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Reliability In a Half-Triangle Distribution and a Skew-Symmetric Distribution

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Abstract

We consider estimation of the right-tail probability in a half-triangle distribution, and also consider inference on reliability, and derive the k-th moment of ratio of two independent half-triangle distributions with different supports. As we define a skew-symmetric random variable from a symmetric triangle distribution about origin, we derive its k-th moment.

Keywords: Half-Triangle Distribution, Reliability, Skew-Symmetric Distribution

1. Introduction

For two independent random variables X and Y and a real number c, the probability P(X<cY) induces the following facts, (i) the probability P(X<cY) is the reliability when the real number c equals one, (ii) the probability P(X<cY) is the distribution of the ratio X/(X+Y) when c=t/(1-t) for 0<t<1, and (iii) the probability P(X<cY) induces the density of a skewed-symmetric random variable if X and Y are symmetric random variable about origin.

Many authors have considered inferences on the reliability in several continuous distributions. And in recent Kim(2006), Lee(2006), and Lee & Won(2006) studied inferences on the reliability in an exponentiated uniform distribution and an exponential distribution. Ali, et al(2006) studied distribution of the ratio of generalized uniform variates.

A triangle distribution was applied to a kernel function in non-parametric density estimation, but here we shall consider a half-triangle random variable X

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with its support $(0,\theta)$. The pdf of a half-triangle random variable is given by:

$$f(x;\theta) = \frac{2}{\theta^2}(\theta - x), \quad 0 < x < \theta \quad \text{(see Rohatgi(1976))}. \qquad (1.1)$$

For example. when a gas station derives its supply of oil-gas once per θ -days, the sales quantity X of the gas during the term of θ -days follows a half-triangle distribution. For a special day, supposing we'll get the probability which the supplied oil-gas has been drained is small, what capacity of oil-gas tank does the gas station need? So, in the case we can consider the right tail probability of a half-triangle distribution and also consider the reliability P(Y<X) of two independent half-triangle distributions. For another example, if an selling income(Y) of a selling agency per a new car at a motor vehicle shop is $Y = X^2$, then X is assumed by a half-triangle random variable. And if X is a half-triangle random variable with support, $(0,\theta)$, $1 - X/\theta$ follows a power function distribution over $(0,1)$.

In this paper we consider estimation of the right-tail probability in a half-triangle distribution, and also consider inference on reliability P(Y<X), and derive k-th moment of ratio X/(X+Y) for two independent half-triangle distributions with different supports. As we define a skew-symmetric random variable from a symmetric triangle distribution about origin which is based on a half-triangle random variable, and hence we derive its k-th moment.

2. Estimating Reliability

2.1 The right-tail probability

From the density (1.1) of the half-triangle random variable X, the followings are well-known:

k-th moment: $E(X^k) = \frac{2}{(k+1)(k+2)}\theta^k, k = 1,2,\ldots$

the cdf: $F(x) = \frac{2}{\theta^2}x(\theta - \frac{x}{2}), \quad 0 < x < \theta$.

From which, its right-tail probability of the half-triangle random variable is given by: $R(t) = P(X > t) = 1 - 2(\theta \cdot t - t^2/2)/\theta^2. \quad 0 < t < \theta$ (2.1)

Assume $X_1, X_2, \ldots, X_m$ be a sample from the half-triangle distribution with density (1.1). Then, since the MLE $\hat{\theta}$ of θ is the greatest order statistics $X_{(m)}$, and hence, by a power of the binomial and the density of $X_{(m)}$, k-th moment of $X_{(m)}$ can be obtained as;

$$E(X^k_{(m)}) = c(m,k) \cdot \theta^k, k \text{ is an integer} \tag{2.2}$$

where $c(m,k) \equiv m \sum_{i=0}^{m-1} (-1)^i 2^{m-i} \binom{m-1}{i} / ((m+k+i)(m+k+i+1))$, if $m > -k$.

