{\hat{\theta}}\right)^{\hat{\beta}}\right]^{\frac{1}{\hat{\beta}}}\exp(-y)dt - t_n. \tag{3}$$

where $\hat{\theta}, \hat{\beta}$ and $\hat{k}$ are the MLE of θ,β,k, respectively.

4. Examples

Two numerical examples illustrate the proposed estimation procedure. The first is the failure times of an aircraft generator taken from Duane (1964). These failure times have been read from a plot in Duane's paper by Black and Rigdon (1996 b). For this system there were n=14 failures, and these failure times are shown in Table 4.1.

Table 4.1. Failure Times for Aircraft Generator in Duane (1964)

10	55	166	205	341	488	567	731	1308	2050	2453	3115	4017	4595

The point estimates for the parameters are $\hat{\theta}=0.218$, $\hat{\beta}=0.420$ and $\hat{k}=4.800$. Thus, from (3), the estimate of MTBF is $\widehat{MTBF}(4596)=672$ and the estimated mean time to the next failure at $t=4596$ is 5268.

The second example consists of the failure times from the second aircraft airconditioning unit from Proschan (1963). The failure times are shown in Table 4.2.

Table 4.2. Failure Times of Aircraft Airconditioning Equipment given by Proschan (1963)

413	427	485	522	622	687	696	865	1496	1532	1733
1851	1885	1916	1934	1952	2019	2076	2145	2167	2201	

The point estimates for the parameters are $\hat{\theta}=338.7$, $\hat{\beta}=1.72$ and hat k = 1.1. Thus, $\widehat{MTBF}(2201)=54$ that is, the mean time to the next failure at $t=2201$ is 2255.

References

1. Bean, M. (1981). Inhomogeneous and modulated gamma processes. *Biometrika*, 68, pp. 143-152.
2. Black, S. E. and Rigdon, S. E. (1996). Statistical inference for a modulated power law process, *Journal of Quality Technology*, 28, 1, 81-90.
3. Crow, L. (1974). Reliability Analysis of Complex, Repairable Systems, *In Reliability and Biometry* ,(Eds., F. Proschan and R. J. Serfling), 379-410, SIAM, Philadelpia.
4. Duane, J. T. (1964). Learning Curve Approach to Reliability, *IEEE Transactions on Aerospace*, V2, 563-566.
5. Lakey, M. J. and Rigdon, S. E. (1992). The modulated power law process, *Proc. 45th Annual Quality Congress*, Milwaukee 1992, 559-563.
6. Proschan, F. (1963). Theoritical Explanation of Observed Decreasing Failure Rate, *Technometrics*, 5, pp. 375-383.
7. Rigdon, S. E. and Basu, A. P.(1989). The power law process: A model for the reliability of repairable systems, *Journal of Quality Technology*, 20, pp. 251-260.

[received date : Apr. 2007, accepted date : May. 2007]