From the moment (2.2), we define an unbiased estimator of θ as follows:

$$\tilde{\theta} = X_{(m)}/c(m,1) \text{ is an unbiased estimator of } \theta \ . \tag{2.3}$$

From the MLE $\hat{\theta}$ and an unbiased estimator $\tilde{\theta}$ of θ in (2.3), the MLE $\widehat{R(t)}$ and a proposed estimator $\widetilde{R(t)}$ of the right-tail probability R(t) in the half-triangle distribution are given by:

$$\widehat{R(t)} = 1 - 2t\,(1/X_{(m)} - t/(2X^2_{(m)}))$$
$$\widetilde{R(t)} = 1 - 2t\,(c(m,1).X_{(m)} - t(c(m,1))^2/(2X^2_{(m)})) \tag{2.4}$$

From two right-tail probability estimators (2.4) and k-th moment (2.2), we obtain expectations and variances of $\widehat{R(t)}$ and $\widetilde{R(t)}$:
For m>2,

$$E(\widehat{R(t)}) = 1 - 2t\,[c(m,-1) \cdot \theta^{-1} - t \cdot c(m.-2) \cdot \theta^{-2}/2] \ .$$
$$E(\widetilde{R(t)}) = 1 - 2t\,[c(m,1) \cdot c(m,-1) \cdot \theta^{-1} - t \cdot c(m,1)^2 \cdot c(m.-2) \cdot \theta^{-2}/2] \ ,$$
$$VAR(\widehat{R(t)}) = 4t^2[t^2 \cdot c(m,-4) \cdot \theta^{-4}/4 - t \cdot c(m,-3) \cdot \theta^{-3} + c(m,-2) \cdot \theta^{-2} - (c(m,-1) \cdot \theta^{-1} - t \cdot c(m,-2) \cdot \theta^{-2}/2)^2] \ .$$
$$VAR(\widetilde{R(t)}) = 4t^2[\theta^{-4}t^2 \cdot c^4(m,1)/(4 \cdot c(m,-4)) - t \cdot c^3(m,1) \cdot \theta^{-3}/c(m,-3) + \frac{c^2(m,1)}{c(m,-2)} \cdot \theta^{-2} - (c(m,1) \cdot c(m,-1) \cdot \theta^{-1} - t \cdot c^2(m,1) \cdot c(m,-2) \cdot \theta^{-2}/2)^2] \ .$$

From means and variances of $\widehat{R(t)}$ and $\widetilde{R(t)}$, Table 1 shows numerical values of mean squared errors(MSE) of $\widehat{R(t)}$ and $\widetilde{R(t)}$ when n=10,20,30, $\theta = 10$, t=(8,9) and $\theta = 20$, $t = (18,19)$.

Table 1. Mean squared errors of $\widehat{R(t)}$ and $\widetilde{R(t)}$

m / t		$\theta=10$		$\theta=20$	
		8	9	18	19
10	$\widehat{R(t)}$	0.06807	0.17161	0.17161	0.25645
	$\widetilde{R(t)}$	0.00763	0.01655	0.01655	0.02490
20	$\widehat{R(t)}$	0.00259	0.00911	0.00909	0.01763
	$\widetilde{R(t)}$	0.00117	0.00103	0.00103	0.00158
30	$\widehat{R(t)}$	0.00113	0.00184	0.00184	0.00446
	$\widetilde{R(t)}$	0.00075	0.00035	0.00035	0.00042

From Table 1, we observe the following:
Fact 1. The proposed estimator $\widetilde{R(t)}$ performs better than the MLE $\widehat{R(t)}$ in a sense of MSE when n=10,20,30, $\theta=10$ and $t=(8,9)$, and $\theta=20$ and $t=(18,19)$.

2.2. Estimating reliability P(Y<X)

Here assume we consider an inference on reliability P(Y<X) of two independent half-triangle random variables X, Y with different supports $(0,\theta_1)$ and $(0.\theta_2)$, respectively, then the reliability R=P(Y<X) can be obtained as:
If $0<\rho\equiv\theta_1/\theta_2<1$, then $R(\rho)\equiv R=P(Y<X)=-\rho^2/6+2\rho/3$,
and hence its reliability $R(\rho)$ is a monotone increasing function of $0<\rho<1$.

Because $R(\rho)$ is an increasing function of ρ on $(0,1)$, an inference on the reliability is equivalent to an inference on ρ (see McCool(1991)). After hence we consider inference on $\rho\equiv\theta_1/\theta_2$ when the $\theta_i's$ is parameter in the density (1.1), instead of estimating R=P(Y<X).

Remark 1. If $\rho>1$, then, for $\eta\equiv\theta_2/\theta_1$ $P(X<Y)=-\eta^2/6+2\eta/3$ is a monotone increasing function of $0<\eta<1$, and hence we can consider inference on reliability on $\eta\equiv\theta_2/\theta_1$ by the same method.

Assume $X_1,X_2,\ldots,X_m$ and $Y_1,Y_2,\ldots,Y_n$ be two independent samples from X and Y with the density (1.1) having its supports $(0,\theta_1)$ and $(0,\theta_2)$, respectively. Then we define the following three estimators of ρ as:

$$\text{The MLE of } \rho : \hat{\rho}=\hat{\theta}_1/\hat{\theta}_2=X_{(m)}/Y_{(n)}$$

An unbiased estimator of ρ : $\tilde{\rho}=X_{(m)}/(c(m,1)\cdot c(n,-1)\cdot Y_{(n)})$.
A proposed estimator of ρ : $\bar{\rho}=(c(n,1)\cdot X_{(m)})/(c(m,1)\cdot Y_{n)})$,

where $c(m,k) \equiv m \cdot \sum_{i=0}^{m-1} (-1)^i 2^{m-i} \binom{m-1}{i} \frac{1}{(m+k+i)(m+k+i+1)}$,if $m > -k$.

Based on three estimators and densities of the greatest order statistics $X_{(m)}$ and $Y_{(n)}$, we can obtain the following expectations and mean squared errors(MSE′s) of $\hat{\rho}, \tilde{\rho}, and\ \bar{\rho}$:

Fact 2. If n>2. then expectations and MSE′s of $\hat{\rho}, \tilde{\rho}, and\ \bar{\rho}$ are :
(a) $E(\hat{\rho}) = c(m,1) \cdot c(n,-1) \cdot \rho$.
(b) $E(\bar{\rho}) = c(n,1) \cdot c(n,-1) \cdot \rho$.
(c) $MSE(\hat{\rho}) = [c(m,2) \cdot c(n,-2) - 2 \cdot c(m,1) \cdot c(n,-1) + 1]\rho^2$.
(d) $MSE(\tilde{\rho}) = [c(m,2) \cdot c(n,-2)/(c^2(m,1) \cdot c^2(n,-1)) - 1]\rho^2$.
(e) $MSE(\bar{\rho}) = [c^2(n,1) \cdot c(m,2) \cdot c(n,-2)/c^2(m,1) - 2 \cdot c(m,1) \cdot c(n,-1) + 1]\rho^2$.

From expectations and MSE of $\hat{\rho}, \tilde{\rho}, \bar{\rho}$ in Fact 3, Table 2 shows the numerical values of MSE′s of $\hat{\rho}, \tilde{\rho}$, and $\bar{\rho}$, when m and n are 10, 20, 30.

Table 2. Mean squared errors of $\hat{\rho}, \tilde{\rho}$, and $\bar{\rho}$ (unit: ρ^2)

m	n	$\hat{\rho}$	$\tilde{\rho}$	$\bar{\rho}$
10	10	0.09477	0.08592	0.09477
	20	0.05199	0.05325	0.05535
	30	0.05024	0.04569	0.04666
20	10	0.10996	0.06650	0.07373
	20	0.03588	0.03442	0.03589
	30	0.02632	0.02699	0.02760
30	10	0.12676	0.06104	0.06781
	20	0.03637	0.02912	0.03039
	30	0.02224	0.02174	0.02224

From Table 2, we observe the following:
Fact 3. An unbiased estimator $\tilde{\rho}$ performs better in a sense of MSE than other two estimators $\hat{\rho}$ and $\bar{\rho}$ unless (m, n)=(10,20) and (m, n)=(20,30), when m and n are 10, 20, 30.

From Fact 3, because three proposed estimators can′t dominate each other, we can recommend another biased estimator which mean squared error is minimized. A biased estimator $\hat{\hat{\rho}}$ is defined by:

$$\hat{\hat{\rho}} \equiv \frac{c(m,1) \cdot c(n,-1)}{c(m,2) \cdot c(n,-2)} \cdot \frac{X_{(m)}}{Y_{(n)}} ,$$

and, hence $MSE(\hat{\hat{\rho}}) = \dfrac{c(m,2)c(n,-2) - c(m,1)c(n,-1)}{c(m,2)c(n,-2)} \cdot \rho^2$,

Therefore, we can obtain:
Fact 4. The biased estimator $\hat{\hat{\rho}}$ has less MSE than other three estimators.

From the quotient density in Rohatgi(1976, p.141), independence of X′s and Y′s, and the densities of the greatest order statistics $X_{(m)}$ and $Y_{(n)}$, the quantity $Q = \dfrac{X_{(m)}/\theta_1}{Y_{(n)}/\theta_2}$ is a pivot quantity having the following density:

$$f_Q(x) = \begin{cases} \frac{2}{3} - \frac{1}{3}x. & \text{if } 0 < x < 1 \\ \frac{2}{3}x^{-2} - \frac{1}{3}x^{-3}, & \text{if } x \geqq 1 \end{cases} .$$

From the density of the quantity Q, $(1-p_1-p_2)100\%$ confidence interval of ρ is given by: $\left(\dfrac{1}{u(p_2)} \cdot \dfrac{X_{(m)}}{Y_{(n)}}, \dfrac{1}{l(p_1)} \cdot \dfrac{X_{(m)}}{Y_{(n)}}\right)$,

where $\int_0^{l(p_1)} \frac{2}{3} - \frac{1}{3}x\,dx = p_1, \quad \int_{u(p_2)}^{\infty} \frac{2}{3}x^{-2} - \frac{1}{3}x^{-3}\,dx = p_2$.

Remark 2. For given small positive numbers p_1 and p_2 with $0 < 1 - p_1 - p_2 < 1$,

$$l(p_1) = 2 - \sqrt{4 - 6p_1} \text{ and } u(p_2) = \frac{1}{6p_2}(2 + \sqrt{4 - 6p_2}) .$$

Next we consider the following null hypothesis :

$$H_0 ; \theta_1 = \theta_2 \quad against \quad H_1 : \theta_1 \neq \theta_2 (\text{or } \theta_1 < \theta_2 . \text{or } \theta_1 > \theta_2) .$$

As applying the likelihood ratio test of size α in Rohatgi(1976, p.436), we apply the following corresponding critical region:

$$\frac{X_{(m)}}{Y_{(n)}} < l\left(\frac{\alpha}{2}\right) \equiv 2 - \sqrt{4 - 3\alpha} \quad \text{or} \quad \frac{X_{(m)}}{Y_{(n)}} > u\left(\frac{\alpha}{2}\right) \equiv \frac{1}{3\alpha}(2 + \sqrt{4 - 3\alpha})$$

3. Distribution of a ratio

In this section we consider distributions of product, quotient, and ratio of two independent half-triangle random variables X and Y having the density (1.1) with different support θ_1 and θ_2, respectively .

From the product density, quotient density, and independence of X and Y in

Rohatgi(1976, p.141), the densities of V=XY and W=Y/X can be obtained by:

$$f_V(v) = \frac{4}{\theta_1^2\theta_2^2}[(\theta_1\theta_2 + v)\cdot \ln\frac{\theta_1\theta_2}{v} - 2(\theta_1\theta_2 - v)], \text{ if } 0 < v < \theta_1\theta_2 \ .$$

$$f_W(w) = \begin{cases} -\frac{1}{3}\cdot\frac{\theta_1^2}{\theta_2^2}\cdot w + \frac{2}{3}\cdot\frac{\theta_1}{\theta_2}, \text{if } 0 < w < \frac{\theta_2}{\theta_1} \\ \frac{2}{3}\cdot\frac{\theta_2}{\theta_1}\cdot\frac{1}{w^2} - \frac{1}{3}\cdot\frac{\theta_2^2}{\theta_1^2}\cdot\frac{1}{w^3}. \text{ if } \frac{\theta_2}{\theta_1} \leqq w \end{cases} . \tag{4.1}$$

Since R=X/(X+Y)=1/(1+W) is a function of W, by using the quotient density (4.1) we derive the density of ratio R=X/(X+Y) of two independent half-triangle random variables X and Y having the density (1.1) with different supports θ_1 and θ_2, respectively.

$$f_R(x) = \begin{cases} -\frac{1}{3}\cdot\frac{\theta_1^2}{\theta_2^2}\cdot(1-t)t^{-3} + \frac{2}{3}\cdot\frac{\theta_1}{\theta_2}t^{-2}, \text{ if } \frac{\theta_1}{\theta_1+\theta_2} < t < 1 \\ \frac{2}{3}\cdot\frac{\theta_2}{\theta_1}(1-t)^{-2} - \frac{1}{3}\cdot\frac{\theta_2^2}{\theta_1^2}\cdot t\cdot(1-t)^{-3}, \text{if } 0 < t \leqq \frac{\theta_1}{\theta_1+\theta_2} \end{cases} \tag{4.2}$$

From the density (4.2) of the ratio R=X/(X+Y), we can obtain mean and variance of ratio R=X/(X+Y) of two independent half-triangle random variables X and Y having the density (1.1) with different support θ_1 and θ_2, respectively .

For $\rho = \theta_1/\theta_2$,

$$E(R) = (\frac{1}{3}\rho^2 + \frac{2}{3}\rho)\ln\frac{1+\rho}{\rho} - (\frac{1}{3}\rho^{-2} + \frac{2}{3}\rho^{-1})\ln(1+\rho) + \frac{1}{3}(\rho^{-1} - \rho) + \frac{1}{2} \ ,$$

and

$$Var(R) = \frac{1}{3}\rho^2\cdot\ln\frac{\rho}{1+\rho} - (\rho^{-2} + \frac{4}{3}\rho^{-1})\cdot\ln(1+\rho) + (\frac{1}{3}\rho + \frac{1}{2})\frac{\rho}{1+\rho} - (\frac{1}{2}\rho^{-2} + \frac{2}{3}\rho^{-1})\cdot\frac{1}{1+\rho} + (\frac{1}{2}\rho^{-1} + \frac{2}{3})\cdot\frac{1+\rho}{\rho} - E^2(R) \ .$$

4. A skew-symmetric distribution.

Based on the density (1.1) of a half-normal random variable, two independent random variables X and Y are defined as the same density:

$$G'(x) = g(x) = (\theta - |x|)/\theta^2, \quad -\theta < x < \theta, \ \theta > 0, \tag{5.1}$$

which is a symmetric triangle density about zero, and it has mean zero and variance $\theta^2/6$.

From our introduction of deriving a skewed density based on a symmetric density about zero in section 1, for any real number c,

$$g(z;\, c) = 2 \cdot g(z) \cdot G(cz)\ , \tag{5.2}$$

which is a skewed density g(z;c) of a continuous random variable Z (see in Ali and Woo(2006)).

From the symmetric density (5.1) we derive the following cdf G(x) of X:

$$G(x) = (-sgn(x) \cdot x^2/2 + \theta \cdot x + \theta^2/2)/\theta^2, \quad -\theta < x < \theta, \tag{5.3}$$

where $sgn(x) = \begin{cases} 1. & \text{if } x \geqq 0 \\ -1, & \text{if } x < 0 \end{cases}$.

Lemmas 2 & 3 in Ali and Woo(2006) will be introduced as follow:.

Lemma 1. If g(z;c)= 2g(z)G(cz) is a skewed density of a continuous random variable Z which is denoted by $Z \sim SD(c)$ and G(z;c) is the cdf of Z, then
(a) $Z \sim SD(c) \Leftrightarrow -Z \sim SD(-c)$ for every real number c. For the skew-symmetric distribution G(z;c), G(z;-c) =1-G(-z;c).
(c) Let $S(z;c) \equiv \int_z^{\infty} \int_0^{ct} g(t)g(s)dsdt$. Then G(z;c)=G(z)-2S(z;c), S(z;c)=-S(z;-c), and S(-z;c) =S(z;c).

From the densities (5.1) & (5.2) and the distribution function (5.3), we get the following skewed density g(z;c) of a random variable Z:

$$g(z;\, c) = (\theta - |\, z\, |)(-sgn(cz)c^2z^2 + 2c\theta \cdot z + \theta^2)/\theta^4, \quad -\theta < z < \theta.\ c \in R^1 \tag{5.4}$$

which becomes a skewed density.

From the above Lemma 1, it's sufficient for us to consider the following skew-symmetric distribution function of Z:

$$G(z;c) = G(z) - 2 \cdot S(z;c), \tag{5.5}$$

From the above iterated integral (5.5) in S(z;c) and the density (5.1), we obtain the S(z;c):
For any z>0 and c>0,

$$S(z;c) = [(4-c)c\theta^4/24 - c\theta^2 z^2 + (2+c)c\theta z^3/6 - c^2 z^4/8]/\theta^4 \qquad (5.6)$$

From the cdf G(x) in (5.3) and the integral S(z;c) in (5.6), the skew-symmetric distribution G(z;c) of the skew-symmetric random variable Z is obtained as;

$$G(z;c) = (-sgn(z) \cdot z^2/2 + \theta \cdot z + \theta^2/2)/\theta^2 - 2[(4-c)c\theta^4/24 - c\theta^2 z^2 + (2+c)c\theta z^3/6 - c^2 z^4/8]/\theta^4, \quad z > 0,\ c > 0.$$

Remark 3, If Z<0 and c<0, then we can obtain the skew-symmetric distribution of Z from the results in Lemma 1.

For c>0, from the skewed density (5.4) we can obtain k-th moment of the skew-symmetric random variable Z as below:

$$E(Z^k;c) = \theta^k[(1+(-1)^k)/(k+1) - (1+(-1)^k)/(k+2) + 2c(1-(-1)^k)/(k+2) + \\ + 2c(-1+(-1)^k)/(k+3) + c^2(-1+(-1)^k)/(k+3) + c^2(1-(-1)^k)/(k+4)].$$

Remark 4. If c<0, then $E(Z^k\,;c) = (-1)^k E(z^k\,;-c)$, from Lemma 1(a).

From k-th moment of Z, we can derive mean and variance of a skew-symmetric random variable Z:

$$E(Z\,;c) = \theta(-c^2/10 + c/3), \quad c > 0, \qquad (5.7)$$

$Var(Z\,;c) = \theta^2[1/6 - (-c^2/10 + c/3)^2]$, if $0 \leqq c < 5/3 + \sqrt{25 + 90/\sqrt{6}}\,/3 \fallingdotseq 4.2858766$
From the k-th moment (5.7) and Remark 4, we evaluate means, variances and the coefficient of skewness as below:

a. If c=±1/2, then mean=±0.1467, variance=0.14660, and skewness=∓2.56049
b. If c=±3/5, then mean=±0.1640, variance=0.13977, and skewness=∓3.25945
c. If c=±1.0, then mean=±0.2333, variance=0.11222, and skewness=∓7.24083
d. If c=±5/3, then mean=±0.2778, variance=0.08906, and skewness=∓16.8242
e. If c=±2.0, then mean=±0.2667, variance=0.09556, and skewness=∓18.7063

where the signs of c-values and skewness preserve the same order.

Hence if c-values are positive, then the density g(z;c) in (5.4) is skewed on the left. And if c-values are negative, then the density g(z;c) in (5.4) is skewed on the right.

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